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CONVERGENCE CRITERION OF THE FAMILY OF EULER-HALLEY TYPE METHODS FOR SECTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. The family of Euler-Halley type methods is extended for sections on Riemannian manifolds. Its convergence criterion is established under the assumption that the sections' covariant derivatives satisfy a kind of Lipschitz condition. Applications to special cases such as the classical Kantorovich's type condition, the γ-condition, Smale's analysis condition are provided.

1. INTRODUCTION

Newton's method and its variations are the most efficient methods known for solving systems of nonlinear equations in Banach space setting. One of the main results on Newton's method is the well-known Kantorovich's theorem ([17, 18]), which has the advantage that Newton's sequence converges to a solution under very mild conditions. Another important result on Newton's method is the Smale's point estimate theory in [42] (see also [40]). In this theory, the notion of an approximation zero was introduced and the rule to judge an initial point to be an approximation zero was provided, depending only on the information of the nonlinear operator at the initial point. Other results on Newton's method such as the estimates of the radii of convergence balls were given by Traub and Wozniakowski [27] and Wang [32] independently. A big step in this direction was made by Wang in [34, 45], where Kantorovich's theorem and Smale's theory were unified and extended. To extend

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and improve the Smale's γ-theory and α-theory of Newton's method for operators in Banach spaces, Wang introduced in [33, 36] the notion of the γ-condition, which is weaker than the Smale′s assumption in [42] (see also [40]) for analytic operators.

Recall that there are several kinds of cubic generalizations for Newton's method. The most important two are the Euler method and the Halley method; see for example, [2, 5, 6, 15, 16]. Another more general family of the cubic extensions is the family of Euler-Halley type methods in Banach spaces, which includes the Euler method and the Halley method as its special cases and has been studied extensively in [13, 14, 38]. In particular, Han established in [14] the cubic convergence of this family for operators satisfying the γ-condition. Furthermore, in [38], a unified convergence criterion is presented under a kind of Lipschitz condition, which includes the γ-condition as a special case and extends the corresponding results in [14].

Recently, the main interests are focused on numerical problems posed on manifolds arising in many natural contexts such as eigenvalue problems, symmetric eigenvalue problems, invariant subspace computations, optimization problems with equality constraints and so on; see for example [10, 12, 23, 25, 26, 28, 31]. For such problems, one often has to compute solutions of a system of equations or to find zeros of a vector field on a Riemannian manifold. Newton's method is one of the most famous methods to approximately solve these problems. The Kantorovich's theorem [17, 18] has been extended for Newton's method on Riemannian manifolds in [11]. The extensions of the famous Smale's α-theory and γ-theory in [42] to analytic vector fields on Riemannian manifolds were done in [8]. In the recent paper [21], we extended the notion of the γ-condition to vector fields on Riemannian manifolds and then established the γ-theory and α-theory of Newton's method for the vector fields on Riemannian manifolds satisfying the γ-condition, which consequently extends the results in [8]. Other extensions about Newton's method on Riemannian manifolds have been done in [20, 22, 29].

In particular, in [22], one kind of the L-average Lipschitz condition is introduced for covariant derivatives of sections on Riemannian manifolds which includes mappings and vector fields as special cases. Then, a convergence criterion of Newton's method and the radii of the uniqueness balls of the singular points for sections on Riemannian manifolds which is independent of the curvatures, are established under the assumption that the covariant derivatives of the sections satisfy this kind of the L-average Lipschitz condition. Some applications to special cases including the Kantorovich's condition and the γ-condition as well as the Smale's α-theory are provided. In particular, the result due to Ferreira and Svaiter in [11] is extended while the results due to Dedieu, Priouret and Malajovich in [8] are improved significantly. Moreover, Alvarez, Bolte and Munier introduced in [1] a Lipschitz-type radial function for the covariant derivative of vector fields and mappings on Riemannian manifolds, and established a unified convergence criterion of Newton's method on Riemannian manifolds, applications of which to analytic vector fields and mappings give first a curvature-free generalization of Smale's α-theory in Euclidean space setting, which improves significantly the corresponding results in [8] and [21]. On the other hand, the Kantorovich's theorem and Smale's α-theory and γ-theory for Newton's method on Lie groups have also been given in [31] and [23], respectively.

Very recently, in [30] the family of Euler-Halley type methods are extended to vector fields on Riemannian manifolds, and the cubic convergence criterion of this family for vector fields is established under the assumption that its covariant derivatives satisfy the γ-condition.

Motivated by the works of [22] and [30], we study the family of Euler-Halley type methods for sections on Riemannian manifolds. A convergence criterion of the family of Euler-Halley type methods is established under the assumption that the sections' covariant derivatives satisfy a kind of Lipschitz condition. Applications to special cases such as the classical Kantorovich's type condition, the γ-condition, Smale's analysis condition are provided. In particular, in the case when the sections are vector fields, the corresponding results due to [30] are extended.

The rest of the paper is organized as follows. In section 2 we present some basic notions and preliminaries on Riemannian manifolds which will be used in the sequel. The convergence criterion of the family of Euler-Halley type methods for the sections on Riemannian manifolds whose covariant derivatives satisfy a kind of Lipschitz condition is established in Section 3, and applications to special cases are given in Section 4.

2. Notions and Preliminaries

Throughout this paper, M denotes a real complete connected m-dimensional Riemannian manifold. Let $p \in M$ and let T_pM denote the tangent space at p to M. Let $\langle \cdot, \cdot \rangle_p$ be the scalar product on T_pM with the associated norm $\| \cdot \|_p$. The subscript p is usually deleted whenever there is no possibility of confusion. For any two distinct elements $q, p \in M$, let $c : [0, 1] \to M$ be a piecewise smooth curve connecting q and p. Then the arc length of c is defined by $l(c) := \displaystyle\int_0^1 \| c'(t) \| \, \mathrm{d}t$, and the Riemannian distance from q to p by $d(q, p) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \to M$ connecting q and p. Thus (M, d) is a complete metric space by the Hopf-Rinow Theorem (*cf.* [4, 9, 19]).

Noting that M is complete, the exponential map at p, $\exp_p : T_pM \to M$ is well-defined on T_pM. Recall that a geodesic in M connecting q and p is called a minimizing geodesic if its arc length equals its Riemannian distance between q and p. Note that there is at least one minimizing geodesic connecting q and p. In particular, the curve $c : [0, 1] \to M$ connecting q and p is a minimizing geodesic if and only if there exists a vector $v \in T_qM$ such that $\|v\| = d(q, p)$ and $c(t) = \exp_q(tv)$ for each $t \in [0, 1]$.

