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KOTTWITZ-RAPOPORT STRATA IN THE SIEGEL MODULI SPACES

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Abstract. In this note we give a survey on results concerning the Siegel moduli spaces with parahoric level structure and the Kottwitz-Rapoport stratification, due to many people. We also report some aspects of KR strata in higher dimensional cases, which are obtained jointly with U. Görtz.

1. Introduction

This is the note of the talk the author gave in the conference "Géométrie arithmétique, représentations galoisiennes et formes modulaires" held in June of 2007, at Université Paris-Nord. The main purpose of this note is to introduce the geometry of the reduction modulo p of some Siegel modular varieties with a small level at p.

Siegel modular varieties and Siegel modular forms are vastly investigated in the past decades. A lot of deep results and finer properties among automorphic forms, Galois representations, and the cohomologies were obtained. There are still many which are in progress. In this note we limit ourselves to the geometry of the special fiber of the Siegel moduli spaces. Studying geometry of reduction modulo p of Siegel moduli spaces with level at p is very fundamental on its own, as this is a direct generalization of modular curves. Another main motivation of these works is to hope for a more direct and explicit description of the Langlands correspondence through the geometry, especially when the ramification of associated local Galois representations occurs.

The following are the contents of this note.

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(1) We introduce the Siegel moduli spaces with parahoric level structure and the Kottwitz-Rapoport (KR) stratification. For further information, the reader is referred to Chai-Norman [1, 2], de Jong [4], Kottwitz-Rapoport [14], Görtz [6], Ngô-Genestier [15], Haines [11, 12], Tilouine [17], the author [18], and the references therein.

(2) We describe the supersingular locus of the Siegel 3-folds with parahoric structures of paramodular type and Klingen type. We describe how to characterize the KR strata in the moduli spaces with Iwahori level structure using geometry. For details, references are [19, 20].

(3) We give a description of the KR strata in the Siegel 3-folds with any parahoric level structure, their relationship under the transition maps, and their relation with p-rank strata.

(4) We report some results on the KR strata in higher dimensional cases. Those include a numerical characterization for KR strata, a method that enables us to reduce some geometric problems to that on p-rank zero strata, and a description of the supersingular KR strata in the case of genus $g = 3$. This is joint work with U. Görtz.

The proof of results in (3) and (4) will be given elsewhere.

2. Moduli Spaces

2.1. Moduli spaces with parahoric level structure

Let $g \geq 1$ be an integer, p a rational prime, $N \geq 3$ an integer with $(p, N) = 1$. Choose $\zeta_N \in \overline{\mathbb{Q}} \subset \mathbb{C}$ a primitive Nth root of unity and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Put $I := \{0, 1, \dots, g\}$. Let $\mathcal{A}_I$ be the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} A_g, \lambda_0, \lambda_g, \eta),$$

where

- each A_i is a g-dimensional abelian variety,
- α is an isogeny of degree p,
- λ_0 and λ_g are principal polarizations on A_0 and A_g, respectively, such that $(\alpha^g)^* \lambda_g = p\lambda_0$.
- η is a symplectic level-N structure on A_0 w.r.t. ζ_N.

Put $\eta_0 := \eta$, $\eta_i := \alpha_* \eta_{i-1}$ for $i = 1, \dots, g$, and $\lambda_{i-1} := \alpha^* \lambda_i$ for $i = g, \dots, 2$. Let $\underline{A_i} := (A_i, \lambda_i, \eta_i)$. Then $\mathcal{A}_I$ parametrizes equivalence classes of objects

$$(\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_g),$$

where $\underline{A}_0 \in \mathcal{A}_{g,1,N}$, and for $i \neq 0$,

$$\underline{A}_i \in \mathcal{A}'_{g,p^{g-i},N} := \{\underline{A} \in \mathcal{A}_{g,p^{g-i},N} \mid \ker \lambda \subset A[p]\,\}.$$

For any non-empty subset $J = \{i_0, \ldots, i_r\} \subset J$, let $\mathcal{A}_J$ be the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(\underline{A}_{i_0} \xrightarrow{\alpha} \underline{A}_{i_1} \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_{i_r}),$$

where $\underline{A}_{i_0} \in \mathcal{A}_{g,1,N}$ if $i_0 = 0$, and $\underline{A}_{i_j} \in \mathcal{A}'_{g,p^{g-i_j},N}$ for others. The moduli space $\mathcal{A}_I$ is the Siegel moduli space (over $\overline{\mathbb{F}}_p$) with Iwahori level structure, while $\mathcal{A}_J$ is the Siegel moduli space with parahoric level structure of type J.

For $J_1 \subset J_2$, let $\pi_{J_1,J_2} : \mathcal{A}_{J_2} \to \mathcal{A}_{J_1}$ be the natural projection. The transition morphism π_{J_1,J_2} is proper and dominant. We have

Theorem 2.1.

(1) *The ordinary locus* $\mathcal{A}_J^{\mathrm{ord}} \subset \mathcal{A}_J$ *is dense*

(2) $\mathcal{A}_J$ *is equi-dimensional of dimension* $g(g+1)/2$

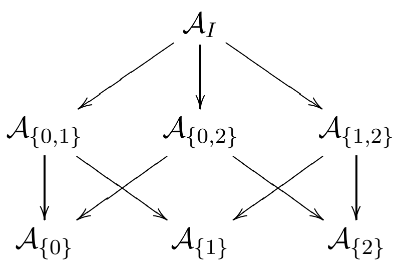
(3) $\mathcal{A}_J$ *is irreducible if* $|J| = 1$, *and for* $|J| \geq 2$, $\mathcal{A}_J$ *has* $(k_1+1)\ldots(k_r+1)$ *irreducible components, where* $k_j := i_j - i_{j-1}$.

(1) See Ngô-Genestier [15] and the author [18]. (2) This follows from the flatness of the integral model; see Görtz [6]. This also follows from (1). (3) See [18]. The case $|J| = 1$ is also obtained in de Jong [3].

2.2. Some results of the Siegel 3-folds with Klingen or paramodular level structure

When $g = 2$, we have the following diagram of transition maps:

(2.1)



Note that there is an involution $\theta_{\mathcal{A}} : \mathcal{A}_I \to \mathcal{A}_I$ which sends

$$(A_0 \to A_1 \to \cdots \to A_g, \lambda_0, \lambda_g, \eta) \mapsto (A_g^t \to \cdots \to A_0^t, \lambda_g^{-1}, \lambda_0^{-1}, \lambda_{g*}\eta_g).$$

Therefore, one may ignore the cases $\mathcal{A}_{\{1,2\}}$ and $\mathcal{A}_{\{2\}}$ as they are included as $\mathcal{A}_{\{0,1\}}$ and $\mathcal{A}_{\{0\}}$. We know that $\mathcal{A}_{\{1\}} = \mathcal{A}_{2,p,N}$ is a 3-dimensional, irreducible variety with isolated singularities. Let

$$\Lambda^*_{2,p,N} := \{\underline{A} \in \mathcal{A}_{2,p,N}\,;\, \ker\lambda = \alpha_p \times \alpha_p\,\}.$$

Proposition 2.2.

(1) *The singular locus* $\mathcal{A}^{\rm sing}_{\{1\}}$ *of* $\mathcal{A}_{\{1\}}$ *is equal to* $\Lambda^*_{2,p,N}$.

(2) *When* $p > 2$, *if* $x \in \Lambda^*_{2,p,N}$, *then one has*

$$\mathcal{A}^{\wedge}_{\{1\},x} \simeq k[[X_1, X_2, X_3, X_4]]/(X_2X_3 - X_1X_4).$$

Using the crystalline theory, one can show that $\mathcal{A}^{\wedge}_{\{1\},x} \simeq k[[X_1, X_2, X_3, X_4]]/(f)$ with $f \equiv X_2X_3 - X_1X_4$ modulo $(X_1, X_2, X_3, X_4)^p$. By change of variables, one can eliminate the higher terms.

