RINGS WITH INDECOMPOSABLE RIGHT MODULES LOCAL

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Abstract. Every indecomposable module over a generalized uniserial ring is uniserial, hence local. This motivates one to study rings \( R \) satisfying the condition (*): \( R \) is a right artinian ring such that every finitely generated, indecomposable right \( R \)-module is local. The rings \( R \) satisfying (*) have been recently studied by Singh and Al-Bleahed (2004), they have proved some results giving the structure of local right \( R \)-modules. In this paper some more structure theorems for local right \( R \)-modules are proved. Examples given in this paper show that a rich class of rings satisfying condition (*) can be constructed. Using these results, it is proved that any ring \( R \) satisfying (*) is such that \( \text{mod-} R \) is of finite representation type. It follows from a theorem by Ringel and Tachikawa that any right \( R \)-module is a direct sum of local modules. If \( M \) is a right module over a right artinian ring such that any finitely generated submodule of any homomorphic image of \( M \) is a direct sum of local modules, it is proved that it is a direct sum of local modules. This provides an alternative proof for that any right module over a right artinian ring \( R \) satisfying (*) is a direct sum of local modules.

0. INTRODUCTION

It is well known that an artinian ring \( R \) is generalized uniserial if and only if every finitely generated indecomposable right \( R \)-module is uniserial. Every uniserial module is local. This motivated Tachikawa [10] to study a ring \( R \) satisfying the condition (*): \( R \) is a right artinian ring such that every finitely generated indecomposable right \( R \)-module is local. Consider the dual condition (**): \( R \) is left artinian such that every finitely generated indecomposable left \( R \)-module is uniform. If a ring \( R \) satisfies (*), it is proved by Tachikawa that \( R \) admits a finitely generated...
injective cogenerator $Q_R$, then $B = \text{End}(Q_R)$ satisfies (**). Tachikawa had studied a ring $R$ satisfying (*) through the corresponding ring $B$, but he did not give structure of local right $R$-modules. Singh and Al-Bleahed [8] have studied rings $R$ satisfying (*) without using the duality, and they have proved some structure theorems on local right $R$-modules. In section 2, structure of a local right $R$-modules is further investigated. By using these results it is proved in Theorem 2.14 that $R$ is of finite representation type. In section 3, general right $R$-modules are investigated. It is well known that exceptional rings as defined by Dlab and Ringel (see [2] or [3]) are balanced ring, and any right module over an exceptional $(1,2)$-ring is a direct sum of local modules. It follows from [2, Proposition 3] and also from [8, Theorem 2.13] that any exceptional $(1,2)$-ring also satisfies (*). It follows from [9, Corollary 4.4], that any right $R$-module is a direct sum of local modules. A direct proof of this result is given, by proving the following: If $M$ is a right module over a right artinian ring, such that any finitely generated submodule of any homomorphic image of $M$ is a direct sum of local modules, then $M$ is a direct sum of local modules (Theorem 3.4). As there is no known duality that can tell that a ring $R$ satisfies (*) if and only if it satisfies (**), it would be interesting to examine condition (***) by itself. In section 4, some examples illustrating various results are given.

1. Preliminaries

All rings considered here are with identity $1 \neq 0$ and all modules are unital right modules unless otherwise stated. Let $R$ be a ring and $M$ be an $R$-module. $J(M)$, $E(M)$ and $\text{socle}(M)$ denote radical, injective hull and socle of $M$ respectively, however $J(R)$ will be denoted by $J$. If $R$ is right artinian, then $J(M) = MJ$. Further, $N \leq M$ denotes that $N$ is a submodule of $M$. A ring $R$ is called a local ring, if $R/J$ is a division ring. Given two positive integers $n$, $m$, a ring $R$ is called an $(n,m)$-ring if it is a local ring, $J^2 = 0$ and for $D = R/J$, $\dim_D J = n$ and $\dim J_D = m$. Any $(1,2)$ (or $(2,1)$) ring $R$ is called an exceptional ring if $E(R)$ (respectively $E(RR)$) is of composition length 3 [4, p 446]. A module in which the lattice of submodules is linearly ordered under inclusion, is called a uniserial module, and module that is a direct sum of uniserial modules is called a serial module [5, Chapter V]. If for a ring $R$, $RR (RR)$ is serial, then $R$ is called a left (right) serial ring. A ring that is local, both serial and artinian, is called a chain ring. A ring $R$ is said to be of finite right representation type, if it admits only finitely many non-isomorphic indecomposable right $R$-modules [5, p 109]. If a module $M$ has finite composition length, then $d(M)$ denotes the composition length of $M$. For definitions of $M$-injective and $M$-projective modules one may refer to [1, p 184].
2. LOCAL MODULES

Consider the following condition on a ring $R$: (*) $R$ is a right artinian ring such that any finitely generated indecomposable right $R$-module is local.

The following is proved in [8, Proposition 2.2]

**Proposition 2.1.** Let $R$ be a right artinian ring. Then $R$ satisfies (*) if and only if for any two non-simple local right $R$-modules $A$, $B$, simple submodules $S$, $T$ of $A$, $B$ respectively, and any $R$-isomorphism $\sigma: S \to T$, either $\sigma$ or $\sigma^{-1}$ extends to an $R$-homomorphism from $A$ to $B$ or from $B$ to $A$ respectively.

**Proposition 2.2.** ([8]). Let $R$ be a ring satisfying (*).

(i) Any uniform right $R$-module is uniserial.

(ii) $R$ is left serial.

(iii) Let $A$, $B$ be two uniserial right $R$-modules each of composition length at least three. Then $M = A \oplus B$ does not contain any local, non-uniserial submodule of composition length 3.

(iv) Let $C_1$, $C_2$ be two uniserial $R$-modules such that for some $k \geq 2$, $C_1/C_1J^k \cong C_2/C_2J^k$, and $C_1J^k$, $C_2J^k$ are non-zero, then $C_1/C_1J^{k+1} \cong C_2/C_2J^{k+1}$.

(v) Let $A_R$, $B_R$ be two local modules such that $d(A) = d(B)$, $AJ^2 = 0 = BJ^2$. For any simple submodule $S$ of $A$, any $R$-monomorphism $\sigma: S \to B$ extends to an $R$-isomorphism from $A$ onto $B$.

For any local module $A_R$, $AJ$ is a direct sum of uniserial modules [8, Lemma 2.7].

**Theorem 2.3.** ([8, Theorem 2.10]). Let $R$ be a ring satisfying (*) and $A_R$ be a local module such that $AJ = C_1 \oplus C_2 \oplus \cdots \oplus C_t$ for some uniserial modules $C_i$. Then the following hold.