Let $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$. Let ∇ denote the Levi-Civita connection on M and let $c : \mathbb{R} \to M$ be a C^κ-curve, where C^κ means smooth or analytic in the case when $\kappa = \infty$ or ω, respectively. Then we use $P_{c,\cdot,\cdot}$ to denote the parallel transport on tangent bundle TM along c with respect to ∇.

In the remainder of this section, we shall describe simply the notions of sections, connections and parallel transports as well as some relative facts. For the details, the readers are refereed to [22] and some text books, for example, [7, 39]. Throughout the whole paper, we shall always assume that E and M are C^κ-manifolds. Let $\pi : E \to M$ be a C^κ-vector bundle. The set of all C^κ-sections of the C^κ-vector bundle π is denoted by $C^\kappa(M, E)$. In the particular cases when $\kappa = \infty$, or ω, a C^κ-section ξ is a smooth section or an analytic section, respectively. Let $C^\kappa(TM)$ denote the set of all the C^κ-vector fields on M and $C^\kappa(M)$ the set of all C^κ-mappings from M to $\mathbb{R}$, respectively. Let $\mathrm{D} : C^\kappa(M, E) \times C^\kappa(TM) \to C^{\kappa-1}(M, E)$ be a connection on the vector bundle π, i.e., for every $X, Y \in C^\kappa(TM)$, $\xi, \eta \in C^\kappa(M, E)$, $f \in C^\kappa(M)$ and $\lambda \in \mathbb{R}$, the following conditions are satisfied:

$$\begin{aligned} &\mathrm{D}_{X+fY}\xi = \mathrm{D}_X\xi + f\mathrm{D}_Y\xi, \quad \mathrm{D}_X(\xi + \lambda\eta) = \mathrm{D}_X\xi + \lambda\mathrm{D}_X\eta \\ &\quad \text{and} \quad \mathrm{D}_X(f\xi) = X(f)\xi + f\mathrm{D}_X\xi. \end{aligned} \tag{2.1}$$

Note that connections on the vector bundle π exist because M is a C^κ-Riemannian manifold with countable bases (cf. [39] for the case when $\kappa = \infty$ and its proof for the general case is similar). For any $(\xi, X) \in C^\kappa(M, E) \times C^\kappa(TM)$, $\mathrm{D}_X\xi$ is called the covariant derivative of ξ with respect to X. Since D is tensorial in X, the value of $\mathrm{D}_X\xi$ at $p \in M$ only depends on the tangent vector $v = X(p) \in T_pM$. Hence, the mapping $\mathrm{D}\xi(p) : T_pM \to \pi^{-1}(p)$ given by

$$\mathrm{D}\xi(p)v := \mathrm{D}_X\xi(p) \quad \text{for each } v \in T_pM \tag{2.2}$$

is well-defined and is a linear map from T_pM to $\pi^{-1}(p)$.

Let $c:\mathbb{R}\to M$ be a C^κ-curve. For any $a,b\in\mathbb{R}$, define the mapping $\mathcal{P}_{c,c(b),c(a)}:\pi^{-1}(c(a))\to\pi^{-1}(c(b))$ by

$$\mathcal{P}_{c,c(b),c(a)}(v)=\eta_v(c(b)) \text{ for each } v\in\pi^{-1}(c(a)),$$

where η_v is the unique C^κ-section such that $\mathrm{D}_{c'(t)}\eta_v=0$ and $\eta_v(c(a))=v$. Then $\mathcal{P}_{c,\cdot,\cdot}$ is called the parallel transport on vector bundle E along c. In particular, we write $\mathcal{P}_{q,p}$ for $\mathcal{P}_{c,q,p}$ in the case when c is a minimizing geodesic connecting p and q. Moreover, for a positive integer i, $\mathcal{P}^i_{p,q}$ stands for the map from $(\pi^{-1}(q))^i$ to $(\pi^{-1}(p))^i$ defined by

$$\mathcal{P}^i_{p,q}(v_1\cdots v_i)=\mathcal{P}_{p,q}v_1\cdots\mathcal{P}_{p,q}v_i,\quad \forall(v_1,\cdots,v_i)\in(T_qM)^i.$$

The next definition of higher order covariant derivative for sections follows from [22]. Let $k\le\kappa$ be a positive integer and let ξ be a C^κ-section. Recall that D is a connection on the vector bundle $\pi:E\to M$ and ∇ is the Levi-Civita connection on M. Then the covariant derivative of order k can be inductively defined as follows.

Define the map $\mathcal{D}^1\xi=\mathcal{D}\xi:(C^\kappa(TM))^1\to C^{\kappa-1}(M,E)$ by

$$\mathcal{D}\xi(X)=\mathrm{D}_X\xi\quad\text{for each } X\in C^\kappa(TM), \tag{2.3}$$

and define the map $\mathcal{D}^k\xi:(C^\kappa(TM))^k\to C^{\kappa-k}(M,E)$ by

$$\begin{aligned}&\mathcal{D}^k\xi(X_1,\cdots,X_{k-1},X_k)=\mathrm{D}_{X_k}(\mathcal{D}^{k-1}\xi(X_1,\cdots,X_{k-1}))\\ &-\sum_{i=1}^{k-1}\mathcal{D}^{k-1}\xi(X_1,\cdots,\nabla_{X_k}X_i,\cdots,X_{k-1})\end{aligned} \tag{2.4}$$

for each $X_1,\cdots,X_{k-1},X_k\in C^\kappa(TM)$. Then, in view of the definition and thanks to (2.1), one can use mathematical induction to prove easily that $\mathcal{D}^k\xi(X_1,\cdots,X_k)$ is tensorial with respect to each component X_i, that is, k multi-linear map from $(C^\kappa(TM))^k$ to $C^{\kappa-k}(M,E)$, where the linearity refers to the structure of $C^k(M)$-module. This implies that the value of $\mathcal{D}^k\xi(X_1,\cdots,X_k)$ at $p\in M$ only depends on the k-tuple of tangent vectors $(v_1,\cdots,v_k)=(X_1(p),\cdots,X_k(p))\in(T_pM)^k$. Consequently, for a given $p\in M$, the map $\mathcal{D}^k\xi(p):(T_pM)^k\to E_p$, defined by

$$\mathcal{D}^k\xi(p)v_1\cdots v_k:=\mathcal{D}^k\xi(X_1,\cdots,X_k)(p)\ \text{ for any } (v_1,\cdots,v_k)\in(T_pM)^k, \tag{2.5}$$

is well-defined, where $X_i\in C^\kappa(TM)$ satisfy $X_i(p)=v_i$ for each $i=1,\cdots,k$. Let $p_0\in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Thus, for any piece-geodesic curve c

connecting p_0 and p, $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ is a k-multilinear map from $(T_pM)^k$ to $T_{p_0}M$. We define the norm of $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ by

$$\| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p) \| = \sup \| \mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)v_1v_2\cdots v_k \|_{p_0},$$

where the supremum is taken over all k-tuple of vectors $(v_1,\cdots,v_k)\in(T_pM)^k$ with each $\|v_j\|_p=1$. Furthermore, for any geodesic $c:\mathbb{R}\to M$ on M, since $\nabla_{c'(s)}c'(s)=0$, it follows from (2.4) that

$$\mathcal{D}^k\xi(c(s))(c'(s))^k = \mathrm{D}_{c'(s)}(\mathcal{D}^{k-1}\xi(c(s))(c'(s))^{k-1}) \quad \text{for each } s\in\mathbb{R}. \tag{2.6}$$

The following lemma plays a key role in this paper.