Note that the set $\Lambda^*_{2,p,N}$ is used by Katsura-Oort [13] to construct the supersingular locus $\mathcal{S}_{\{0\}}$ of $\mathcal{A}_{2,1,N}$. We recall the construction as follows. For each $\xi \in \Lambda^*_{2,p,N}$, let S_ξ parametrize the isogenies $(\varphi : \underline{A}_1 \to \underline{A}_2)$ of degree p with $\underline{A}_1 = \xi$. One has $S_\xi \simeq \mathbf{P}^1$ and has a projection map ${\rm pr}_2 : S_\xi \to \mathcal{S}_{\{0\}}$ which sends $(\varphi : \underline{A}_1 \to \underline{A}_2) \mapsto \underline{A}_2$. One shows that

- The map $\coprod_{x\in\Lambda^*_{2,p,N}} S_\xi \to \mathcal{S}_{\{0\}}$ is surjective, and there are $p+1$ branches passing through each superspecial point of $\mathcal{S}_{\{0\}}$.
- This induces an isomorphism $\coprod_{x\in\Lambda^*_{2,p,N}} S_\xi \simeq \widetilde{\mathcal{S}}_{\{0\}}$, where $\widetilde{\mathcal{S}}_{\{0\}}$ is the normalization of $\mathcal{S}_{\{0\}}$.

In fact, if one considers the supersingular locus $\mathcal{S}_{\{0,1\}}$ of $\mathcal{A}_{\{0,1\}}$, then the picture is clearer. We have [19, Proposition 4.5]

$$\mathcal{S}_{\{0,1\}} = \left(\coprod_{\xi\in\Lambda^*_{2,p,N}} S'_\xi\right) \cup \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S'_\gamma\right), \tag{2.2}$$

where

$$\begin{aligned} S'_\xi &= \{(\varphi : \underline{A}_0 \to \underline{A}_1) \in \mathcal{A}_{\{0,1\}}\,;\underline{A}_1 = \xi\,\} \simeq \mathbf{P}^1, \\ S'_\gamma &= \{(\varphi : \underline{A}_0 \to \underline{A}_1) \in \mathcal{A}_{\{0,1\}}\,;\underline{A}_0 = \gamma\,\} \simeq \mathbf{P}^1. \end{aligned} \tag{2.3}$$

We call S'_ξ a horizontal component of $\mathcal{S}_{\{0,1\}}$, and call S'_γ a vertical component of $\mathcal{S}_{\{0,1\}}$. If one has an isogeny $\varphi : \underline{A}_0 \to \underline{A}_1$ of supersingular abelian surfaces, then either $\underline{A}_0 \in \Lambda_{2,1,N}$ or $\underline{A}_1 \in \Lambda^*_{2,p,N}$. We have natural projections

$$\mathcal{S}_{\{0\}} \xleftarrow{\mathrm{pr}_0} \mathcal{S}_{\{0,1\}} \xrightarrow{\mathrm{pr}_1} \mathcal{S}_{\{1\}}.$$

The Katsura-Oort construction uses the projection pr_0. Using the other projection pr_1, one gives a description of the supersingular locus $\mathcal{S}_{\{1\}}$; see [19, Theorem 4.7] for more details.

3. Local Model Diagrams and the KR Stratification

3.1. Local models

Let $V := \mathbb{Q}_p^{2g}$, $L_0 := \mathbb{Z}_p^{2g}$, and $e_1, \dots, e_{2g}$ the standard basis. Let ψ be the standard alternating pairing. One has

$$\psi = \begin{pmatrix} 0 & \widetilde{I}_g \\ -\widetilde{I}_g & 0 \end{pmatrix}, \quad \widetilde{I}_g = \text{anti-diag}(1, \dots, 1).$$

Put $\Lambda_{-i} = \mathbb{Z}_p^{2g}$ for $0 \le i \le 2g$. Let ψ_0 be the standard alternating pairing on Λ_0, same as ψ on L_0. Define, for each $1 \le i \le 2g$, a map $\alpha : \Lambda_{-2g+i-1} \to \Lambda_{-2g+i}$ by $\alpha(e_i) = pe_i$ and $\alpha(e_j) = e_j$ if $j \ne i$. Let ψ_{-g} on Λ_{-g} be $\frac{1}{p}$ times the pull-back of ψ_0; it is a perfect pairing. We get a lattice chain

$$\Lambda_I : \Lambda_{-g} \xrightarrow{\alpha} \dots \xrightarrow{\alpha} \Lambda_{-1} \xrightarrow{\alpha} \Lambda_0.$$

Denote by $\mathbf{M}_I^{\mathrm{loc}}$ the local model associated to the lattice chain Λ_I. It is a projective scheme over $\mathbb{Z}_p$ which parametrizes the objects $(\mathcal{F}_{-i})_{i \in I}$, where

- each $\mathcal{F}_{-i} \subset \Lambda_{-i} \otimes \mathcal{O}_S$ is a locally free $\mathcal{O}_S$-submodule of rank g, locally a direct summand,
- $\mathcal{F}_0$ and $\mathcal{F}_{-g}$ are isotropic w.r.t. the pairings ψ_0 and ψ_{-g}, respectively, and
- $\alpha(\mathcal{F}_{-i}) \subset \mathcal{F}_{-i+1}$ for all $i \in I$.

We write $\mathbf{M}_{I,\overline{\mathbb{F}}_p}^{\mathrm{loc}}$ for the reduction $\mathbf{M}_I^{\mathrm{loc}} \otimes \overline{\mathbb{F}}_p$ modulo p.

3.2. Local model diagrams

Let $\widetilde{\mathcal{A}}_I$ be the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects $(\underline{A}_\bullet, \xi)$, where $\underline{A}_\bullet \in \mathcal{A}_I$ and $\xi : H^1_{\mathrm{DR}}(A_\bullet/S) \simeq \Lambda_I \otimes \mathcal{O}_S$ is an isomorphism of chains which is compatible with α and preserves the polarizations up to scalars. We have the local model diagram (see de Jong [4] and Rapoport-Zink [16]):

$$\begin{array}{ccc} & \widetilde{\mathcal{A}}_I & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathcal{A}_I & & \mathbf{M}_{I,\overline{\mathbb{F}}_p}^{\mathrm{loc}}, \end{array}$$

where ψ is the morphism that sends each object $(\underline{A}_\bullet, \xi)$ to the image $\xi(\omega_\bullet)$ of the Hodge submodule $\omega_\bullet \subset H^1_{\rm DR}(A_\bullet)$, and φ is the morphism that forgets the trivialization ξ.

Let $\mathcal{G}_I$ be the group scheme over $\mathbb{Z}_p$ representing the functor $S \mapsto \mathrm{Aut}(\Lambda_I \otimes \mathcal{O}_S, [\psi_0], [\psi_{-g}])$. This group acts on $\widetilde{\mathcal{A}}_I$ and $\mathbf{M}^{\rm loc}_I$ from the left. One has that

- the morphism ψ is $\mathcal{G}_I$-equivalent, surjective and smooth, and
- the morphism $\varphi : \widetilde{\mathcal{A}}_I \to \mathcal{A}_I$ is a $\mathcal{G}_I$-torsor.

We can also define the local model $\mathbf{M}^{\rm loc}_J$ for each non-empty subset $J \subset I$, and have the local model diagram between $\mathcal{A}_J$, $\widetilde{\mathcal{A}}_J$ and $\mathbf{M}^{\rm loc}_{J,\overline{\mathbb{F}}_p}$ as above.