(i) Either all $C_i/C_iJ$ are isomorphic or $t \leq 2$.

(ii) Any local submodule of $AJ$ is uniserial.

(iii) If $d(C_1) \geq 2$, then either $t \leq 2$ or any $C_i$ is simple for $i \geq 2$.

**Proposition 2.4.** Let $R$ be a ring satisfying (*).

(i) Let $A_1$ and $A_2$ be any two uniserial right $R$-modules. Then $A_1J \oplus A_2J$ does not contain a submodule that is local but not uniserial.

(ii) If a non-zero homomorphic image of a uniserial right $R$-module $L$ is injective, then $L$ is injective.

(iii) Let $A_R$ be a local module, and $AJ = C_1 \oplus D$, where $C_1$ is uniserial. Let $\sigma$ be an $R$-endomorphism of $A$ such that $\ker \sigma \cap C_1 = 0$, and $\sigma$ is not an automorphism. Then $\sigma(A)$ is a uniserial module of composition length more than $d(C_1)$. $A/D$ embeds in $\sigma(A)$ and no homomorphic image of $A/D$ is injective. If a module $B_R$ embeds in $C_1$, then no non-zero homomorphic image of $B$ is injective.
Let \( A_R \) be a local module, and \( AJ = C_1 \oplus C_2 \oplus \ldots \oplus C_t \) for some uniserial submodules \( C_i \). Let \( s = \max\{d(C_i) : 1 \leq i \leq t\} \). Then for any simple submodule \( S \) of \( A \), and any uniserial submodule \( B \) of \( A \) of composition length \( s \), any \( R \)-homomorphism \( \sigma : S \to B \) extends to an \( R \)-endomorphism of \( A \); if in addition \( S \) is contained in a uniserial submodule of composition length \( s \), then \( \sigma \) is an automorphism.

**Proof.**

(i) On the contrary suppose that \( A_1J \oplus A_2J \) contains a non-uniserial local submodule \( uR \). Then \( u = u_1 + u_2 \), \( 0 \neq u_i \in A_iJ \), and \( uJ \) is a direct sum of two non-zero uniserial submodules. As \( A_i \) are uniserial, without loss of generality we take \( u_iR = A_iJ \). Then \( uJ^2 = A_1J^3 \oplus A_2J^3 \). This gives that \((A_1 \oplus A_2)/uJ^2 = B_1 \oplus B_2 \) for some uniserial modules with \( d(B_i) \geq 3 \). But \( uR/uJ^2 \) is local, non-uniserial of composition length 3, and it embeds in \( B_1 \oplus B_2 \). This contradicts (2.2)(iii). Hence \( A_1J \oplus A_2J \) does not contain a non-uniserial, local submodule.

(ii) It is immediate from the fact that any uniform right \( R \)-module is uniserial.

(iii) By (2.3)(ii), \( \sigma(A) \) is uniserial. As \( B = ker\sigma \) embeds in \( D \), it is immediate that \( d(\sigma(A)) \geq d(A/D) = d(C_1) + 1 \). As \( B \cap C_1 = 0 \), \( C_1 \) embeds in \( \sigma(A) \). Also, \( C_1 \) embeds in \( A/D \), it also follows that \( A/D \) embeds in \( \sigma(A) \). As \( \sigma(A) \) is not injective, by (ii) no homomorphic image of \( A/D \) is injective. The last part also follows from (ii).

(iv) Let \( C = socle(B) \) and \( \sigma : S \to B \) be an \( R \)-homomorphism. Suppose the contrary. As every uniserial \( R \)-module is quasi-injective, \( t \geq 2 \), \( d(A) \geq s + 2 \) and \( AJ \) contains no uniserial submodule of composition length more than \( s \). By (2.1), \( \sigma^{-1} : C \to S \) extends to an \( R \)-endomorphism \( \lambda \) of \( A \). Then \( \lambda \) is not an automorphism, and \( \lambda(A) \subseteq AJ \). As \( \lambda \) is one-to-one on \( B \), we get \( d(\lambda(A)) \geq s + 1 \). But by (2.3)(ii), \( \lambda(A) \) is uniserial, so we have a contradiction. The last part again follows from (2.3)(ii).

(2.1) gives the following.

**Proposition 2.5.** Let a ring \( R \) satisfy (*). Let \( A_R, B_R \) be two local, modules such that \( A \) is \( B \)-projective and \( B \) is \( A \)-projective. Let \( A_1 < A_2 < A \), \( B_1 < B_2 < B \) be such that \( A_2/A_1 \) is simple and there exists an \( R \)-isomorphism \( \sigma : A_2/A_1 \to B_2/B_1 \). Then either there exists an \( R \)-homomorphism \( \lambda \) of \( A \) to \( B \) inducing \( \sigma \) or there exists an \( R \)-homomorphism \( \lambda : B \to A \) inducing \( \sigma^{-1} \).

Henceforth, throughout this section \( R \) is a ring satisfying (*).

**Lemma 2.6.** Let \( A_R \) be a local module.
(i) If \( AJ = C_1 \oplus C_2 \), where \( C_i \) are minimal submodules, then either \( A/C_1 \) or \( A/C_2 \) is injective.

(ii) If \( AJ = C_1 \oplus C_2 \), where \( C_i \) are uniserial, then either \( A/C_1 \) or \( A/C_2 \) is such that its every non-simple homomorphic image is injective.

(iii) Suppose \( AJ = C_1 \oplus C_2 \oplus \ldots \oplus C_t \) such that each \( C_i \) is uniserial and \( t \geq 3 \).

For each \( 1 \leq i \leq t \), let \( L_i \) be the direct sum of all \( C_j \) with \( j \neq i \). Then every non-simple homomorphic image of any \( A/L_i \) is injective.

(iv) Let \( AJ = C_1 \oplus C_2 \oplus D \) with \( C_1 \) and \( C_2 \) both uniserial. Suppose for some \( k, l, C_1 J^k/C_1 J^{k+1} \) and \( C_2 J^l/C_2 J^{l+1} \) are isomorphic, and for some \( u \geq 1 \), \( C_1 J^{k+u} \neq 0 \neq C_2 J^{l+u} \), then \( C_1 J^{k+u}/C_1 J^{k+u+1} \) and \( C_2 J^{l+u}/C_2 J^{l+u+1} \) are isomorphic.

Proof.

(i) If none of \( A/C_i \) is injective, then \( A \) embeds in \( E(A/C_1)J \oplus E(A/C_2)J \), which contradicts (2.4)(i). This proves (i).

(ii) By applying (i) to \( A/AJ^2 \) and by using Proposition (2.4)(ii), it follows.