Lemma 2.1. *Let $c:\mathbb{R}\to M$ be a geodesic and Y a C^k vector field on M such that $\nabla_{c'(s)}Y(c(s))=0$. Then, for each $k=0,1,2$,*

$$\begin{aligned}\mathcal{P}_{c,c(0),c(t)}\mathcal{D}^k\xi(c(t))Y(c(t))^k &= \mathcal{D}^k\xi(c(0))Y(c(0))^k\\ &\quad+\int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathcal{D}^{k+1}\xi(c(s))(c(s))^kc'(s))\mathrm{d}s.\end{aligned} \tag{2.7}$$

In particular,

$$\begin{aligned}\mathcal{P}_{c,c(0),c(t)}\mathcal{D}^k\xi(c(t))c'(t)^k &= \mathcal{D}^k\xi(c(0))c'(0)^k\\ &\quad+\int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathcal{D}^{k+1}\xi(c(s))(c'(s))^{k+1})\mathrm{d}s.\end{aligned} \tag{2.8}$$

Proof. Equality (2.7) results from [22] for the case when $k=0$. Below, we will show that equality (2.7) is true for the case when $k=1$, that is,

$$\begin{aligned}\mathcal{P}_{c,c(0),c(t)}\mathcal{D}\xi(c(t))Y(c(t)) &= \mathcal{D}\xi(c(0))Y(c(0))\\ &\quad+\int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathcal{D}^{2}\xi(c(s))Y(c(s))c'(s))\mathrm{d}s,\end{aligned} \tag{2.9}$$

while the proof for the case when $k=2$ is similar and so is omitted here. To this end, let $\eta=\mathcal{D}\xi(Y)$. Since (2.7) is true for $k=0$, it follows that

$$\mathcal{P}_{c,c(0),c(t)}\eta(c(t)) = \eta(c(0)) + \int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathcal{D}\eta(c(s))c'(s))\mathrm{d}s. \tag{2.10}$$

By (2.4) (with $k=2$), one has

$$\begin{aligned}(\mathcal{D}^2\xi(c(s))Y(c(s))c'(s) &= \mathrm{D}_{c'(s)}((\mathcal{D}\xi(c(s))Y(c(s))) - \mathcal{D}\xi(c(s))(\nabla_{c'(s)}Y(c(s)))\\ &= \mathrm{D}_{c'(s)}((\mathcal{D}\xi(c(s))Y(c(s)))\\ &= \mathrm{D}_{c'(s)}\xi(c(s))\\ &= \mathcal{D}\xi(c(s))c'(s)\end{aligned}$$

thanks to the assumption that $\nabla_{c'(s)}Y(c(s)) = 0$. This combining with (2.10) yields (2.9). The proof of the lemma is complete. ■

We conclude this section by extending the family of Euler-Halley type methods from vector fields in [30] to sections on M. Let $\xi \in C^2(M, E)$ and $p_0 \in M$. Then the family of Euler-Halley iterations with parameter $\lambda \in [0, 2]$ and initial point p_0 for solving $\xi(p) = 0$ is defined as follows.

$$p_{n+1} = T_{\xi,\lambda}(p_n) = \exp_{p_n}(u_\xi(p_n) + v_{\xi,\lambda}(p_n)), \quad n = 0, 1, 2, \cdots, \tag{2.11}$$

where

$$\begin{aligned} u_\xi(p) &= -\mathcal{D}\xi(p)^{-1}\xi(p), \\ v_{\xi,\lambda}(p) &= -\frac{1}{2}\mathcal{D}\xi(p)^{-1}\mathcal{D}^2\xi(p)u_\xi(p)Q_{\xi,\lambda}(p)u_\xi(p), \\ Q_{\xi,\lambda}(p) &= \{\mathbf{I}_{T_pM} + \tfrac{\lambda}{2}\mathcal{D}\xi(p)^{-1}\mathcal{D}^2\xi(p)u_\xi(p)\}^{-1}, \end{aligned}$$

and $\mathbf{I}_{T_pM}$ is the identity on T_pM.

3. Convergence Criterion

For a Banach space or a Riemannian manifold Z, we use $\mathbf{B}_Z(p, r)$ and $\overline{\mathbf{B}_Z(p, r)}$ to denote respectively the open metric ball and the closed metric ball at p with radius r, that is,

$$\mathbf{B}_Z(p, r) := \{q \in Z|\ d(p, q) < r\} \quad \text{and} \quad \overline{\mathbf{B}_Z(p, r)} := \{q \in Z|\ d(p, q) \leq r\}.$$

We often omit the subscript Z if no confusion caused.

Let L be a positive nondecreasing integrable function on $[0, R]$, where R is a positive number large enough such that $\int_0^R (R - u)L(u)\mathrm{d}u \geq R$. The notion of Lipschitz condition with the L average for operators from Banach spaces to Banch spaces was introduced in [45] by Wang for the study of Smale's point estimate theory. The following definition extends this notion to sections on Riemannian manifold M, which is taken from [22]. Let $\pi : E \to M$ be a C^1-vector bundle and ξ a C^1-section of this vector bundle. Throughout the whole paper, we always assume that $p_0 \in M$ is such that $\mathcal{D}\xi(p_0)^{-1}$ exists. In the remainder, for each $p, q \in M$ we use $\Gamma_{p,q}$ to denote the set of all geodesics in M connecting p and q.