3.3. The KR stratification

Consider the decomposition into $\mathcal{G}_I$-orbits:

$$\mathbf{M}^{\rm loc}_{I,\overline{\mathbb{F}}_p} = \coprod_x \mathbf{M}^{\rm loc}_{I,x}, \quad \widetilde{\mathcal{A}}_I = \coprod_x \widetilde{\mathcal{A}}_{I,x}.$$

Since φ is a $\mathcal{G}_I$-torsor, the stratification on $\widetilde{\mathcal{A}}_I$ descends to a stratification

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}_I(\mu)} \mathcal{A}_{I,x}.$$

This is called the Kottwitz-Rapoport (KR) stratification. Here the index set $\mathrm{Adm}_I(\mu)$, which is called the set of μ-admissible elements, is a finite subset of $\widetilde{W}$, the extended Weyl group for GSp_{2g}, and $\mu = (1,\dots,1,0,\dots,0)$ (with $|\mu| = g$) is the minuscule dominant coweight. One has

$$\widetilde{W} = X_*(T) \rtimes W \subset \mathbf{A}(\mathbb{R}^{2g}),$$

where $T \subset \mathrm{GSp}_{2g}$ is the diagonal subgroup, $W = W(\mathrm{GSp}_{2g})$ the linear Weyl group, and $\mathbf{A}(\mathbb{R}^{2g})$ is the group of affine transformations on $\mathbb{R}^{2g}$. Let $\theta := (1,2g)(2,2g-1)\dots(g,g+1)$. Then

$$W \simeq \{\sigma \in S_{2g} = W(\mathrm{GL}_{2g})\,;\, \theta\sigma = \sigma\theta\}.$$

By definition,

$$\mathrm{Adm}_I(\mu) = \{x \in \widetilde{W}\,;\, x \le t_{w(\mu)} \text{ for some } w \in W\}.$$

$$\mathrm{Perm}_I(\mu) = \{x \in \widetilde{W} \subset \mathbf{A}(\mathbb{R}^{2g})\,;\, \mathbf{0} \le x(w'_i) - w'_i \le \mathbf{1},\ \forall\, 1 \le i \le 2g\},$$

where $w'_i = (0,\dots,0,1,\dots,1)$ with $|w'_i| = i$, and $\le$ is the Bruhat order on $\widetilde{W}$.

Kottwitz and Rapoport [14] have shown that $\mathrm{Adm}_I(\mu) = \mathrm{Perm}_I(\mu)$.

In fact, the set $\mathrm{Adm}_I(\mu)$ is contained in a smaller subset $W_a\tau \subset \widetilde{W}$, where

- τ is the element that is less than μ and fixes the base alcove

  $\mathbf{a} = \{u \in \mathbb{R}^{2g}; u_1+u_{2g} = \dots u_g+u_{g+1},\ 1+u_1 > u_{2g} > \dots > u_{g+1} > u_g\}$, and

- W_a is the affine Weyl group, which is $< s_0, s_1, \dots, s_g >$.

We can write down these elements explicitly:

$$\begin{aligned} s_i &= (i, i+1)(2g+1-i, 2g-i), \quad i = 1, \dots, g-1, \\ s_g &= (g, g+1), \quad s_0 = (-1, 0, \dots, 0, 1), (1, 2g), \\ \tau &= (0, \dots, 0, 1, \dots, 1), (1, g+1)(2, g+2) \dots (g, 2g). \end{aligned}$$

We also have the following results

Proposition 3.3.

(1) *Each stratum $\mathcal{A}_{I,x}$ is smooth of pure dimension $\ell(x)$.*

(2) *The p-rank function is constant on each KR stratum. Furthermore, one has*

$$p\text{−rank}(x) = \frac{1}{2}\#\mathrm{Fix}(w),$$

where we write $w = (\nu, w)$ and $\mathrm{Fix}(w) := \{i; w(i) = i\}$.

(1) This follows from the local model diagram and the dimensions of the strata in the $\mathbf{M}^{\mathrm{loc}}_{I,\overline{\mathbb{F}}_p}$; see Haines [11]. (2) See Ngô-Genestier [15].

3.4. Number of μ-admissible elements

We find the following formula in Haines [10, p.1272]:

$$N_g := \#\mathrm{Adm}_I(\mu, g) = \sum_{d=0}^{g} N_g^{g-d},$$

where N_g^{g-d} is the number of x with p-rank=$g-d$:

$$N_g^{g-d} = \binom{g}{d} 2^{g-d} \sum_{k=0}^{d} \binom{d}{k} 2^k a_k.$$

Here $a_0 = 1$ and for $n \geq 1$, $a_n := \#\{\sigma \in S_n\, ; \sigma(i) \neq i\ \forall\, i\}$. One also has the formula $1 + \sum_{k=1}^{n} \binom{n}{k} a_k = n!$. From these, we get

n	0	1	2	3	4
a_n	1	0	1	2	9

$g = 2$

p-rank	0	1	2	total
#	5	4	4	13

$g = 3$

p-rank	0	1	2	3	total
#	29	30	12	8	79

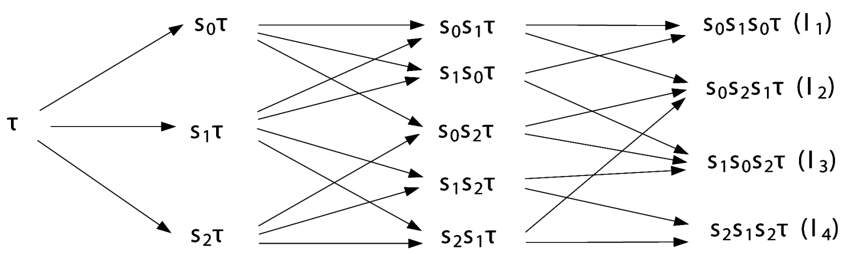
$g = 4$

p-rank	0	1	2	3	4	total
#	233	232	120	32	16	633

4. KR Strata in Siegel 3-Folds with Parahoric Level Structure

4.1. The Iwahori case

The following are the elements (called KR-types) in the set $\mathrm{Adm}(\mu)$ together with the Bruhat order.



Put $\mathrm{Adm}^i(\mu) := \{x \in \mathrm{Adm}(\mu);\ p\text{-rank}(x) = i\}$. We have

$$\begin{aligned}\mathrm{Adm}^2(\mu) &= \{s_0s_1s_0\tau,\ s_1s_0s_2\tau,\ s_2s_1s_2\tau,\ s_0s_2s_1\tau\},\\ \mathrm{Adm}^1(\mu) &= \{s_0s_1\tau,\ s_1s_2\tau,\ s_2s_1\tau,\ s_1s_0\tau\},\\ \mathrm{Adm}^0(\mu) &= \{\tau,\ s_1\tau,\ s_0\tau,\ s_2\tau,\ s_0s_2\tau\}.\end{aligned} \tag{4.1}$$

For each $0 \le f \le 2$, let $\mathcal{A}_I^f \subset \mathcal{A}_I$ (resp. $\mathcal{A}_I^{\le f} \subset \mathcal{A}_I$) be the subvariety consisting of objects with p-rank f (resp. p-rank less or equal to f). We conclude (see [20])

- The p-rank stratum $\mathcal{A}_I^1 \subset \mathcal{A}_I^{\le 1}$ is not dense. This implies that p-rank strata do not form a stratification on $\mathcal{A}_I$.

- The supersingular locus $\mathcal{S}_I \subset \mathcal{A}_I$ consists of one-dimensional components and two-dimensional components. This rules out the possibility of equi-dimensionality of p-rank strata.
- The morphism $\mathcal{S}_I \to \mathcal{S}_{\{0\}}$ is not finite. This limits the method of using p-adic monodromy to conclude an irreducibility result for p-rank strata in $\mathcal{A}_I$; see [20] for more details.

4.2. Geometric characterization for KR strata

Let $a = (\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2) \in \mathcal{A}_I(k)$. One wants to determine the KR-type $KR(a)$ of a in $\mathrm{Adm}(\mu)$. Let $(\overline{M}_2 \to \overline{M}_1 \to \overline{M}_0)$ be the chain of de Rham cohomology groups, and let $\omega_i \subset \overline{M}_i$ be the Hodge filtration. Put

(4.2) $$G_0 := \ker(A_0 \to A_1), \quad G_1 := \ker(A_1 \to A_2).$$

From the Dieudonné theory, we have

(4.3) $$\omega_i/\alpha(\omega_{i+1}) = \mathrm{Lie}\, G_i{}^*, \quad \overline{M}_i/\omega_i + \alpha(\overline{M}_{i+1}) = \mathrm{Lie}(G_i^D).$$

Define

(4.4) $$\sigma_i(a) := \dim \omega_i/\alpha(\omega_{i+1}), \quad \sigma_i'(a) := \dim \overline{M}_i/\omega_i + \alpha(\overline{M}_{i+1}).$$

Clearly, the invariants (σ_i, σ_i'), $i = 0, 1$, characterize the KR-types in $\mathrm{Adm}^1(\mu) \cup \mathrm{Adm}^2(\mu)$ because if $(G_0, G_1) \neq (\alpha_p, \alpha_p)$ then the group $\ker(A_0 \to A_2)$ is determined by (G_0, G_1), which is determined by (σ_i, σ_i').