(iii) For \( t \geq 3 \), as all \( C_i/C_iJ \) are isomorphic by (2.3), the result follows from (i).

(iv) It is enough to prove the result for \( u = 1 \). Suppose that \( C_1 J^{k+1}/C_1 J^{k+2} \) and \( C_2 J^{l+1}/C_2 J^{l+2} \) are not isomorphic. For some indecomposable idempotent \( e \in R \), \( C_1 J^k/C_1 J^{k+2} \) and \( C_2 J^l/C_1 J^{l+2} \) both are homomorphic images of \( eR \). This gives a local, non-uniserial module \( B_R \) of composition length 3 with \( BJ = L_1 \oplus L_2 \) such that \( B/L_1 \cong C_1 J^k/C_1 J^{k+2} \) and \( B/L_2 \cong C_2 J^l/C_1 J^{l+2} \). Let \( A = A/(C_1 J^{k+2} \oplus C_2 J^{l+2}) \). Then \( B \) embeds in the radical of the direct sum of \( A/C_1 \oplus D \) and \( A/C_2 \oplus D \), which is a contradiction to (2.4)(i). This proves the result.

Lemma 2.7. Let \( A_R \) be a local module and \( B_R \) any module. For some \( C \subseteq B \), let \( \sigma : A \to B/C \) be an \( R \)-homomorphism.

(i) There exists a local submodule \( D \) of \( A \times B \) such that \( D = (a, b)R \) with \( aR = A \) and \( \sigma(a) = b + C \). If \( D \) is uniserial and \( d(B) \leq d(A) \) then \( \sigma \) can be lifted to some \( R \)-homomorphism \( \eta : A \to B \).

(ii) If \( A \times B \) does not contain a local submodule \( D_1 \) with \( d(D_1) > d(A) \), then \( A \) is \( B \)-projective.

(iii) If \( A \times B \) has no non-uniserial local submodule and \( d(B) \leq d(A) \), then \( A \) is \( B \)-projective.

Proof. (i) Let \( N = \{(x, y) \in A \times B : \sigma(x) = y + C\} \). Let \( \pi : A \times B \to A \) be the natural projection. Then \( \pi(N) = A \). There exists a local submodule \( D \) of \( N \) such that \( \pi(D) = A \). Clearly \( D = (a, b)R \) with \( A = aR \) and \( \sigma(a) = b + C \). Now
\[ d(D) \geq d(A). \] Suppose \( d(D) = d(A) \). Then \( D \cong A \) and \( \eta : A \to B \) given by \( \eta(ar) = br \) lifts \( \sigma \). In case \( D \) is uniserial and \( d(B) \leq d(A) \), then \( D \cong A \), so once again \( \sigma \) can be lifted. After this (ii) is immediate. Under the hypothesis in (iii), the hypothesis in (ii) holds, so \( A \) is \( B \)-projective.

**Lemma 2.8.** Let \( A_R \) and \( B_R \) be two uniserial modules and \( \sigma : A \to B/C \) be an \( R \)-epimorphism for some \( C < B \).

(i) If \( A \) is not injective and \( d(B) \leq d(A) \), then either \( B \) is injective or \( \sigma \) can be lifted to some \( R \)-homomorphism \( \eta : A \to B \).

(ii) If \( d(B) \leq d(A) \) and neither \( A \) nor \( B \) is injective, then \( A \) is \( B \)-projective.

(iii) Any uniserial right \( R \)-module is either injective or quasi-projective.

(iv) Let \( C = \socle(A) \), and \( C < D < A \) with \( D/C \) a simple module. If \( C \cong D/C \), then all the composition factors of \( A \) are isomorphic.

**Proof.** By Lemma 2.7, there exists a local submodule \( D = (a,b)R \subseteq A \times B \) such that \( A = aR \) and \( \sigma(a) = b + C \). Suppose \( d(B) \leq d(A) \). If \( D \) is uniserial, it follows from (2.7)(i) that \( \sigma \) lifts to an \( R \)-homomorphism \( \eta : A \to B \). Suppose \( D \) is not uniserial. Then \( DJ = C_1 \oplus C_2 \) for some non-zero uniserial submodules \( C_i \). Let \( \pi_A \) and \( \pi_B \) be the natural projections of \( A \times B \) onto \( A \) and \( B \) respectively. Then for one of the \( C_i \) say \( C_1 \), \( \pi_A(C_1) = AJ \). But \( \pi_B(C_1) \subseteq BJ \) and \( d(B) \leq d(A) \), it follows that \( C_1 \) is isomorphic to \( AJ \) under \( \pi_A \). Therefore \( AJ \times BJ = C_1 \oplus (0 \times BJ) \), \( DJ = C_1 \oplus (DJ \cap (0 \times BJ)) \) and \( C_2 \cong DJ \cap (0 \times BJ) \). Suppose that neither \( A \) nor \( B \) is injective, then \( D \subseteq E(A)J \oplus E(B)J \), therefore by (2.4), \( D \) is uniserial. Then, by using (2.7)(i), we get \( A \) is \( B \)-projective. From this (i), (ii) and (iii) follow. (iv) is immediate from the fact that the injective hull of \( A \) is uniserial.

**Lemma 2.9.** Let \( A_R \) be a local module such that \( AJ = A_1 \oplus A_2 \) for some uniserial submodules \( A_i \) and there exists an \( R \)-isomorphism \( \sigma : \socle(A_1) \to \socle(A_2) \). Let there exists an \( R \)-endomorphism \( \mu \) of \( A \) that extends \( \sigma \). Let \( M_i \) be the maximal submodule of \( A_i \). Then:

(i) \( d(A_1) \leq d(A_2) \), \( A/A_1 \) is injective and \( A/M_2 \oplus A_1 \) is injective.

(ii) If \( d(A_1) < d(A_2) \), then \( A/A_2 \) is quasi-projective, \( A/(M_1 \oplus A_2) \) is not injective and \( A/(M_2 \oplus A_1) \) is injective.

(iii) If \( d(A_1) = d(A_2) \), then \( A/M_2 \oplus A_1 \cong A/M_1 \oplus A_2 \) and both are injective.

(iv) If \( A_1/A_1J \cong A_2/A_2J \), then \( A_1 \cong A_2 \).