Definition 3.1. Let $R > r > 0$. Then $\mathcal{D}\xi(p_0)^{-1}\mathcal{D}\xi$ is said to satisfy the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$, if, for any two points $p, q \in \mathbf{B}(p_0, r)$, $c_1 \in \Gamma_{p_0,p}$ a minimizing geodesic and $c_2 \in \Gamma_{p,q}$, we have

$$\| \mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}\xi(q)P_{c_2,q,p} - \mathcal{D}\xi(p)) \| \le \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)\mathrm{d}u. \tag{3.1}$$

The majorizing function h defined in the following, which was first introduced and studied by Wang (cf. [45]), is a powerful tool in our study. Let $r_0 > 0$ and $b > 0$ be such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1 \quad \text{and} \quad b = \int_0^{r_0} L(u)u\mathrm{d}u. \tag{3.2}$$

For $\beta > 0$, define the majorizing function h by

$$h(t) = \beta - t + \int_0^t L(u)(t-u)\mathrm{d}u \quad \text{for each } 0 \le t \le R. \tag{3.3}$$

Some useful properties are described in the following proposition; see [45].

Proposition 3.1. *The function h is monotonic decreasing on $[0, r_0]$ and monotonic increasing on $[r_0, R]$. Moreover, if $\beta \le b$, h has a unique zero respectively in $[0, r_0]$ and $[r_0, R]$, which are denoted by r_1 and r_2.*

The following lemma is taken from [22] and about estimation of the norm of the inverse $\mathcal{D}\xi(q)^{-1}$ around the point p_0 which is useful in this paper.

Lemma 3.1. *Let $0 < r \le r_0$ and suppose that $\mathcal{D}\xi(p_0)^{-1}\mathcal{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$. Let $p, q \in \mathbf{B}(p_0, r)$, let $c_1 \in \Gamma_{p_0,p}$ be a minimizing geodesic and $c_2 \in \Gamma_{p,q}$ such that $l(c_1) + l(c_2) < r$. Then, $\mathcal{D}\xi(q)^{-1}$ exists and*

$$\begin{aligned} &\|\mathcal{D}\xi(q)^{-1}\mathcal{P}_{c_2,q,p}\mathcal{P}_{c_1,p,p_0}\mathcal{D}\xi(p_0)\| \\ &\le \frac{1}{1 - \int_0^{l(c_1)+l(c_2)} L(u)\mathrm{d}u} = \frac{-1}{h'(l(c_1)+l(c_2))}. \end{aligned} \tag{3.4}$$

We still need the following two lemmas. Below, we always assume that ξ is a C^2-section.

Lemma 3.2. *Let $p, q \in M$, $v \in T_pM$ and $c \in \Gamma_{p,q}$ be such that $c(t) = \exp_p tv$ for each $t \in [0, 1]$. Then*

$$\begin{aligned} &\mathcal{P}_{c,p,q}\mathcal{D}\xi(q)P_{c,q,p} = \mathcal{D}\xi(p) + \mathcal{D}^2\xi(p)v \\ &+ \int_0^1 (\mathcal{P}_{c,p,c(s)}\mathcal{D}^2\xi(c(s))P^2_{c,c(s),p} - \mathcal{D}^2\xi(p))v\mathrm{d}s. \end{aligned} \tag{3.5}$$

Proof. By (2.8) (with $k = 1$), we have

$$\mathcal{P}_{c,p,q}\mathcal{D}\xi(q)c'(1) = \mathcal{D}\xi(p)c'(0) + \int_0^1 \mathcal{P}_{c,p,c(s)}\mathcal{D}^2\xi(c(s))(c'(s))^2\mathrm{d}s. \tag{3.6}$$

Since $c'(1) = P_{c,q,p}v$, $c'(s) = P_{c,c(s),p}v$ and $c'(0) = v$, it follows from (3.6) that

$$\mathcal{P}_{c,p,q}\mathcal{D}\xi(q)P_{c,q,p} = \mathcal{D}\xi(p) + \int_0^1 \mathcal{P}_{c,p,c(s)}\mathcal{D}^2\xi(c(s))P^2_{c,c(s),p}v\mathrm{d}s,$$

which implies (3.5). This completes the proof of the lemma. ∎

Lemma 3.3. *Let* $0 < r \le r_0$. *Suppose that*

$$\|\mathcal{D}\xi(p_0)^{-1}\mathcal{D}^2\xi(p_0)\| = L(0) \tag{3.7}$$

and

$$\begin{aligned} &\| \mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)P^2_{c_2,q,p} - \mathcal{D}^2\xi(p)) \| \\ &\le L(l(c_1) + l(c_2)) - L(l(c_1)) \end{aligned} \tag{3.8}$$

holds for all $p, q \in \mathbf{B}(p_0, r)$, $c_1 \in \Gamma_{p_0,p}$ *a minimizing geodesic and* $c_2 \in \Gamma_{p,q}$ *such that* $l(c_1) + l(c_2) < r$. *Then the following assertions hold.*

(i)

$$\|\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \le h''(l(c_1) + l(c_2)). \tag{3.9}$$

(ii) *For each* $q \in \mathbf{B}(p_0, r)$, $\mathcal{D}\xi(q)^{-1}$ *exists and*

$$\|\mathcal{D}\xi(q)^{-1}\mathcal{P}_{c_2,q,p}\mathcal{P}_{c_1,p,p_0}\mathcal{D}\xi(p_0)\| \le \frac{-1}{h'(l(c_1) + l(c_2))}.$$

Proof. Note that

$$\begin{aligned} &\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)P^2_{c_2,q,p}P^2_{c_1,p,p_0} \\ = \; &\mathcal{D}\xi(p_0)^{-1}\mathcal{D}^2\xi(p_0) + \mathcal{D}\xi(p_0)^{-1}(\mathcal{P}_{c,p_0,p}\mathcal{D}^2\xi(p)P^2_{c,p,p_0} - \mathcal{D}^2\xi(p_0)) \\ &+\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)P^2_{c_2,q,p} - \mathcal{D}^2\xi(p))P^2_{c_1,p,p_0} \end{aligned} \tag{3.10}$$