Here is the correspondence:

p-rank(a)	2	2	2	2	1	1	1	1
$(\sigma_0(a), \sigma_0'(a))$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$
$(\sigma_1(a), \sigma_1'(a))$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$
$KR(a)$	$s_0s_1s_0\tau$	$s_0s_2s_1\tau$	$s_1s_0s_2\tau$	$s_2s_1s_2\tau$	$s_0s_1\tau$	$s_1s_2\tau$	$s_2s_1\tau$	$s_1s_0\tau$

Note that when the point a is supersingular the invariant $(\sigma_i(a), \sigma_i'(a))$ is $(1,1)$ for $i = 0, 1$, but there are 5 such KR strata. We define a new invariant:

$$\sigma_{02}(a) := \omega_0/\alpha^2(\omega_2), \quad \sigma_{02}'(a) := \dim \overline{M}_0/\omega_0 + \alpha^2(\overline{M}_2),$$

where $\alpha^2 : \overline{M}_2 \to \overline{M}_0$ is the composition. We get

p-rank(a)	0	0	0	0
$(\sigma_0(a), \sigma_0'(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_1(a), \sigma_1'(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_{02}(a), \sigma_{02}'(a))$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$KR(a)$	$s_0s_2\tau$	$s_0\tau$	$s_2\tau$	$s_1\tau$, τ

Note that unlike the invariants (σ_i, σ_i'), the invariant $(\sigma_{02}, \sigma_{02}')$ does not determine the isomorphism classes of finite subgroups $\ker(A_0 \to A_2)$; the latter has finer information than the invariant $(\sigma_{02}, \sigma_{02}')$. Now it remains to distinguish $s_1\tau$ and τ. For this, we study the supersingular locus $\mathcal{S}_I$ of $\mathcal{A}_I$.

Suppose that $a = (\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2) \in \overline{\mathcal{A}_{s_1\tau}}$, that is, $(\sigma_{02}(a), \sigma_{02}'(a)) = (2,2)$. Then from the description of $\mathcal{S}_I$ (see [20]), one shows that

$$a \in \mathcal{A}_\tau \iff \underline{A}_1 \in \Lambda^*_{2,p,N}.$$

Let $\underline{A}_0$ be any superspecial point, $A_0 \to A_1$ an isogeny of degree p, and $M_1 \subset M_0$ their Dieudonné modules. Then we have

$$\underline{A}_1 \in \Lambda^*_{2,p,N} \iff \text{In } \overline{M}_0 = M_0/pM_0,\ \langle \overline{M}_1, V\overline{M}_1 \rangle = 0.$$

Translating this property in terms of chains of de Rham cohomology groups, we have

Lemma 4.1. *Let $a = (\underline{A}_\bullet) \in \overline{\mathcal{A}_{s_1\tau}}$ and $\overline{M}_\bullet$ the chain of de Rham cohomology groups. Then $KR(a) = \tau \iff \langle \alpha(\overline{M}_1), \alpha(\omega_1) \rangle_0 = 0$.*

This completes the geometric characterization of KR strata.

4.3. KR strata under the transition maps

Recall that we have

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}_I(\mu)} \mathcal{A}_{I,x}, \quad \mathrm{Adm}_I(\mu) \subset W_a\tau, \quad W_a =< s_0, \ldots, s_g >,$$
$$\mathcal{A}_J = \coprod_{x \in \mathrm{Adm}_J(\mu)} \mathcal{A}_{J,x}, \quad \mathrm{Adm}_J(\mu) \subset W_J\backslash\widetilde{W}/W_J,$$

where $\mathrm{Adm}_J(\mu)$ is the image of $\mathrm{Adm}_I(\mu)$ in $W_J\backslash W_a\tau/W_J \subset W_J\backslash\widetilde{W}/W_J$ and $W_J =< s_i \,|\, i \notin J >$, a finite group. In the situation where the genus $g = 2$, we consider the cases $J = \{0,1,2\}, \{0,1\}, \{0,2\}, \{1\}$, or $\{0\}$ as mentioned before.

For $x \in \mathrm{Adm}_I(\mu)$, let

$$[x]_J = \{ y \in \mathrm{Adm}_I(\mu) \,|\, [y] = [x] \text{ in } W_J\backslash W_a\tau/W_J \}.$$

Let $\mathcal{A}_{[x]_J}$ be the corresponding KR stratum in $\mathcal{A}_J$, regarding $[x]_J$ as an element in $W_J\backslash\widetilde{W}/W_J$.

(1) $J = \{0,1\}$ (Klingen level) and $W_J =< s_2 >$. Using $\tau s_2 = s_0\tau$, we compute that

$$[\tau]_J = \{\tau, s_2\tau, s_0\tau, s_{02}\tau\}, \qquad \text{dim=1},$$
$$[s_1\tau]_J = \{s_1\tau, s_{10}\tau, s_{21}\tau\}, \qquad \text{dim=2},$$
$$[s_{12}\tau]_J = \{s_{12}\tau, s_{120}\tau, s_{212}\tau\}, \quad \text{dim=3},$$
$$[s_{01}\tau]_J = \{s_{01}\tau, s_{010}\tau, s_{201}\tau\}, \quad \text{dim=3}.$$

We have

Theorem 4.2.

(i) *There are 2 ordinary irreducible components; they are (properly) contained in $\mathcal{A}_{[s_{01}\tau]_J}$ and $\mathcal{A}_{[s_{12}\tau]_J}$ respectively.*

(ii) *There are 3 p-rank one irreducible components; they are (properly) contained in $\mathcal{A}_{[s_1\tau]_J}$, $\mathcal{A}_{[s_{01}\tau]_J}$ and $\mathcal{A}_{[s_{12}\tau]_J}$ respectively.*

(iii) *The closure $\overline{\mathcal{A}_{[s_1\tau]_J}}$ is a smooth surface, which is the intersection of $\overline{\mathcal{A}_{[s_{01}\tau]_J}}$ and $\overline{\mathcal{A}_{[s_{12}\tau]_J}}$.*

(iv) *The stratum $\mathcal{A}_{[\tau]_J}$ consists of "horizontal" components of the supersingular locus $\mathcal{S}_J$ (see (2.3)).*

(v) *The intersection $\mathcal{S}_J \cap \mathcal{A}_{[s_1\tau]_J}$ consists of open "vertical" components of $\mathcal{S}_J$ (see (2.3)).*

(vi) *The union $\mathcal{A}_{[s_{01}\tau]_J} \cup \mathcal{A}_{[s_{12}\tau]_J}$ is the smooth locus of $\mathcal{A}_J$.*

Question. Is $\pi_{\{0\},J} : \overline{\mathcal{A}_{[s_1\tau]_J}} \to \mathcal{A}_{\{0\}}^{\text{non-ord}}$ the blow-up of $\mathcal{A}_{\{0\}}^{\text{non-ord}}$ at the singular (superspecial) points? We expect it has the affirmative answer.