**Proof.** Suppose \( f : eR \to A \) is the projective cover of \( A \). We take \( A = eR/B \) and \( A_i = C_i/B \) for some right ideals \( B < C_i < eR \). Suppose there exists
an $R$-endomorphism $\mu$ of $A$ that extends $\sigma$. We can find an $R$-endomorphism $\lambda$ of $eR$ that lifts $\mu$. Then $\lambda(B) \subseteq B$, $\lambda(\text{soc}(C_1)) + B = \text{soc}(C_2) + B \not\subseteq C_1 + B$. Hence $C_1$ is not invariant under the endomorphisms of $eR$, $eR/C_1$ is not quasi-projective, therefore $A/A_1$ being isomorphic to $eR/C_1$ is not quasi-projective. By (2.8)(iii), $A/A_1$ is injective. As $\mu(A_1) \cong A_1$ and $\mu(A_1) \cap A_1 = 0$, it follows that $d(A_1) \leq d(A_2)$ and $A_1$ embeds in $A_2$. Let $M_i \leq A_i$. Suppose $d(A_1) < d(A_2)$, it follows that $A/A_1$ is isomorphic to a submodule $A_2$, and hence $A/(A_2 + M_1)$ is not injective. Therefore by (2.6)(i), $A/(A_1 + M_2)$ is injective. As $A/A_2$ is not injective, by (2.8)(iii), it is quasi-projective. If $d(A_1) = d(A_2)$, then the isomorphism $\sigma$ gives that $A/A_1 + M_2$ and $A/A_2 + M_1$ are isomorphic, so once again, by (2.6)(ii), both are injective. The hypothesis in (iv) gives that $A/(M_2 + A_1) \cong A/(M_1 + A_2)$, so they are injective by (i). By (ii), $d(A_1) = d(A_2)$. Hence $A_1 \cong A_2$.

**Theorem 2.10.** Let $R$ be a local ring satisfying (*). If $J^2 \neq 0$, then $R$ is a chain ring.

**Proof.** By (2.2), $R$ is a left serial ring. If $R$ is not right serial, we get a local, right $R$-module $A$ such that $A AJ = C_1 \oplus C_2$ with each $C_i$ uniserial, $d(C_1) = 2$, $d(C_2) = 1$. As every composition factor of $A$ is isomorphic to $R/J$, it contradicts (2.9)(iv). Hence $R$ is a chain ring.

**Lemma 2.11.** Let $A_R$ be a local module such that $A AJ = A_1 \oplus A_2 \oplus L$ for some uniserial modules $A_i$, with $d(A_1) > 1$, and $L \neq 0$. Then no two composition factors of $A_1$ are isomorphic.

**Proof.** By (2.3), $A J/AJ$ is homogeneous. Suppose, $A_1$ has two isomorphic composition factors. Then for some $s \geq 1$, $A_1/A_1 J \cong A_1 J^s/A_1 J^{s+1}$. Let $B = A/(A_1 J^{s+1} + L)$. Then $B$ contradicts (2.9)(ii).

**Theorem 2.12.** Let $A_R$ be a local module over a ring $R$ satisfying (*) such that $AJ = C_1 \oplus C_2 \oplus \ldots \oplus C_t$ for some uniserial modules $C_i$ such that $t \geq 2$, and $d(C_1) \geq 2$. Let $C_i/C_i J \cong C_t/C_t J$ for some $i > 1$, then $t = 2$. If $A$ is projective, then $C_1 \cong C_2$.

**Proof.** To start with, we take $A = eR$ for some indecomposable idempotent $e$. Suppose $C_1/C_1 J \cong C_2/C_2 J$. So there exists an indecomposable idempotent $f \in R$, such that for some $u, v \in eJf, C_1 = uR, C_2 = vR$. Then $u, v \in eJf \setminus J^2$. As $R$ is left serial, $Rf = Ru = Rv$. We get $v = bu$ for some unit $b$ in $eRe, C_2 = bC_1, d(C_1) = d(C_2)$. This contradicts (2.3)(iii) unless $t = 2$. By (2.6)(iv), $\text{soc}(C_1) \cong \text{soc}(C_2)$, hence $C_1 \cong C_2$. In general, as $A$ is a homomorphic image of an $eR$, where $e = e^2$ is indecomposable, the result follows.
Theorem 2.13. Let $A_R$ be a local module such that $AJ = C_1 \oplus C_2$, where $C_i$ are uniserial, and $C_1J^k/C_1J^{k+1} \cong C_2J^l/C_2J^{l+1} \neq 0$, for some $k < l$.

(i) $A/C_1$ has all its non-simple homomorphic images injective.
(ii) No two composition factors of $C_2$ are isomorphic.
(iii) No composition factor of $C_2$ is isomorphic to a composition factor of $C_1$.
(iv) $A, A/C_1$ and $A/C_2$ are all quasi-projective.

Proof. Let $\lambda : eR \to A$ give the projective cover of $A$. Then $eJ = D_1 \oplus D_2 \oplus L$, where $D_1, D_2$ are uniserial and $C_1 = \lambda(D_1)$. If $L \neq 0$, by (2.11), $D_1$ has no two composition factors isomorphic, which is a contradiction. Hence $L = 0$, and $eJ = D_1 \oplus D_2$. For some $s \geq 1$, $D_1/D_1J \cong D_1J^s/D_1J^{s+1}$. Thus $eR/(D_2 \oplus D_1J)$ embeds in $D_1/D_1J^{s+1}$, therefore it is not injective. Consequently, by (2.6)(i), $eR/(D_1 \oplus D_2J)$ is injective. Then, by (2.4)(ii), every non-simple homomorphic image of $eR/D_1$ is injective. If $D_2$ has two isomorphic composition factors, the interchange of the roles of $D_1, D_2$ will give that every non-simple homomorphic image of $eR/D_2$ is injective, in particular, $eR/(D_2 \oplus D_1J)$ is injective, which is a contradiction. Hence $D_2$ has no two composition factors isomorphic.

Suppose $eR/D_2$ is not quasi-projective. Then $D_2$ is not invariant under the $R$-endomorphisms of $eR$, consequently, there exists a non-zero homomorphism of $D_2$ into $D_1$. Therefore $D_2/D_2J \cong D_1J^v/D_1J^{v+1}$ for some $v \geq 0$. If $v > 0$, we get $eR/D_1 \oplus D_2J$ is not injective, which is a contradiction to (i) for $eR$. Hence $v = 0$. Then $eR/D_2 \oplus D_1J$ is isomorphic to $eR/D_1 \oplus D_2J$, so once again it is injective, which is a contradiction. Hence $eR/D_2$ is quasi-projective.