Since $P^2_{c_2,q,p}$ and $P^2_{c_1,p,p_0}$ are isometries, by (3.7) and (3.8), we have from (3.10) that

$$\begin{aligned} &\|\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \\ \le \; &L(0) + L(l(c_1)) - L(0) + L(l(c_1) + l(c_2)) - L(l(c_1)) \\ = \; &L(l(c_1) + l(c_2)) \\ = \; &h''(l(c_1) + l(c_2)), \end{aligned}$$

which implies (i). To prove (ii), by Lemma 3.1, it is sufficient to show that $\mathcal{D}\xi(p_0)^{-1}\mathcal{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$. To do this, let $p, q \in \mathbf{B}(p_0, r)$, let $c_1 \in \Gamma_{p_0,p}$ be a minimizing geodesic and $c_2 \in \Gamma_{p,q}$ such that $l(c_1) + l(c_2) < r$. Let $v_2 \in T_pM$ be such that $c_2(t) = \exp_p tv_2$ for each $t \in [0, 1]$, $q = \exp_p v_2$, and $\|v_2\| = l(c_2)$. Hence, it follows from Lemma 3.2 that

$$\begin{aligned} &\mathcal{P}_{c_2,p,q}\mathcal{D}\xi(q)P_{c_2,q,p} = \mathcal{D}\xi(p) + \mathcal{D}^2\xi(p)v_2 \\ &+ \int_0^1 (\mathcal{P}_{c_2,p,c_2(s)}\mathcal{D}^2\xi(c_2(s))P^2_{c_2,c_2(s),p} - \mathcal{D}^2\xi(p))v_2 \mathrm{d}s. \end{aligned} \tag{3.11}$$

Noting that by (3.9) one has

$$\|\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\mathcal{D}^2\xi(p)\| \le h''(l(c_1)) = L(l(c_1)).$$

This, together with (3.8) and (3.11) implies that

$$\begin{aligned} &\|\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}\xi(q)P_{c_2,q,p} - \mathcal{D}\xi(p))\| \\ \le\ & L(l(c_1))\|v_2\| + \int_0^1 \|\mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,c_2(s)} \\ &\mathcal{D}^2\xi(c_2(s))P^2_{c_2,c_2(s),p} - \mathcal{D}^2\xi(p))\|\|v_2\|\mathrm{d}s \\ \le\ & L(l(c_1))\|v_2\| + \int_0^1 (L(l(c_1) + s\|v_2\|) - L(l(c_1)))\|v_2\|\mathrm{d}s \\ =\ & \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)\mathrm{d}u, \end{aligned} \tag{3.12}$$

which implies that $\mathcal{D}\xi(p_0)^{-1}\mathcal{D}\xi$ satisfies the 2-piece L-average Lipschitz condition in $\mathbf{B}(p_0, r)$. This completes the proof of the lemma. ∎

Let $\{t_n\}$ denote the sequence generated by the Euler-Halley iteration (with parameter $\lambda \in [0, 2]$) for $h(t)$ with initial point $t_0 = 0$, that is,

$$t_{n+1} = T_{h,\lambda}(t_n) = t_n + u_h(t_n) + v_{h,\lambda}(t_n), \quad n = 0, 1, 2, \cdots,$$

where

$$\begin{aligned} u_h(t) &= -h'(t)^{-1}h(t) \\ v_{h,\lambda}(t) &= -\frac{1}{2}h'(t)^{-1}h''(t)u_h(t)Q_{h,\lambda}(t)u_h(t) \\ Q_{h,\lambda}(t) &= (1 + \frac{\lambda}{2}h'(t)^{-1}h''(t)u_h(t))^{-1}. \end{aligned}$$

Then the following lemma holds from [38].

Lemma 3.4. *Suppose that $\beta \le b$. Then, for each $t \in [0, r_1]$,*

(*i*) $0 < H_h(t) = h'(t)^{-2}h''(t)h(t) < 1$;

(*ii*) $T_{h,\lambda}(t) \in [0, r_1]$ *and* $T_{h,\lambda}(t)$ *is monotonically increasing on* $[0, r_1]$ *for each* $\lambda \in [0, 2]$;

(*iii*) $t \le T_{h,\lambda}(t)$.

(*iv*) $\{t_n\}$ *is increasing monotonically and convergent to* r_1.

The following lemma is taken from [13]; see also [38].

Lemma 3.5. *For any* $n = 0, 1, 2, \cdots$,

$$\begin{aligned} h(t_{n+1}) = &\ \frac{1}{2}h''(t_n)\{(2-\lambda)u_h(t_n) + v_{h,\lambda}(t_n)\}v_{h,\lambda}(t_n) \\ &+ \int_0^1 \int_0^\tau \{h''(t_n + s(t_{n+1} - t_n)) - h''(t_n)\}\mathrm{d}s\mathrm{d}\tau (t_{n+1} - t_n)^2. \end{aligned}$$

The similar expression for sections is described in the following lemma, whose proof is similar to that of [30, Lemma 3.2] where the expression for vector fields is presented, and so is omitted hear. Recall that

$$p_{n+1} = \exp_{p_n}(u_\xi(p_n) + v_{\xi,\lambda}(p_n)), \quad n = 0, 1, 2, \cdots.$$

Lemma 3.6. *Let* n *be a nonnegative integer and write*

$$w_n = u_\xi(p_n) + v_{\xi,\lambda}(p_n). \tag{3.13}$$

Let c_n *be the curve defined by* $c_n(t) := \exp_{p_n}(tw_n)$ *for each* $t \in [0, 1]$. *Then*

$$\begin{aligned} &\mathcal{P}_{c_n,p_n,p_{n+1}}\xi(p_{n+1}) \\ =&\ \frac{1}{2}\mathcal{D}^2\xi(p_n)\{(2-\lambda)u_\xi(p_n) + v_{\xi,\lambda}(p_n)\}v_{\xi,\lambda}(p_n) \\ &+ \int_0^1 \int_0^\tau (\mathcal{P}_{c_n,p_n,c_n(s)}\mathcal{D}^2\xi(c_n(s))P^2_{c_n,c_n(\tau),p_n} - \mathcal{D}^2\xi(p_n))w_n^2\mathrm{d}s\mathrm{d}\tau. \end{aligned}$$

In the remainder of this paper, we always assume that ξ is a C^2 section and that $p_0 \in M$ such that $\mathcal{D}\xi(p_0)^{-1}$ exists. Recall that $\beta = \| \mathcal{D}\xi(p_0)^{-1}\xi(p_0) \|$, b and r_1 are given by (3.2) and Proposition 3.1, respectively. Then the main theorem of this paper is stated as follows.