(2) $J = \{0, 2\}$ (Siegel parahoric level) and $W_J =< s_1 >$. Using $\tau s_1 = s_1\tau$, we compute that

$$[\tau]_J = \{\tau, s_1\tau\}, \qquad \text{dim=0}, \quad H_2 = \alpha_p \times \alpha_p,$$
$$[s_2\tau]_J = \{s_2\tau, s_{12}\tau, s_{21}\tau\}, \qquad \text{dim=2}, \quad H_2(\eta) = \mu_p \times \alpha_p,$$
$$[s_0\tau]_J = \{s_0\tau, s_{10}\tau, s_{01}\tau\}, \qquad \text{dim=2}, \quad H_2(\eta) = \mathbb{Z}/p \times \alpha_p,$$
$$[s_{02}\tau]_J = \{s_{02}\tau, s_{201}\tau, s_{120}\tau\}, \quad \text{dim=3}, \quad H_2(\eta) = \mathbb{Z}/p \times \mu_p,$$
$$[s_{212}\tau]_J = \{s_{212}\tau\}, \qquad \text{dim=3}, \quad H_2 = \mu_p \times \mu_p,$$
$$[s_{010}\tau]_J = \{s_{010}\}, \qquad \text{dim=3}, \quad H_2 = \mathbb{Z}/p \times \mathbb{Z}/p.$$

Here $H_2(\eta)$ means $\ker(A_{0,\eta} \to A_{2,\eta})$ for a generic point η of this KR stratum. We have

Theorem 4.3.

(i) *There are 3 ordinary irreducible components. Two are $\mathcal{A}_{[s_{212}\tau]_J}$ and $\mathcal{A}_{[s_{010}\tau]_J}$, and the other is properly contained in the stratum $\mathcal{A}_{[s_{02}\tau]_J}$.*

(ii) *There are 2 p-rank one irreducible components. They are properly contained in $\mathcal{A}_{[s_0\tau]_J}$ and $\mathcal{A}_{[s_2\tau]_J}$, respectively.*

(iii) *The supersingular locus $\mathcal{S}_J$ has pure dimension 2. It is contained in the 3-dimensional closure $\overline{\mathcal{A}_{[s_{02}\tau]_J}}$.*

(iv) *The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ consists of points $(\underline{A}_0 \xrightarrow{F} \underline{A}_0^{(p)})$, where A_0 is superspecial.*

(v) *The union $\mathcal{A}_{[s_{212}\tau]_J} \cup \mathcal{A}_{[s_{010}\tau]_J} \cup \mathcal{A}_{[s_{02}\tau]_J}$ is the smooth locus of $\mathcal{A}_J$.*

In fact, in the module space $\mathcal{A}_I$ with Iwahori level structure, we have

$$\mathcal{S}_I = \overline{\mathcal{A}_{s_{021}\tau}} \cap \overline{\mathcal{A}_{s_{102}\tau}}.$$

These two components are mapped, through the transition map $\pi_{J,I}$, onto the component $\overline{\mathcal{A}_{[s_{02}\tau]_J}}$.

(3) $J = \{1\}$ (paramodular level) and $W_J =< s_0, s_2 >$. Using $\tau s_0 = s_2\tau$ and $\tau s_2 = s_0\tau$, we compute that

$$[\tau]_J = \{\tau, s_0\tau, s_2\tau, s_{02}\tau\}, \quad \text{dim=0},$$
$$[s_1\tau]_J = \{\text{ the rest }\}, \quad \text{dim=3}.$$

We have

Theorem 4.4.

(i) *There is 1 ordinary irreducible component.*

(ii) *There is 1 p-rank one irreducible component.*

(iii) *The supersingular locus has pure dimension 1. Each component is isomorphic to $\mathbf{P}^1$. The intersection $S_J \cap \mathcal{A}_{[s_1\tau]_J}$ is the smooth locus of S_J.*

(iv) *The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ is the singular locus of $\mathcal{A}_J$, also the singular locus of $\mathcal{S}_J$, which is equal to the set $\Lambda^*_{2,p,N}$.*

(v) *The stratum $\mathcal{A}_{[s_1\tau]_J}$ is the smooth locus.*

(4) $J = \{0\}$ (smooth base) and $W_J =< s_1, s_2 >$. We compute that $[\tau]_J$ is everything. The whole moduli space $\mathcal{A}_{\{0\}}$ is a single KR stratum.

5. Some Aspects in Higher Dimensional Cases (Joint with Ulrich Görtz)

In this section we will restrict ourselves to the Iwahori level case, but for higher genus.

5.1. Numerical characterization

Let $a = (\underline{A}_0 \to \cdots \to A_g) \in \mathcal{A}_I(k)$. Let

$$M_\bullet : \ M_{-g} \to M_{-g+1} \to \cdots \to M_0, \quad VM_\bullet : VM_{-g} \to VM_{-g+1} \to \cdots \to VM_0.$$

be the associated chain of Dieudonné modules. Then we have

$$KR(a) = \mathrm{inv}(M_\bullet, VM_\bullet) \in \mathrm{Iw}\backslash \mathrm{GSp}_{2g}(L)/\mathrm{Iw} \simeq \widetilde{W},$$

where $L = \mathrm{Frac}\, W(k)$, Iw is the standard Iwahori open compact subgroup (whose reduction mod p is the Borel subgroup $B_\triangle$ of upper triangular matrices in GSp_{2g}), and $\widetilde{W}$ is the extended Weyl group of GSp_{2g}.

Another way to think about KR types is as follows. Let

$$\overline{M}_\bullet : \ \overline{M}_{-g} \to \overline{M}_{-g+1} \to \cdots \to \overline{M}_0$$

be the chain of de Rham cohomology groups, together with Hodge filtrations. We ignore the F and V structures, and just consider the isomorphism classes of these chains of vector spaces over k, together with Hodge filtration as subspaces. Then the isomorphism classes give rise to the KR types.

Just as flag varieties, on the one hand, we have a *group-theoretic description* for the cell decomposition (coming from the Bruhat decomposition). On the other hand, we use the *incidence relation* to construct the Schubert cells. The latter description is used to compute the Chow rings of flag varieties in the intersection theory.

Definition. Let $a = (\underline{A}_0 \to \cdots \to A_g) \in \mathcal{A}_I(k)$ and let $\overline{M}_{-g} \to \overline{M}_{-g+1} \to \cdots \to \overline{M}_0$ be the chain of de Rham cohomologies with Hodge filtration $\omega_{-i} \subset \overline{M}_{-i}$. Let $\alpha_{i,j} : \overline{M}_{-j} \to \overline{M}_{-i}$ be the composition for $0 \le i < j \le g$. Define

$$\sigma_{ij}(a) := \dim \omega_{-i}/\alpha_{ij}(\omega_{-j}), \quad \sigma'_{ij}(a) := \dim \overline{M}_{-i}/\omega_{-i} + \alpha_{ij}(\overline{M}_{-j}).$$

For $0 \le i, j \le g$, define

$$d_{ij}(a) := \dim \alpha_{0i}(\omega_{-i}) + \alpha_{0j}(\overline{M}_{-j})^\perp.$$

Clearly, the function

$$\underline{\sigma} : a \mapsto (\sigma_{ij}(a), \sigma'_{ij}(a), d_{ij}(a))$$

is constant on each KR stratum. This particularly implies that the function

$$p - \mathrm{rank}(a) = \sum_{i=0}^{g-1} 2 - \sigma_{i,i+1}(a) - \sigma'_{i,i+1}(a)$$

is constant on each KR stratum. Conversely, we prove (see [8]).

Theorem 5.1. *The KR strata are distinguished by the invariant $\underline{\sigma}$. That is, if $x \neq x' \in \mathrm{Adm}(\mu)$, then $\underline{\sigma}(\mathcal{A}_{I,x}) \neq \underline{\sigma}(\mathcal{A}_{I,x'})$.*

5.2. The shuffle construction

The goal is to reduce geometric problems on KR strata $\mathcal{A}_x$ to those on KR strata with *p-rank zero* and on KR strata of moduli spaces of *lower genus* g.

Observation. An Iwahori level structure on (A, λ) is a flag of finite group schemes

$$H_\bullet: \ 0 \subset H_1 \subset \cdots \subset H_g \subset A[p]$$

satisfying certain conditions. This structure is defined through the p-torsion subgroup $(A[p], \lambda)$ with polarization.

Let $\mathrm{BT}^1_{h,I}$ be the set of isomorphism classes of $(G, \lambda, H_\bullet)$ over k, where

- (G, λ) is a principally polarized BT^1 of height $2h$,
- $H_\bullet : \ H_1 \subset \cdots \subset H_h \subset G$ a flag of finite flat group schemes such that $< \lambda(H_h), H_h >= 0$ (Note that $\lambda : G \to G^D$).