Suppose there exists an $R$-isomorphism $\sigma : D_1J^i/D_1J^{i+1} \to D_2J^j/D_2J^{j+1}$ for some $i$ and $j$, with $D_1J^i \neq 0$. If $j \leq i$, then $D_2/D_2J \cong D_1J^v/D_1J^{v+1}$ for some $v$, and as in the above paragraph, we get a contradiction. Hence $i < j$. Then $D_1J^u/D_1J^{u+1} \cong D_2/D_2J \cong D_2J^v/D_2J^{v+1}$ for some $u \geq 1$. Then $eR/eJ \cong D_2J^v/D_2J^{v+1}$ is isomorphic to the top and bottom composition factors of $eR/D_2 \oplus D_1J^u$, and to the top and bottom composition factors of $eR/D_2 \oplus D_2J^v$. At the same time $D_2/D_2J$ is isomorphic to a composition factor of $eR/D_2 \oplus D_2J^u$. The periodicity of the composition factors gives that $D_2/D_2J$ is also isomorphic to a composition factor of $eR/D_2 \oplus D_1J^u$. Thus $D_2/D_2J$ is either isomorphic to a composition factor of $D_1/D_1J^u$ or it is isomorphic to $eR/eJ$. In the former case, we get a contradiction to $i < j$, and in the later case, every composition factor of $eR/D_2 \oplus D_2J^u$ and of $eR/D_2 \oplus D_1J^u$ is isomorphic to $eR/eJ$, and therefore $D_1/D_1J \cong D_2/D_2J$, which is a contradiction. Hence $D_1$ has no composition factor isomorphic to a composition factor of $D_2$. Hence $C_2 = \lambda(D_2)$. It follows that any submodule of $D_1 \oplus D_2$ is invariant under any $R$-endomorphism of $eR$. Consequently, $A, A/C_1$ and $A/C_2$ are all quasi-projective.
Theorem 2.14. If a ring $R$ satisfies (*), then there exist only finitely many non-isomorphic, local right $R$-modules.

Proof. All indecomposable finitely generated right $R$-modules are local. As $R$ is right artinian, there exists a bound on the composition lengths of the local modules and on the number of possible semi-simple modules that occur as socles of the local right $R$-modules. To prove the result it is enough to prove that given a triple $(S_R, n, T_R)$, where $S_R$ is simple, $T_R$ is semi-simple and $n$ is a positive integer, there do not exist more than two local modules $A_R$ such that $S \cong A/AJ$, $d(A) = n$ and socle$(A) \cong T$.

Fix a local module $A_R$. Let $B_R$ be another local module such that $A/AJ \cong B/BJ$, $d(A) = d(B)$ and socle$(A) \cong$ socle$(B)$. If $A$ is uniserial, then so is $B$, and obviously $A_R \cong B_R$. So we shall suppose that $A$ is not uniserial. Now $A$, $B$ admit same projective cover, say $eR$.

Suppose $AJ$ is semi-simple. Then $BJ$ is also semi-simple. By (2.2)(v), $A$ and $B$ are isomorphic.

Henceforth we shall suppose that $AJ$ is not semi-simple. Then $AJ = D_1 \oplus D_2 \oplus \ldots \oplus D_u$, $BJ = H_1 \oplus H_2 \oplus \ldots \oplus H_u$ and $eJ = C_1 \oplus C_2 \oplus \ldots \oplus C_l$ for some uniserial modules $D_i$, $H_j$, $C_k$, with $u \leq t$. We take $d(D_1) \geq 2$, $d(H_1) \geq 2$ and $D_1$ a homomorphic image of $C_1$.

Suppose $t \geq 3$. Then all other $C_j$ for $j \geq 2$ are simple. As $D_1$ and $H_1$ have same composition length, and by (2.11), no two composition factors of $C_i$ are isomorphic, we get an isomorphic $\sigma :$ socle$(D_1) \rightarrow$ socle$(H_1)$. Because of (2.1), we can take $\sigma$ such that it extends to an $R$-homomorphism $\lambda : A \rightarrow B$. As in (2.4)(iv), $\lambda$ is an isomorphism. Hence $A_R \cong B_R$.

Henceforth, we take $t = 2$. Then $u = 2$. It follows that $A/(D_1 \oplus D_2J)$ is either isomorphic to $eR/C_1 \oplus C_2J$ or to $eR/C_1 \oplus C_1J$. As socle$(A) \cong$ socle$(B)$, we take socle$(D_1) \cong$ socle$(H_1)$ for $i = 1, 2$. Suppose $d(D_1) = d(H_1)$. By using (2.1), we can suppose that there exists an $R$-homomorphism $\lambda : A \rightarrow B$ such that $\lambda(\text{socle}(D_1)) = \text{socle}(H_1)$. If $\lambda$ is not an isomorphism, then $\lambda(A)$ is a uniserial module contained in $BJ$ such that $\lambda(A) \cap H_2 = 0$, and $d(\lambda(A)) > d(H_1)$. Therefore $d(\lambda(A) + H_2) > d(BJ)$, which is a contradiction. Hence $A_R \cong B_R$.

Suppose $d(D_1) \neq d(H_1)$. Because of (2.6)(ii), we take $D_1$ such that every non-simple homomorphic image of $A/D_1$ is injective. If $d(D_2) < d(H_2)$, then as socle$(D_2) \cong$ socle$(H_2)$, $A/D_1$ embeds in $H_2$, so $A/D_1$ is not injective, which is a contradiction. Hence $d(H_2) < d(D_2)$. Then $B/H_1$ embeds in $D_2$, therefore $B/H_1$ has no non-zero homomorphic image injective. Hence every non-simple homomorphic image of $B/H_2$ is injective. Therefore, $A/D_1 \oplus D_2J$ and $B/H_2 \oplus H_1J$ are isomorphic, that gives $D_2/D_2J \cong H_1/H_1J$ and $D_1/D_1J \cong H_2/H_2J$. Now $d(D_1) < d(H_1)$, so $D_1$ embeds in $H_1$. Therefore $D_1/D_1J$ is isomorphic to a composition factor of $H_1$. Thus $D_1/D_1J$ is isomorphic to a composition factor of $H_1$. Therefore
as well as of $H_2$. Then by (2.13), no two composition factors of $H_1$ are isomorphic and no two composition factors of $H_2$ are isomorphic. So there exists unique positive integer $t$ such that $D_1/D_1J \cong H_1J^t/H_1J^{t+1}$. That gives $D_1 \cong H_1J^t$. Thus $d(D_1) = d(H_1) - t$ and $d(D_2) = d(H_2) + t$. Hence by the cases discussed above, the result follows.

\section{3. Decomposition Theorem}

\textbf{Lemma 3.1.} Let $M$ be any right module over a ring $R$.