Theorem 3.1. *Let* $\beta \leq b$*. Suppose that conditions* (3.7) *and* (3.8) *hold in* $\mathbf{B}(p_0, r_1)$*, i.e.,*

$$\|\mathcal{D}\xi(p_0)^{-1}\mathcal{D}^2\xi(p_0)\| = L(0)$$

and

$$\| \mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)P^2_{c_2,q,p} - \mathcal{D}^2\xi(p)) \| \leq L(l(c_1)+l(c_2)) - L(l(c_1))$$

holds for all $p, q \in \mathbf{B}(p_0, r_1)$*,* $c_1 \in \Gamma_{p_0,p}$ *a minimizing geodesic and* $c_2 \in \Gamma_{p,q}$ *such that* $l(c_1) + l(c_2) < r_1$*. Then the sequence* $\{p_n\}$ *generated by* (2.11) *with initial point* p_0 *is well-defined for all* $\lambda \in [0, 2]$ *and converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$*. Moreover,*

$$d(p^*, p_n) \leq r_1 - t_n.$$

Proof. It is sufficient to show that the sequence $\{p_n\}$ generated by (2.11) with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and satisfies

$$d(p_n, p_{n+1}) \leq \|u_\xi(p_n)\| + \|v_{\xi,\lambda}(p_n)\| \leq t_{n+1} - t_n$$

for each $n = 0, 1, \cdots$. To do this, we will use mathematical induction to prove that the generated sequence $\{p_n\}$ is well-defined and the following statements hold for each $n = 0, 1, \cdots$:

- (a) $\|u_\xi(p_n)\| \leq u_h(t_n)$;
- (b) $Q_{\xi,\lambda}(p_n)$ exits and $\|Q_{\xi,\lambda}(p_n)\| \leq Q_{h,\lambda}(t_n)$;
- (c) $\|v_{\xi,\lambda}(p_n)\| \leq v_{h,\lambda}(t_n)$;
- (d) $d(p_n, p_{n+1}) \leq \|w_n\| \leq \|u_\xi(p_n)\| + \|v_{\xi,\lambda}(p_n)\| \leq t_{n+1} - t_n$, where w_n is defined by (3.13).

Indeed, in the case when $n = 0$, (a) results from

$$\|u_\xi(p_0)\| = \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| = \beta = u_h(t_0). \tag{3.14}$$

Since $t_0 = 0$, one has $h'(t_0) = -1$ and $h''(t_0) = L(0)$. This, together with (3.7), (3.14) and Lemma 3.4(i) implies that

$$\| -\frac{\lambda}{2}\mathrm{D}\xi(p_0)^{-1}\mathrm{D}^2\xi(p_0)u_\xi(p_0)\| \leq -\frac{\lambda}{2}h'(t_0)^{-1}h''(t_0)u_h(t_0) < 1.$$

Then, using the Banach Lemma, $Q_{\xi,\lambda}(p_0)$ exits and

$$\|Q_{\xi,\lambda}(p_0)\| \leq \frac{1}{1 + \frac{\lambda}{2}h'(t_0)^{-1}h''(t_0)u_h(t_0)} = Q_{h,\lambda}(t_0).$$

Thus, (b) and (c) follow. As $\|w_0\| \leq \|u_\xi(p_0)\| + \|v_{\xi,\lambda}(p_0)\| \leq t_1 - t_0 \leq \beta$ due to the fact that (a) and (c) hold for $n = 0$, and $p_1 = \exp_{p_0}(w_0)$, we have

$$d(p_0, p_1) \leq \|w_0\| \leq \|u_\xi(p_0)\| + \|v_{\xi,\lambda}(p_0)\| \leq t_1 - t_0.$$

Therefore, (d) holds for $n = 0$. Now assume that $p_1, \cdots, p_{k+1}$ are well-defined and that (a)-(d) are true for $n = 0, 1, \cdots, k$. Then,

$$d(p_k, p_{k+1}) \leq \|w_k\| \leq \|u_\xi(p_k)\| + \|v_{\xi,\lambda}(p_k)\| \leq t_{k+1} - t_k \tag{3.15}$$

and

$$d(p_0, p_{k+1}) \leq t_{k+1} < r_1. \tag{3.16}$$

Below, we will show that (a)-(d) are true for $n = k + 1$. Let $c \in \Gamma_{p_0,p_k}$ be a minimizing geodesic. Define the curve c_k by $c_k(t) := \exp_{p_k}(tw_k), t \in [0, 1]$. By (3.16) and Lemma 3.3(ii), $\mathrm{D}\xi(p_{k+1})^{-1}$ exists and

$$\|\mathrm{D}\xi(p_{k+1})^{-1} P_{c_k,p_{k+1},p_k} \circ P_{c,p_k,p_0} \mathrm{D}\xi(p_0)\| \leq -h'(t_{k+1})^{-1}. \tag{3.17}$$

By Lemma 3.6, one has

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1} P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}} \xi(p_{k+1})\| \\
\leq\ & \frac{1}{2}\|\mathrm{D}\xi(p_0)^{-1} P_{c,p_0,p_k} \mathrm{D}^2\xi(p_k)\|\{(2-\lambda)\|u_\xi(p_k)\| \\
& +\|v_{\xi,\lambda}(p_k)\|\}\|v_{\xi,\lambda}(p_k)\| \\
& + \int_0^1 \int_0^\tau \mathrm{D}\xi(p_0)^{-1} P_{c,p_0,p_k} \\
& \quad (P_{c_k,p_k,c_k(s)} \mathrm{D}^2\xi(c_k(s)) P^2_{c_k,c_k(\tau),p_k} - \mathrm{D}^2\xi(p_k)) w_k^2 \mathrm{d}s\mathrm{d}\tau.
\end{aligned} \tag{3.18}$$

Noting that by Lemma 3.3(i), we have

$$\|D\xi(p_0)^{-1} P_{c,p_0,p_k} D^2\xi(p_k)\| \leq h''(t_c) \leq h''(t_k) \tag{3.19}$$

By induction assumptions, one has

$$\|u_\xi(p_k)\| \leq u_h(t_k), \quad \|v_{\xi,\lambda}(p_k)\| \leq v_{h,\lambda}(t_k) \quad \text{and} \quad \|w_k\| \leq t_{k+1} - t_k.$$

Combining this with (3.19), (3.18) and (3.8) gives that

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1} P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}} \xi(p_{k+1})\| \\
\leq\ & \frac{1}{2} h''(t_n)\{(2-\lambda)u_h(t_n) + v_{h,\lambda}(t_n)\} v_{h,\lambda}(t_n) \\
& + \int_0^1 \int_0^\tau \{h''(t_n + s(t_{n+1} - t_n)) - h''(t_n)\} \mathrm{d}s\mathrm{d}\tau (t_{n+1} - t_n)^2,
\end{aligned}$$

which implies

$$\|\mathrm{D}\xi(p_0)^{-1}P_{c,p_0,p_k}\circ P_{c_k,p_k,p_{k+1}}\xi(p_{k+1})\|\le h(t_{k+1}), \tag{3.20}$$