We may formulate $\mathrm{BT}^1_{h,I}$ as a category of groupoids with objects as above. But let us regard it simply as a set for simplicity. Clearly, we have a surjective map

$$\mathrm{BT}^1_{h,I} \xrightarrow{KR} \mathrm{Adm}_I(\mu).$$

For two integers $s \geq 1$ and $t \geq 1$ with $s + t = g$, denote by $Sh(s,t)$ the set of maps

$$\varphi : \{0, 1, \ldots, g\} \to \{0, 1, \ldots, s\}$$

such that

$$\varphi(0) = 0, \ \varphi(g) = s, \ \text{and} \ \varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1, \ \forall\, i = 0, \ldots, g-1.$$

It is called the set of shuffle maps of s letters and t letters.

For example, let $\varphi \in Sh(4,3)$, we use φ to shuffle **123** into 1234 as follows. Suppose

$$\varphi \ : \ 0\ 1\ \underline{1}\ 2\ \underline{2}\ 3\ 4\ \underline{4}.$$

We underline the repeated numbers, remove them, and replace by **123**:

$$\varphi \ : \ 0\ 1\ \mathbf{1}\ 2\ \mathbf{2}\ 3\ 4\ \mathbf{3}.$$

For $\varphi \in sh(s,t)$, define $\varphi' : \{0,1,\ldots,g\} \to \{0,1,\ldots,t\}$, called the complement of φ, as follows.

$$\varphi'(0) = 0, \quad \varphi'(i+1) + \varphi(i+1) = \varphi'(i) + \varphi(i) + 1, \quad \forall\, i = 0,\ldots,g-1.$$

With information above, we construct a map

$$\mathrm{sh}_\varphi : \mathrm{BT}^1_{s,I} \times \mathrm{BT}^1_{t,I} \to \mathrm{BT}^1_{g,I}$$

by

$$\big((G,\lambda,H_\bullet),(G'\lambda'_,H'_\bullet)\big) \mapsto (G\times G', \lambda\times\lambda', \varphi(H_\bullet,H'_\bullet)),$$

where

$$\varphi(H_\bullet,H'_\bullet):\ K_1 \subset K_2 \subset \cdots \subset K_g \subset G\times G', \quad K_i = H_{\varphi(i)} \times H_{\varphi'(i)}.$$

The shuffle map sh_φ descends to the set $\mathrm{Adm}_I(\mu)$:

$$\begin{array}{ccc} \mathrm{BT}^1_{s,I}\times \mathrm{BT}^1_{t,I} & \xrightarrow{\mathrm{sh}_\varphi} & \mathrm{BT}^1_{g,I} \\ \Big\downarrow{\scriptstyle (KR,KR)} & & \Big\downarrow{\scriptstyle KR} \\ \mathrm{Adm}_I(\mu,s)\times \mathrm{Adm}_I(\mu,t) & \xrightarrow{\mathrm{sh}_\varphi} & \mathrm{Adm}_I(\mu,g). \end{array}$$

In general, the map sh_φ is not injective. But we have

- The restriction $\mathrm{sh}_\varphi : \mathrm{Adm}^0_I(\mu,g-f)\times \mathrm{Adm}^f_I(\mu,f) \to \mathrm{Adm}^f_I(\mu,g)$ is injective.
- $$\mathrm{Adm}^f_I(\mu,g) = \coprod_{\varphi\in Sh(g-f,f)} \mathrm{sh}_\varphi(\mathrm{Adm}^0_I(\mu,g-f)\times \mathrm{Adm}^f_I(\mu,f)).$$

These follow easily from the canonical decomposition $G = G^{\mathrm{et,m}} \oplus G^{\mathrm{loc,loc}}$.

For any $x_1 \in \mathrm{Adm}_I(\mu,s)$, $x_2\in \mathrm{Adm}_I(\mu,t)$ and $\varphi\in Sh(s,t)$, we get a shuffle morphism

$$\mathrm{sh}_\varphi : \mathcal{A}_{s,x_1}\times \mathcal{A}_{t,x_2} \to \mathcal{A}_{g,x},$$

where $x = \mathrm{sh}_\varphi(x_1,x_2)$. This produces various subvarieties in a KR stratum $\mathcal{A}_{g,x}$ which may give enough information about what we want to know on $\mathcal{A}_{g,x}$. For example, let x be any element say in $\mathrm{Adm}^f_I(\mu,g)$. Then there exist a unique

$x_1 \in \mathrm{Adm}_I^0(\mu, g-f)$, $x_2 \in \mathrm{Adm}_I^f(\mu, f)$, and $\varphi \in Sh(g-f, f)$ such that $x = \mathrm{sh}_\varphi(x_1, x_2)$. So we have a morphism

$$\mathrm{sh}_\varphi : \mathcal{A}_{g-f,x_1} \times \mathcal{A}_{f,x_2} \to \mathcal{A}_{g,x}.$$

Geometric information on $\mathcal{A}_{g,x}$, for example possible Newton polygons, can be read from those on $\mathcal{A}_{g-f,x}$.

5.3. Admissible elements: g=3 and p-rank zero

We list all 29 μ-admissible elements with p-rank zero in the extended Weyl group $\widetilde{W} = X_*(T) \rtimes W(\mathrm{GSp}_6)$. Below

$$\tau = (0,0,0,1,1,1), (14)(25)(36), \quad s_0 = (-1,0,0,0,0,1), (16),$$

$$s_1 = (12)(56), \quad s_1 = (23)(45) \quad \text{and} \quad s_3 = (34).$$

Write $s_{i_1 i_2 \ldots i_r}$ for the element $s_{i_1} s_{i_2} \cdots s_{i_r}$ in the affine Weyl group W_a.

KR	$(\nu, w) \in X_*(T) \rtimes W$	KR	$(\nu, w) \in X_*(T) \rtimes W$
(1) τ	(0,0,0,1,1,1), (14)(25)(36)	(16) $s_{310}\tau$	(0,0,1,0,1,1), (132645)
(2)$s_0\tau$	(0,0,0,1,1,1), (1463)(25)	(17) $s_{120}\tau$	(0,0,0,1,1,1), (16)(2453)
(3)$s_1\tau$	(0,0,0,1,1,1), (142635)	(18) $s_{320}\tau$	(0,0,1,0,1,1), (154623)
(4)$s_2\tau$	(0,0,0,1,1,1), (153624)	(19) $s_{230}\tau$	(0,1,0,1,0,1), (124653)
(5)$s_3\tau$	(0,0,1,0,1,1), (1364)(25)	(20) $s_{201}\tau$	(0,0,0,1,1,1), (1562)(34)
(6)$s_{10}\tau$	(0,0,0,1,1,1), (145)(263)	(21) $s_{301}\tau$	(0,0,1,0,1,1), (135642)
(7) $s_{20}\tau$	(0,0,0,1,1,1), (153)(246)	(22) $s_{121}\tau$	(0,0,0,1,1,1), (16)(25)(34)
(8)$s_{30}\tau$	(0,0,1,0,1,1), (13)(25)(46)	(23) $s_{231}\tau$	(0,1,0,1,0,1), (1265)(34)
(9)$s_{01}\tau$	(0,0,0,1,1,1), (142)(356)	(24) $s_{312}\tau$	(0,0,1,0,1,1), (16)(2354)
(10)$s_{21}\tau$	(0,0,0,1,1,1), (15)(26)(34)	(25) $s_{323}\tau$	(0,1,1,0,0,1), (123654)
(11)$s_{31}\tau$	(0,0,1,0,1,1), (135)(264)	(26) $s_{3010}\tau$	(0,0,1,0,1,1), (132)(456)
(12)$s_{12}\tau$	(0,0,0,1,1,1), (16)(24)(35)	(27) $s_{3120}\tau$	(0,0,1,0,1,1), (16)(23)(45)
(13)$s_{32}\tau$	(0,0,1,0,1,1), (154)(236)	(28) $s_{3230}\tau$	(0,1,1,0,0,1), (123)(465)
(14) $s_{23}\tau$	(0,1,0,1,0,1), (124)(365)	(29) $s_{2301}\tau$	(0,1,0,1,0,1), (12)(34)(56)
(15)$s_{010}\tau$	(0,0,0,1,1,1), (145632)		

The partial (Bruhat) order on this finite set is expressed as follows. Two elements x and y have relation $x < y$ in the Bruhat order if and only if there is a chain with $x = x_0 \to x_1 \to \cdots \to x_n = y$.