(i) Let $L$ be a finitely generated submodule of $M$ such that $L$ is a summand of any finitely generated submodule of $M$ containing $L$. Let $S < M$ be such that $S$ is finitely generated and in $\overline{M} = M/L$, $\overline{S}$ is a summand of every finitely generated submodule of $\overline{M}$. Then $L + S$ is a summand of any finitely generated submodule of $M$ containing $L + S$.

(ii) Let $N \leq M$ such that $N$ is finitely generated and is summand of any finitely generated submodule of $M$ containing $N$. Then $NJ = MJ \cap N$.

(iii) If $L$ is a finitely generated submodule of $M$ such that it is a summand of every finitely generated submodule of $M$ containing $L$, then any summand $K$ of $L$ is also a summand of any finitely generated submodule of $M$ containing $K$.

\textbf{Proof.}

(i) Let $L + S \leq T$, where $T$ is a finitely generated submodule of $M$. Then $T = L \oplus C$, $L + S = L \oplus W$ for some $C \leq M, W \leq M$. Therefore $\overline{S} = \overline{W}$ and $\overline{S} \leq \overline{C}$ in $\overline{M} = M/L$. By the hypothesis, $\overline{C} = \overline{S} \oplus \overline{K}$ for some $K \leq M$ containing $L$. Thus $T = S + K = W + K$ and $W \cap K \subseteq L$. As $K$ is finitely generated, $K = L \oplus V$ for some $V \leq K$, $T = (W + L) + V$. Suppose for some $w \in W, x \in L$, and $v \in V, w + x = v$. Then $w \in W \cap K \subseteq L, v \in L \cap V = 0$. Hence $(W + L) \oplus V = T = (S + L) \oplus V$.

(ii) Let $x \in MJ \cap N$. Then $x = \sum_{j} x_j a_j$ for some finitely many $x_j \in M, a_j \in J$.

Set $K = \sum_{j} x_j R + N$. Then $K$ is finitely generated, $x \in KJ$, $K = N \oplus P$ for some $P \leq K$, and $KJ = NJ \oplus PJ$. Hence $x \in NJ$.

(iii) Now $L = K \oplus S$ for some $S \leq L$. Suppose $K \leq T$, a finitely generated submodule of $M$. Then $T + S = L \oplus V = K \oplus (S \oplus V)$. This gives $T = K \oplus W$, where $W = T \cap (S \oplus V)$.

\textbf{Definition 3.2.} A module $M$ is said to satisfy $(\diamond)$ if any finitely generated submodule of any homomorphic image of $M$ is a direct sum of local modules having finite composition lengths.
Lemma 3.3. Let $M_R$ be a module satisfying (◇) and $R$ be right artinian. Let $A = \bigoplus_{\alpha \in \Lambda} A_{\alpha} \subseteq M$ such that, each $A_{\alpha}$ is finitely generated and for any finite subset $X$ of $\Lambda$, $A_X = \sum_{\alpha \in X} A_{\alpha}$ is a summand of any finitely generated submodule of $M$ containing it. Let $S$ be a local submodule of $M$ such that $S$ in $M/A$ is non-zero and is a summand of any finitely generated submodule of $M$ containing $S$.

(a) Let $\Gamma$ be any finite subset of $\Lambda$ such that $S \cap A = S \cap C$, where $C = A_{\Gamma}$. Then $S$ in $M/C$ is also a summand of any finitely generated submodule of $M/C$ containing $S$.

(b) There exists a local submodule $S_1$ of $M$ such that $A \cap S_1 = 0$, $S_1 = S$ in $M/A$, and for any finite subset $\Gamma$ of $\Lambda$, $A_{\Gamma} + S_1$ is a summand of any finitely generated submodule of $M$ containing $S$.

Proof.

(a) It follows from (3.1)(ii) that $AJ = MJ \cap A$. Now $S \cap A = SJ \cap A = SJ \cap (MJ \cap A) = S \cap AJ$. As $S$ is finitely generated, we get a finite subset $\Gamma$ of $\Lambda$ such that $S \cap A = S \cap C \cap J$, where $C = A_{\Gamma}$. In $M = M/C$, let $S$ be contained in a finitely generated submodule $T$, with $C \subseteq T$. Then $T$ is finitely generated. Now $A = C \oplus D$ for some $D \subseteq A$. Consider $T_1 = T + D$. In $M/A, T_1 = T$ and $S \subseteq T_1$. Therefore $T_1 = S \oplus L$ for some $A \subseteq L, S \cap L = S \cap A = S \cap C \cap J$. We get $T = S + (T \cap L)$ with $S \cap (L \cap T) \subseteq C \cap J$. This gives $(S + C) \cap ((L \cap T) + C) = C + [(S + C) \cap (L \cap T)] = C$, as $C \subseteq L \cap T$. Hence, $S$ in $M/C$ is a summand of $T$.

(b) Let $\Gamma$ be a finite subset of $\Lambda$ such that $S \cap A = S \cap C \cap J$, where $C = A_{\Gamma}$. We choose $S$ to be of smallest composition length among those local submodules $S'$ for which $S = S'$. By the hypothesis, $S + C = C \oplus S_1$ for some local submodule $S_1$ of $M$. Then in $M/A$, $S = S_1$ and $d(S_1) \leq d(S)$. That gives $d(S) = d(S_1)$ and $C + S = C \oplus S$. Hence $A \cap S = 0$. Let $X$ be any finite subset of $\Lambda$. Now $A \cap S = A_X \cap S = 0$. Let $T$ be any finitely generated submodule of $M$ containing $A_X$ such that in $M/A_X$, $S \subseteq T$, then by (a), $S$ is a summand of $T$. Now $T = A_X \oplus P$ for some $P \subseteq T$. In $M/A_X$, $S \subseteq P$, $P = S \oplus Q$ for some $Q \subseteq M$ containing $A_X$. Therefore, $T = S \oplus Q$, as $S \cap Q \subseteq A_X \cap S = 0$. But $A_X$ is also a summand of $Q$. Hence $A_X \oplus S$ is a summand of $T$. This proves the result. ■