thanks to Lemma 3.5. Since

$$\begin{aligned}\|u_\xi(p_{k+1})\| &= \|-\mathrm{D}\xi(p_{k+1})^{-1}\xi(p_{k+1})\|\\ &\le \|\mathrm{D}\xi(p_{k+1})^{-1}P_{c_k,p_{k+1},p_k}\circ P_{c,p_k,p_0}\mathrm{D}\xi(p_0)\|\\ &\quad\cdot\|\mathrm{D}\xi(p_0)^{-1}P_{c,p_0,p_k}\circ P_{c_k,p_k,p_{k+1}}\xi(p_{k+1})\|\\ &\le u_h(t_{k+1})\end{aligned}$$

due to (3.17) and (3.20), (a) is true for $k+1$, that is,

$$\|u_\xi(p_{k+1})\|\le u_h(t_{k+1}). \tag{3.21}$$

Note that

$$\begin{aligned}&\|-\frac{\lambda}{2}\mathrm{D}\xi(p_{k+1})^{-1}\mathrm{D}^2\xi(p_{k+1})u_\xi(p_{k+1})\|\\ \le\ &\frac{\lambda}{2}\|\mathrm{D}\xi(p_{k+1})^{-1}P_{c_k,p_{k+1},p_k}\circ P_{c,p_k,p_0}\mathrm{D}\xi(p_0)\|\\ &\cdot\|\mathrm{D}\xi(p_0)^{-1}P_{c,p_0,p_k}\circ P_{c_k,p_k,p_{k+1}}\mathrm{D}^2\xi(p_{k+1})\|\cdot\|u_\xi(p_{k+1})\|\end{aligned}$$

and

$$\|\mathrm{D}\xi(p_0)^{-1}P_{c,p_0,p_k}\circ P_{c_k,p_k,p_{k+1}}\mathrm{D}^2\xi(p_{k+1})\|\le h''(t_{k+1})$$

due to Lemma 3.3(i). Thus, it follows from (3.17) and (3.21) that

$$\begin{aligned}&\|-\frac{\lambda}{2}\mathrm{D}\xi(p_{k+1})^{-1}\mathrm{D}^2\xi(p_{k+1})u_\xi(p_{k+1})\|\\ \le\ &-\frac{\lambda}{2}h'(t_{k+1})^{-1}h''(t_{k+1})u_h(t_{k+1})<1,\end{aligned} \tag{3.22}$$

where the last inequality is because of Lemma 3.4(i). Thus, by the Banach lemma, (3.22) implies that $Q_{\xi,\lambda}(p_{k+1})$ exists and

$$\begin{aligned}Q_{\xi,\lambda}(p_{k+1}) &= \|(\mathbf{I}_{T_{p_k}M}+\frac{\lambda}{2}\mathrm{D}\xi(p_{k+1})^{-1}\mathrm{D}^2\xi(p_{k+1})u_\xi(p_{k+1}))^{-1}\|\\ &\le \frac{1}{1+\frac{\lambda}{2}h'(t_{k+1})^{-1}h''(t_{k+1})u_h(t_{k+1})}\\ &= Q_{h,\lambda}(t_{k+1}).\end{aligned} \tag{3.23}$$

Hence, p_{k+2} is well-defined and (b) is true for $n = k+1$. Since

$$\begin{aligned}\|v_{\xi,\lambda}(p_{k+1})\| &= \| -\frac{1}{2}\mathrm{D}\xi(p_{k+1})^{-1}\mathrm{D}^2\xi(p_{k+1})u_\xi(p_{k+1})Q_{\xi,\lambda}(p_{k+1})u_\xi(p_{k+1})\| \\ &\le \frac{1}{2}\|\mathrm{D}\xi(p_{k+1})^{-1}\mathrm{D}^2\xi(p_{k+1})u_\xi(p_{k+1})\|\|Q_{\xi,\lambda}(p_{k+1})\|\|u_\xi(p_{k+1})\|,\end{aligned}$$

it follows from (3.22), (3.23) and (3.21) that

$$\|v_{\xi,\lambda}(p_{k+1})\| \le -\frac{1}{2}h'(t_{k+1})^{-1}h''(t_{k+1})u_h(t_{k+1})Q_{h,\lambda}(t_{k+1})u_h(t_{k+1}) = v_{h,\lambda}(t_{k+1}),$$

which implies that (c) holds for $k+1$. Consequently,

$$\|w_{k+1}\| \le \|u_\xi(p_{k+1})\| + \|v_{\xi,\lambda}(p_{k+1})\| \le \|u_h(t_{k+1})\| + \|v_{h,\lambda}(t_{k+1})\| = t_{k+2} - t_{k+1}.$$

Thus (d) is true for $n = k+1$. This completes the proof of the theorem. ■

4. Applications

This section is devoted to applications of the result obtained in the previous section. More precisely, applications to special cases such as the classical Kantorovich's type condition, the γ-condition, Smale's analysis condition are provided. In particular, in the case when the sections are vector fields, the corresponding results due to [30] are extended.

4.1. Theorem under Kantorovich's condition

Let C and K be positive constants. Take

$$L(u) = C + Ku \quad \text{for each } u \in [0, R].$$

Then, r_0 is the solution of the equation

$$\int_0^{r_0} L(u)\mathrm{d}u = \int_0^{r_0} (C + Ku)\mathrm{d}u = Cr_0 + \frac{1}{2}Kr_0^2 = 1,$$

i.e.,

$$r_0 = \frac{2}{C + \sqrt{C^2 + 2K}}.$$

Thus,

$$b = \int_0^{r_0} uL(u)\mathrm{d}u = \int_0^{r_0} u(C + Ku)\mathrm{d}u = \frac{2(C + 2\sqrt{C^2 + 2K})}{3(C + \sqrt{C^2 + 2K})^2}.$$

In this case the majorizing function is

$$h(t) = \beta - t + \frac{1}{2}Ct^2 + \frac{1}{6}Kt^3,$$

and $r_1 \leq r_2$ are its two positive solutions when $\beta \leq b$. Hence, the following Kantorovich's type theorem is obtained directly from Theorem 3.1.