(1) $\tau \to s_0\tau,\ s_1\tau,\ s_2\tau,\ s_3\tau$
(2) $s_0\tau \to s_{10}\tau,\ s_{20}\tau,\ s_{30}\tau,\ s_{01}\tau$
(3) $s_1\tau \to s_{10}\tau,\ s_{01}\tau,\ s_{21}\tau,\ s_{31}\tau,\ s_{12}\tau$
(4) $s_2\tau \to s_{20}\tau,\ s_{21}\tau,\ s_{12}\tau,\ s_{32}\tau,\ s_{23}\tau$
(5) $s_3\tau \to s_{30}\tau,\ s_{31}\tau,\ s_{32}\tau,\ s_{23}\tau$
(6) $s_{10}\tau \to s_{010}\tau,\ s_{310}\tau,\ s_{120}\tau$
(7) $s_{20}\tau \to s_{120}\tau,\ s_{320}\tau,\ s_{230}\tau,\ s_{201}\tau$
(8) $s_{30}\tau \to s_{310}\tau,\ s_{320}\tau,\ s_{230}\tau,\ s_{301}\tau$
(9) $s_{01}\tau \to s_{010}\tau,\ s_{201}\tau,\ s_{301}\tau$
(10) $s_{21}\tau \to s_{201}\tau,\ s_{121}\tau,\ s_{231}\tau$
(11) $s_{31}\tau \to s_{310}\tau,\ s_{301}\tau,\ s_{231}\tau,\ s_{312}\tau$
(12) $s_{12}\tau \to s_{120}\tau,\ s_{121}\tau,\ s_{312}\tau$
(13) $s_{32}\tau \to s_{320}\tau,\ s_{312}\tau,\ s_{323}\tau$
(14) $s_{23}\tau \to s_{230}\tau,\ s_{231}\tau,\ s_{323}\tau$
(15) $s_{010}\tau \to s_{3010}\tau$
(16) $s_{310}\tau \to s_{3010}\tau,\ s_{3120}\tau$
(17) $s_{120}\tau \to s_{3120}\tau$
(18) $s_{320}\tau \to s_{3120}\tau,\ s_{3230}\tau$
(19) $s_{230}\tau \to s_{3230}\tau,\ s_{2301}\tau$
(20) $s_{201}\tau \to s_{2301}\tau$
(21) $s_{301}\tau \to s_{3010}\tau,\ s_{2301}\tau$
(22) $s_{121}\tau$ (max.)
(23) $s_{231}\tau \to s_{2301}\tau$
(24) $s_{312}\tau \to s_{3120}\tau$
(25) $s_{323}\tau \to s_{3230}\tau$
(26) $s_{3010}\tau$ (max.)
(27) $s_{3120}\tau$ (max.)
(28) $s_{3230}\tau$ (max.)
(29) $s_{2301}\tau$ (max.)

The following table indicates the possible Newton polygons occurring in each KR stratum. The symbol A represents the supersingular Newton polygon; the symbol B represents the Newton polygon with slopes $\frac{1}{3}$ and $\frac{2}{3}$. Let NP denote the set of the Newton polygons of points in the KR stratum.

KR	NP	KR	NP	KR	NP
(1) τ	A	(11) $s_{31}\tau$	B	(21) $s_{301}\tau$	A,B
(2) $s_0\tau$	A	(12) $s_{12}\tau$	A	(22) $s_{121}\tau$	A
(3) $s_1\tau$	A	(13) $s_{32}\tau$	B	(23) $s_{231}\tau$	A,B
(4) $s_2\tau$	A	(14) $s_{23}\tau$	B	(24) $s_{312}\tau$	A,B
(5) $s_3\tau$	A	(15) $s_{010}\tau$	A,B	(25) $s_{323}\tau$	A,B
(6) $s_{10}\tau$	B	(16) $s_{310}\tau$	A,B	(26) $s_{3010}\tau$	A,B
(7) $s_{20}\tau$	B	(17) $s_{120}\tau$	A,B	(27) $s_{3120}\tau$	A,B
(8) $s_{30}\tau$	A	(18) $s_{320}\tau$	A,B	(28) $s_{3230}\tau$	A,B
(9) $s_{01}\tau$	B	(19) $s_{230}\tau$	A,B	(29) $s_{2301}\tau$	A,B
(10) $s_{21}\tau$	A	(20) $s_{201}\tau$	A,B		

5.4. Numerical invariants for $g = 3$

The following is the result of computation of the invariants $(\sigma_{ij}, \sigma'_{ij})$ and d_{ij}. Recall these invariants. Let $s = (\underline{A}_0 \to \cdots \to \underline{A}_g)$ be a point in $\mathcal{A}_I(k)$. Let $(\overline{M}_{-g} \xrightarrow{\alpha} \overline{M}_{-g+1} \ldots, \xrightarrow{\alpha} \overline{M}_0)$ be the associated chain of de Rham cohomologies. For $0 \le i < j \le g$, write $\alpha_{ij} : \overline{M}_{-j} \to \overline{M}_{-i}$ for the composition. Define

$$\sigma_{ij}(s) := \dim \omega_{-i}/\alpha_{ij}(\omega_{-j}), \quad \sigma'_{ij}(s) := \dim \overline{M}_{-i}/(\omega_{-i} + \alpha_{ij}(\overline{M}_{-j})).$$

For $1 \le i, j \le g-1$, define

$$d_{ij}(s) = \dim \alpha_{0i}(\omega_{-i}) + \alpha_{0j}(\overline{M}_{-j})^{\perp}.$$

Given an element $x \in \mathrm{Adm}(\mu)$, we use the expression $x = (\nu, w)$ to compute the lattice $(\mathcal{L}'_\bullet)$ with $t\Lambda'_{-i} \subset \mathcal{L}_{-i} \subset \Lambda'_{-i}$. Then we use this lattice to compute the invariants $(\sigma_{ij}, \sigma'_{ij})$ and d_{ij}. We first compute the invariants $(\sigma_{ij}, \sigma'_{ij})$ for each (p-rank zero μ-admissible) element x.

KR	$(\sigma_{02}, \sigma'_{02})$	$(\sigma_{13}, \sigma'_{13})$	$(\sigma_{03}, \sigma'_{03})$	KR	$(\sigma_{02}, \sigma'_{02})$	$(\sigma_{13}, \sigma'_{13})$	$(\sigma_{03}, \sigma'_{03})$
(1) τ	(2,2)	(2,2)	(3,3)	(16) $s_{310}\tau$	(2,2)	(1,2)	(2,2)
(2) $s_0\tau$	(2,2)	(2,2)	(2,3)	(17) $s_{120}\tau$	(2,2)	(1,2)	(2,3)
(3) $s_1\tau$	(2,2)	(2,2)	(3,3)	(18) $s_{320}\tau$	(2,2)	(2,1)	(2,2)
(4) $s_2\tau$	(2,2)	(2,2)	(3,3)	(19) $s_{230}\tau$	(2,1)	(2,2)	(2,2)
(5) $s_3\tau$	(2,2)	(2,2)	(3,2)	(20) $s_{201}\tau$	(1,2)	(2,2)	(2,3)
(6) $s_{10}\tau$	(2,2)	(1,2)	(2,3)	(21) $s_{301}\tau$	(1,2)	(2,2)	(2,2)
(7) $s_{20}\tau$	(2,2)	(2,2)	(2,3)	(22) $s_{121}\tau$	(2,2)	(2,2)	(3,3)
(8) $s_{30}\tau$	(2,2)	(2,2)	(2,2)	(23) $s_{231}\tau$	(2,1)	(2,2)	(3,2)
(9) $s_{01}\tau$	(1,2)	(2,2)	(2,3)	(24) $s_{312}\tau$	(2,2)	(2,1)	(3,2)
(10) $s_{21}\tau$	(2,2)	(2,2)	(3,3)	(25) $s_{323}\tau$	(2,1)	(2,1)	(3,1)
(11) $s_{31}\tau$	(2,2)	(2,2)	(3,2)	(26) $s_{3010}\tau$	(1,2)	(1,2)	(1,2)
(12) $s_{12}\tau$	(2,2)	(2,2)	(3,3)	(27) $s_{3120}\tau$	(2,2)	(1,1)	(2,2)
(13) $s_{32}\tau$	(2,2)	(2,1)	(3,2)	(28) $s_{3230}\tau$	(2,1)	(2,1)	(2,1)
(14) $s_{23}\tau$	(2,1)	(2,2)	(3,2)	(29) $s_{2301}\tau$	(1,1)	(2,2)	(2,2)
(15) $s_{010}\tau$	(1,2)	(1,2)	(1,3)				

In the following two tables some KR strata are already distinguished by the invariants $(\sigma_{ij}, \sigma'_{ij})$.