Theorem 3.4. If a module $M_R$ satisfies satisfies (◇), where $R$ is right artinian, then $M$ is a direct sum of local modules. Any module over a ring $R$ satisfying (◇) is a direct sum of local modules.
Proof. Let $xR$ be a local submodule of $M$ of smallest composition length such that $xR \not\subseteq MJ$. Let $T$ be a finitely generated submodule of $M$ containing $xR$. Now $T = \bigoplus_{i=1}^{n} A_i$ for some local submodules $A_i$. Let $\pi_i : T \to A_i$ be the projections giving this decomposition of $T$. If for every $i$, either $\pi_i(xR) \subsetneq A_iJ$ or $A_i \subseteq MJ$, then $xR \subseteq MJ$, which is a contradiction. Thus for some $i$, $\pi_i(xR) \not\subseteq A_iJ$ and $A_i \not\subseteq MJ$. Then $\pi_i(xR) = A_i$, $d(xR) = A_i$. Therefore $\pi_i$ maps $xR$ isomorphically onto $A_i$. Hence $xR$ is a summand of $T$. Let $F$ be the family of all those local submodules of $M$ that are summand of any finitely generated submodule that contains them. Thus $F$ is non-empty. A subfamily $F'$ of $F$ is said to satisfy condition $(S)$, if the sum of the members of $F'$ is direct and the sum of any finite subfamily of $F'$ is a summand of any finitely generated submodule of $M$ containing that sum. The set of all such subfamilies is non-empty. Union of any chain of subfamilies of $F$ satisfying $(S)$ satisfies $(S)$. So, there exists a maximal subfamily $\{A_\alpha\}_{\alpha \in \Lambda}$ of $F$ satisfying $(S)$. Thus $\{A_\alpha\}_{\alpha \in \Lambda}$ satisfies the hypothesis in (3.3). Now $N = \sum_{\alpha \in \Lambda} A_\alpha = \bigoplus_{\alpha \in \Lambda} A_\alpha$. Suppose $M \neq N$. Then as for $M$, $M/N$ has a local submodule $\overline{B}$ that is a summand of any finitely generated submodule of $M/N$ containing $\overline{B}$. As seen in the proof of (3.3)(b), we can choose $B$ such that it is local, $N \cap B = 0$ and the family $\{A_\alpha\}_{\alpha \in \Lambda} \cup \{B\}$ satisfies $(S)$, which is a contradiction to the maximality of $\{A_\alpha\}_{\alpha \in \Lambda}$. Hence $M = N$, a direct sum of local submodules. As any module over a ring satisfying (*), satisfies $(\phi)$, the second part follows.

Theorem 3.5. Let $R$ be a ring satisfying (*), and $M$ be any right $R$-module. Then any local submodule of $MJ$ is uniserial and $MJ$ is a direct sum of uniserial submodules. $R/r.ann(J)$ is a generalized uniserial ring.

Proof. Let $T$ be a finitely generated submodule of $MJ$. Suppose $T$ is not a direct sum of uniserial submodules. So there exists a local submodule $uR$ of $T$ that is not uniserial. There exists a finitely generated submodule $K$ of $M$ such that $T \subseteq KJ$. Now $K = \bigoplus_{i=1}^{n} A_i$ for some local submodules $A_i$. Let $\pi_i : K \to A_i$ be the corresponding projections and $L_i = ker(\pi_i \mid uR)$. As $uR/L_i$ embeds in $A_iJ$, by (2.2), each $uR/L_i$ is uniserial. Therefore $L_i \neq 0$ for any $i$. However, $\bigcap_{i} L_i = 0$, so we get, say $L_1, L_2$ such that $L_1 \not\subseteq L_2$ and $L_2 \not\subseteq L_1$. Let $v = \pi_1(u) + \pi_2(u)$. Then $vR \cong uR/(L_1 \cap L_2)$, it is local but not uniserial. As $\pi_i(u)R \subseteq A_iJ$, by [8, Lemma 2.7], $\pi_i(u)R$ is uniserial. For any local module $A_R$, as $AJ$ is a direct sum of uniserial modules, any uniserial submodule $wR$ of $AJ$ embeds in a uniserial submodule of $AJ$. From this it follows that there exist two uniserial $R$-modules $B_1$ such that $vR$ embeds in $B_1J \oplus B_2J$, which contradicts (2.4)(i). Hence any submodule of $MJ$ is a direct sum of uniserial modules.
Now $R' = R/r.ann(J)$ embeds in a finite direct sum $K$ of copies of $J_R$. As any local submodule of $K$ is uniserial, $R'$ is right serial. As $R'$ is also left serial, is a generalized uniserial ring.

4. Some Examples

Lemma 4.1. Let $A$ be a uniserial module over a generalized uniserial ring $R$, such that no two composition factors of $A$ are isomorphic. Then the module $M = A \oplus A$ has the following properties.

(i) If $L$ is any submodule of $M$, then $L = L_1 \oplus L_2$ and $M = M_1 \oplus M_2$ for some uniserial modules $L_i$, $M_i$ such that $L_1 \subseteq M_i$.

(ii) If $K \leq L \leq M$ such that $K$ is maximal in $L$, then $L = L_1 \oplus L_2$, $K = K_1 \oplus L_2$ for some uniserial modules $L_i$, $K_1 < L_1$.

(iii) Let $L = L_1 \oplus L_2$ be a submodule of $M$ such that $L_1$ are uniserial and $d(L_1) = d(L_2)$. Then $K = L_1 \oplus L'_1$ is fully invariant in $M$.