Theorem 4.1. *Suppose that*

$$\|\mathcal{D}\xi(p_0)^{-1}\mathcal{D}^2\xi(p_0)\| = C$$

and

$$\| \mathcal{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)P^2_{c_2,q,p} - \mathcal{D}^2\xi(p)) \| \leq Kl(c_2)$$

holds for all $p, q \in \mathbf{B}(p_0, r)$*,* $c_1 \in \Gamma_{p_0,p}$ *a minimizing geodesic and* $c_2 \in \Gamma_{p,q}$ *such that* $l(c_1) + l(c_2) < r_1$*. If*

$$\beta = \|\mathcal{D}\xi(p_0)^{-1}\xi(p_0)\| \leq \frac{2(C + 2\sqrt{C^2 + 2K})}{3(C + \sqrt{C^2 + 2K})^2},$$

then the sequence $\{p_n\}$ *generated by* (2.11) *with initial point* p_0 *is well-defined for all* $\lambda \in [0, 2]$ *and converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, r_1)}$*.*

4.2. Theorem under the γ-condition

The γ-condition for operators in Banach spaces was first introduced by Wang [36] for the study of Smale's point estimate theory and extended to vector fields on Riemannian manifolds in [21, 30]. In the remainder of this section, we shall always assume that ξ is a C^3-section. Let $r > 0$ and $\gamma > 0$ be such that $r\gamma < 1$. Definition 4.1 below extends this notion to sections on Riemannian manifolds with a similar version in [30]. Recall that the norm of a k multi-linear operator T on a Banach space E is defined by

$$\|T\| = \sup\{\|T\ v_1 v_2 \cdots v_k\| :\ v_i \in E \text{ and } \|v_i\| \leq 1 \text{ for each } i = 1, 2, \cdots, k\}.$$

Definition 4.1. Let $p_0 \in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Then ξ is said to satisfy the 2-piece γ-condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$, if for any two points $p, q \in \mathbf{B}(p_0, r)$, any $c_1 \in \Gamma_{p_0,p}$ a minimizing geodesic and $c_2 \in \Gamma_{p,q}$ satisfying $l(c_1) + l(c_2) < r$, we have

$$\|\mathrm{D}\xi(p_0)^{-1}\mathrm{D}^2\xi(p_0)\| \leq 2\gamma.$$

and

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p} \circ \mathcal{P}_{c_2,p,q}\mathrm{D}^3\xi(q)\| \leq \frac{6\gamma^2}{(1 - \gamma(l(c_1) + l(c_2)))^4}.$$

Let L be the function defined by

$$L(u) = \frac{2\gamma}{(1-\gamma u)^3} \quad \text{for each } 0 < u < \frac{1}{\gamma}. \tag{4.1}$$

The following lemma shows that the 2-piece γ-condition of order 2 at p_0 implies that conditions (3.7) and (3.8) hold for L given by (4.1). In the case when ξ is a vector field, Lemma 4.1 below has been proved in [30]. The proof for the sections is similar and so is omitted here.

Lemma 4.1. *Let $\gamma > 0$ and $0 < r \le \frac{2-\sqrt{2}}{2\gamma}$. Let $p_0 \in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Suppose that ξ satisfies the 2-piece γ-condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$. Then, for any two points $p, q \in \mathbf{B}(p_0, r)$, $c_1 \in \Gamma_{p_0,p}$ a minimizing geodesic and $c_2 \in \Gamma_{p,q}$ with $l(c_1) + l(c_2) < r$, we have*

$$\begin{aligned}&\|\mathrm{D}\xi(p_0)^{-1}P_{c_1,p_0,p}(P_{c_2,p,q}\mathrm{D}^2\xi(q)P^2_{c_2,q,p} - \mathrm{D}^2\xi(p))\| \\ \le\ & \frac{2\gamma}{(1-\gamma(l(c_1)+l(c_2)))^3} - \frac{2\gamma}{(1-\gamma l(c_1))^3}.\end{aligned}$$

Corresponding to the function L defined by (4.1), r_0 and b in (3.2) are

$$r_0 = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{1}{\gamma} \quad \text{and} \quad b = (3-2\sqrt{2})\frac{1}{\gamma},$$

and the majorizing function given in (3.3) reduces to

$$h(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t} \quad \text{for each } 0 \le t \le R.$$

Hence the condition $\beta \le b$ is equivalent that $\alpha = \gamma\beta \le 3 - 2\sqrt{2}$. Then the following proposition was proved in [37]; see also [45].

Proposition 4.1. *Assume that $\alpha = \gamma\beta \le 3 - 2\sqrt{2}$. Then the zeros of h are*

$$r_1 = \frac{1+\alpha-\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}, \quad r_2 = \frac{1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma} \tag{4.2}$$

and

$$\beta \le r_1 \le \left(1+\frac{1}{\sqrt{2}}\right)\beta \le \left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \le r_2 \le \frac{1}{2\gamma}.$$

Recall that $\beta = \|\mathcal{D}\xi(p_0)^{-1}\xi(p_0)\|$. Thus the following Theorem follows directly from Theorem 3.1 and Lemma 4.1.

Theorem 4.2. *Suppose that*

$$\alpha = \beta\gamma \le 3 - 2\sqrt{2}$$

and X satisfies the 2-piece γ-condition of order 2 at p_0 in $\mathbf{B}(p_0, r_1)$. Then the sequence $\{p_n\}$ generated by (2.11) *with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and converges to a singular point p^* of ξ in $\overline{\mathbf{B}(p_0, r_1)}$.*

4.3. Application to analytic sections

Throughout this subsection, we always assume that M is an analytic complete m-dimensional Riemannian Manifold and ξ is analytic on M. Let $p_0 \in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Following [8], we define

$$\gamma(\xi, p_0) = \sup_{k\ge 2} \left\| \mathrm{D}\xi(p_0)^{-1} \frac{\mathrm{D}^k\xi(p_0)}{k!} \right\|_{p_0}^{\frac{1}{k-1}}. \tag{4.1}$$

Also we adopt the convention that $\gamma(\xi, p_0) = \infty$ if $\mathrm{D}\xi(p_0)$ is not invertible. Note that this definition is justified and in the case when $\mathrm{D}\xi(p_0)$ is invertible, by analyticity, $\gamma(\xi, p_0)$ is finite.

The following lemma shows that an analytic section satisfies the 2-piece γ-condition of order 2 at p_0 in $\mathbf{B}\left(p_0, \frac{2-\sqrt{2}}{2\gamma(\xi,p_0)}\right)$, whose proof is similar to that of [30] and so is omitted here.

Lemma 4.2. *Let $\gamma = \gamma(\xi, p_0)$ and $0 < r \le \frac{2-\sqrt{2}}{2\gamma}$. Then ξ satisfies the 2-piece γ-condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$.*

Recall that $\beta = \|\mathcal{D}\xi(p_0)^{-1}\xi(p_0)\|$. Thus the following theorem follows directly from Theorem 4.2 and Lemma 4.2.

Theorem 4.3. *Let $\gamma = \gamma(\xi, p_0)$. Suppose that*

$$\alpha = \beta\gamma \le 3 - 2\sqrt{2}.$$

Then the sequence $\{p_n\}$ generated by (2.11) *with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and converges to a singular point p^* of ξ in $\overline{\mathbf{B}(p_0, r_1)}$, where r_1 is given by* (4.2).

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