$(\sigma_{03}, \sigma'_{03})$	(1,2)	(2,1)	(2,2)	(2,2)	(1,3)	(3,1)
$(\sigma_{02}, \sigma'_{02})$	(1,2)	(2,1)	(2,2)	(1,1)	(1,2)	(2,1)
$(\sigma_{13}, \sigma'_{13})$	(1,2)	(2,1)	(1,1)	(2,2)	(1,2)	(2,1)
KR	(26) $s_{3010}\tau$	(28) $s_{3230}\tau$	(27) $s_{3120}\tau$	(29) $s_{2301}\tau$	(15) $s_{010}\tau$	(25) $s_{323}\tau$

$(\sigma_{03}, \sigma'_{03})$	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{02}, \sigma'_{02})$	(1,2)	(2,1)	(2,2)	(2,2)	(2,2)
$(\sigma_{13}, \sigma'_{13})$	(2,2)	(2,2)	(2,1)	(1,2)	(2,2)
KR	(21) $s_{301}\tau$	(19) $s_{230}\tau$	(18) $s_{320}\tau$	(16) $s_{310}\tau$	(8) $s_{30}\tau$

The following two tables are given by the invariants $(\sigma_{03}, \sigma'_{03}) = (2,3)$ and $(\sigma_{03}, \sigma'_{03}) = (3,2)$, respectively. There are two classes in the each set of classes with invariants $(\sigma_{ij}, \sigma'_{ij})$ constant. They are distinguished by the invariant d_{12} in the first table (resp. by the invariant d_{21} in the second table). Notice that each pair of classes has the inclusion relation. In the first table, every smaller element is obtained by dropping s_2 from the bigger element. In the second table, every smaller element is obtained by dropping s_1 from the bigger element.

$(\sigma_{03}, \sigma'_{03})$	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)
$(\sigma_{02}, \sigma'_{02})$	(1,2)	(1,2)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{13}, \sigma'_{13})$	(2,2)	(2,2)	(1,2)	(1,2)	(2,2)	(2,2)
d_{12}	2	3	2	3	2	3
KR	(9)$s_{01}\tau$	(20) $s_{201}\tau$	(6) $s_{10}\tau$	(17) $s_{120}\tau$	(2) $s_0\tau$	(7) $s_{20}\tau$

$(\sigma_{03}, \sigma'_{03})$	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)
$(\sigma_{02}, \sigma'_{02})$	(2,1)	(2,1)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{13}, \sigma'_{13})$	(2,2)	(2,2)	(2,1)	(2,1)	(2,2)	(2,2)
d_{21}	1	2	1	2	1	2
KR	(14) $s_{23}\tau$	(23) $s_{231}\tau$	(13) $s_{32}\tau$	(24) $s_{312}\tau$	(5) $s_3\tau$	(11) $s_{31}\tau$

The following is the table for supersingular KR strata (see Subsection for detailed descriptions). Note that $(\sigma_{03}, \sigma'_{03}) = (3,3)$ implies $(\sigma_{02}, \sigma'_{02}) = (2,2)$ and $(\sigma_{13}, \sigma'_{13}) = (2,2)$. Therefore, there is no need to list them.

$(\sigma_{03}, \sigma'_{03})$	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)
d_{12}	2	2	3	3	3	3
d_{21}	1	2	1	2	2	2
d_{11}				2	3	3
d_{22}				3	2	3
KR	(1) τ	(3) $s_1\tau$	(4) $s_2\tau$	(10) $s_{21}\tau$	(12) $s_{12}\tau$	(22) $s_{121}\tau$

5.5. Supersingular KR strata

A KR stratum $\mathcal{A}_x$ is called *supersingular* if it is contained in the supersingular locus $\mathcal{S}_I$. The following are all supersingular KR-types in $\mathrm{Adm}_I(\mu)$:

$$\{\tau, s_1\tau, s_2\tau, s_{12}\tau, s_{21}\tau, s_{121}\tau, s_0\tau, s_3\tau, s_{03}\tau\} = W_{\{0,3\}}\tau \cup W_{\{1,2\}}\tau.$$

Note that the union of all supersingular KR strata is properly contained in the supersingular locus $\mathcal{S}_I$.

Let $\Lambda_{3,1,N} \subset \mathcal{A}_{3,1,N}$ denote the set of superspecial points in $\mathcal{A}_{3,1,N}$.

Theorem 5.2.

(a) (Case: $x \in W_{\{0,3\}}\tau$). *Let $\Lambda_{3,1,N}$ be the set of superspecial principally polarized abelian 3-folds with a level-N structure over $\overline{\mathbb{F}}_p$. Then*

(1) *The closure* $\overline{\mathcal{A}_{s_{121}\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to*

$$\mathrm{GL}_3/B_\triangle = \{(\underline{a},\underline{b}) \in \mathbf{P}^2 \times \mathbf{P}^2 \mid \underline{a}\cdot\underline{b} = 0\} =: X \subset \mathbf{P}^2 \times \mathbf{P}^2,$$

where $B_\triangle \subset \mathrm{GL}_3$ *is the Borel subgroup of upper triangular matrices.*

(2) *The closure* $\overline{\mathcal{A}_{s_{21}\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to* $\{(\underline{a},\underline{b}) \in X \mid \underline{b}\cdot\underline{b}^{(p)} = 0\}$.

(3) *The closure* $\overline{\mathcal{A}_{s_{12}\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to* $\{(\underline{a},\underline{b}) \in X \mid \underline{a}\cdot\underline{a}^{(p)} = 0\}$.

(4) *The closure* $\overline{\mathcal{A}_{s_1\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to*

$$F_{\mathbf{P}^2} \cap X = \{(a, a^{(p)}) \mid a\cdot\underline{a}^{(p)} = 0\}.$$

(5) *The closure* $\overline{\mathcal{A}_{s_2\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to*

$$V_{\mathbf{P}^2} \cap X = \{(b^{(p)}, b) \mid b\cdot\underline{b}^{(p)} = 0\}.$$

(6) $|\mathcal{A}_\tau| = |\Lambda_{3,1,N}|\cdot|U(3)(\mathbb{F}_p)/B_0(\mathbb{F}_p)|$, *where* B_0 *is a Borel subgroup over* $\mathbb{F}_p$.

(b) (Case: $x \in W_{\{1,2\}}\tau$). *Let* $J = \{1,2\}$, *and* $\Lambda_J := \pi_{J,I}(\mathcal{A}_\tau)$, *where* $\pi_{J,I} : \mathcal{A}_I \to \mathcal{A}_J$ *is the natural projection.*

(1) *The closure* $\overline{\mathcal{A}_{s_{30}\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1$.

(2) *The closure* $\overline{\mathcal{A}_{s_3\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1$.

(3) *The closure* $\overline{\mathcal{A}_{s_0\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1$.

(4) $|\mathcal{A}_\tau| = |\Lambda_J|\cdot(p^2+1)$.

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