Example A. Let $F$ be a field admitting an endomorphism $\sigma$ such that $[F : \sigma(F)] = 2$. Consider matrix units $\{e_{ij}, 1 \leq i \leq j \leq n\}$ such that for $i > 1$, $ae_{ij} = e_{ij}a$, $ae_{11} = e_{11}a$, $e_{1k}a = \sigma(a)e_{1k}$ for any $k > 1$ and any $a \in F$. Let $R$ be the set of all upper triangular matrices over $F$. We write its members as $\sum_{i \leq j} a_{ij}e_{ij}$. Two member of $R$ are added componentwise, and multiplication is defined by using the above specified laws for the matrix units. We also look at $R$ as $T_n(F)$ the ring of $n \times n$ upper triangular matrices over $F$. Using the fact that $T_n(F)$ is generalized uniserial, we get that $R$ is left serial. We see that for any $1 < k < n$, $a \in F$, $ae_{1k} = e_{11}(ae_{1k})$. Hence the right ideal $e_{11}R$ is the set of all matrices in $R$, whose last $n - 1$ rows are zero rows. Now $F = \sigma(F) + u\sigma(F)$, where $u \in F \setminus \sigma(F)$. $e_{11}J = A \oplus B$, where $A$, $B$ are right ideals such that any member of $A$ is of the form $\sum_{k>1} \sigma(a_{1k})e_{1k}$, and any member of $B$ is of the form $\sum_{j>1} a_{1j}e_{1j}$. By comparing with the right ideal $\sum_{j>1} a_{1j}F$ in $T_n(F)$, we see that $A$ and $B$ are isomorphic uniserial right ideals of $R$, such that they are quasi-injective and quasi-projective. They can be regarded as modules over $T_n(F)$. No two composition factors of $A$ are isomorphic. For some submodules $K$, $K'$ of $e_{11}J$, consider $M = e_{11}R/K$ and $N = e_{11}R/K'$. Let $L/K$, $L'/K'$ be simple submodules of $M$, $N$ respectively and $\mu : L/K \rightarrow L'/K'$ be an $R$-isomorphism. By (4.1), $L = L_1 \oplus L_2$, $K = K_1 \oplus L_2$, $L' = L'_1 \oplus L'_2$, $K' = K'_1 \oplus L'_2$ for some uniserial modules $L_i$, $L'_i$, $K_1 \leq L_1$ and $K'_1 \leq L'_1$. Let
Let $\eta : L_1/K_1 \rightarrow L'_1/K'_1$ be the $R$-isomorphism induced by $\mu$. Write $e_{11}R = M_1 \oplus M_2 = M'_1 \oplus M'_2$ where each $M_i$, $M'_i$ is uniserial, $L_i \subseteq M_i$, and $L'_i \subseteq M'_i$. Then there exists unique $R$-isomorphism $\lambda : M_1 \rightarrow M'_1$ which induces $\eta$. Now $soc(L_1) = x_1e_{11}F$, $soc(L'_1) = x_1'e_{11}F$, for some $x_1, x_1' \in F$ such that $\lambda(x_1e_{11}) = x_1'e_{11}$. Further $d(L_1) = d(L_2)$. Let $soc(L_2) = x_2e_{11}F$, $soc(L'_2) = x_2'e_{11}F$, $x_2, x_2' \in F$. We can find $w \in F$ such that $wx_2 = x_2$. Let $\lambda_w$ be the $R$-automorphism of $e_{11}R$ given by left multiplication by $w$. If $\lambda_w$ extend $\lambda$, then $\lambda_w$ lifts $\eta$. Otherwise, let $\lambda_w(x_1e_{11}) = x_1'e_{11}a + x_2'e_{11}b$ for some $a, b \in F$. If $a = 0$, then $\lambda_w(soc(e_{11}R)) = x_2'e_{11}F$ which is a contradiction. Hence $a \neq 0$. Then $\phi$ the $R$-automorphism of $e_{11}R$ given by left multiplication by $w\sigma(a)^{-1}$ is such that $\phi(x_1e_{11}) = x_1'e_{11} + x_2'e_{11}c$ for some $c \in F$. Then $\phi$ lifts $\sigma$.

We verify the condition in (2.1) to prove that $R$ satisfies (*). Let $M, N$ be any two local $R$-modules, and $S$ be a simple submodule of $M$. Let $\phi : S \rightarrow N$ be an $R$-monomorphism. We can take $M = e_{rr}R/K$, and $N = e_{ss}R/L$ for some $1 \leq r, s \leq n$, $K < e_{rr}R$, and $L < e_{ss}R$. Now the case for $r = s = 1$, has been discussed above. Notice that the last $n - 1$ rows of $R$ constitute the ring $R'$ of $n - 1 \times n - 1$ upper triangular matrices over $F$, $e_{11}J$ being a direct sum of two copies of the first row of $R'$, is injective as a right $R'$-module. Using this it can be verified that $R$ satisfies the condition given in (2.1). Hence $R$ satisfies (*) on the right.

Example B. Let $F$ be a field, $R = \begin{bmatrix} F & F + Fx \\ 0 & F + Fx \end{bmatrix}$, where $x^2 = 0$. As a left ideal, $J e_{22} = Fxe_{22} + Fe_{12} + Fxe_{12} = C_1 \oplus C_2$, where $C_1 = Fe_{12}$, $C_2 = Fxe_{22} + Fxe_{12} = Rxe_{22}$, $J^2 e_{22} = \begin{bmatrix} 0 & F + Fx \\ 0 & Fx \end{bmatrix}$, $J^3 e_{22} = \begin{bmatrix} 0 & Fx \\ 0 & Fx \end{bmatrix}$, $J^4 e_{22} = \begin{bmatrix} 0 & Fx \\ 0 & 0 \end{bmatrix}$, $\cong Re_{11} \cong C_1$. Observe that $soc(e(Re_{22})) = Fe_{12} \oplus Fxe_{12}$. As $C_2$ is invariant under all endomorphisms of $Re_{22}$, $Re_{22}/C_2$ is quasi-projective. Also $Re_{22}/Fxe_{22}$ is quasi-projective. Let $M = Re_{22}/C_1 = Fxe_{12} + Fe_{22} + Fxe_{22}$. It is uniserial and its proper submodules are $C_2 > B = Fxe_{12}$. Let $\sigma$ be an endomorphism of $B$. Suppose $\sigma(Re_{22}) = x e_{22} z$, $z \in F$. Then the $R$-endomorphism of $M$ given by multiplication by $z$ extends $\sigma$. Similarly for $C_2$, as any endomorphism of $C_2$ is given by multiplication by an element of $F$. This gives $M$ is quasi-injective. As $M$ contains a copy of $Re_{11}$, $M$ is $Re_{11}$-injective. Let $L$ be a left ideal properly contained in $Re_{22}$. If $L = Fxe_{22} + Fxe_{12}$, then $\sigma(xe_{22}) = \alpha e_{22}$ for some $\alpha \in F$ and $\sigma$ is given by right multiplication by $\alpha^{xe_{22}}$ in $M$. If $L = C_1 \oplus C_2$, then $\sigma(xe_{22}) = \alpha x e_{22}$, $\sigma(e_{12}) = \beta e_{12}$ for some $\alpha, \beta \in F$, and $\sigma$ is given by right multiplication by $(\alpha + \beta x)e_{22}$. If $L$ is any of $Fxe_{12}, Fe_{12}$, then $L \cong Re_{11}$, as $M$ is $Re_{11}$-injective, $\sigma$ is given by right multiplication by a member of $M$. If $L = Fxe_{12} + Fe_{12}$, then $\sigma(e_{12}) = \alpha e_{12}$ for some $\alpha \in F$, and $\sigma$ is given by right multiplication by $\alpha xe_{22}$. Hence $M$ is $Re_{22}$-injective. This proves that $M$ is injective. Similarly, one
can prove that any non-simple, uniserial, homomorphic image of $Re_{22}$ is injective. After this one can easily verify that $R$ satisfies (\*) on the left. Then the ring $R'$ anti-isomorphic to $R$ satisfies (\*) on the right. Observe that in $Je_{22} = C_1 \oplus C_2$, $C_1 \cong JC_2$, but $C_1 \not\cong C_2/JC_2$.

We are yet not aware of an example of a local module over a ring $R$ satisfying (\*), for which $t \geq 3$ as in (2.6).

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