ON LOCAL INTEGRATED C-COSINE FUNCTION
AND WEAK SOLUTION OF SECOND ORDER
ABSTRACT CAUCHY PROBLEM

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Abstract. Let $\alpha$ be a nonnegative number, $C : X \rightarrow X$ a bounded linear injection on a Banach space $X$ and $A : D(A) \subset X \rightarrow X$ a closed linear operator in $X$ which satisfies $C^{-1}AC = A$ and may not be densely defined. We prove some equivalence relations between the generation of a local $\alpha$-times integrated C-cosine function on $X$ with generator $A$ and the uniqueness existence of weak solutions of the abstract Cauchy problem:

$$\begin{align*}
ACP_2(A, f, x, y) & \quad \begin{cases}
    u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\
    u(0) = x, u'(0) = y,
\end{cases}
\end{align*}$$

where $x, y \in X$ are given and $f$ is an $X$-valued function defined on a subset of $\mathbb{R}$.

1. INTRODUCTION

Let $X$ be a Banach space over $\mathbb{F}$ with norm $\| \cdot \|$ and dual space $X^*$, and let $B(X)$ denote the set of all bounded linear operators from $X$ into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\begin{align*}
ACP_2(A, f, x, y) & \quad \begin{cases}
    u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\
    u(0) = x, u'(0) = y,
\end{cases}
\end{align*}$$

where $x, y \in X$ are given, $A : D(A) \subset X \rightarrow X$ is a closed linear operator and $f$ is an $X$-valued function defined on a subset of $\mathbb{R}$ containing $(0, T_0)$. A function $u$ is called a strong solution of $ACP_2(A, f, x, y)$, if $u \in C^2((0, T_0), X) \cap C^1([0, T_0), X) \cap C((0, T_0), [D(A)])$ and satisfies $ACP_2(A, f, x, y)$. Here $[D(A)]$ denotes the Banach
space \( D(A) \) equipped with the graph norm \( |x|_A = \|x\| + \|Ax\| \) for \( x \in D(A) \). For each \( \alpha > 0 \) and \( C \in B(X) \), a family \( C(\cdot)(= \{ C(t) \mid 0 \leq t < T_0 \}) \) in \( B(X) \) is called a local \( \alpha \)-times integrated \( C \)-cosine function on \( X \) if it is strongly continuous, \( C(\cdot)C = CC(\cdot) \), and satisfies

\[
2C(t)C(s)x = \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t+s} \left( \int_0^t - \int_0^s \right) (t + s - r)^{\alpha - 1} C(r)Cx \, dr + \int_0^{t+s} (s + t - r)^{\alpha - 1} C(r)Cx \, dr \right\}
\]

(1.1)

for all \( 0 \leq t, s, t + s < T_0 \) and \( x \in X \); or called a local (0-times integrated) \( C \)-cosine function on \( X \) if it is strongly continuous, \( C(\cdot)C = CC(\cdot) \), and satisfies

\[
2C(t)C(s)x = C(t+s)Cx + C(|t-s|)Cx \quad \text{for all } 0 \leq t, s, t+s < T_0 \text{ and } x \in X,
\]

where \( \Gamma(\cdot) \) denotes the Gamma function. Moreover, we say that \( C(\cdot) \) is nondegenerate, if \( x = 0 \) whenever \( C(t)x = 0 \) for all \( 0 \leq t < T_0 \). In this case, its (integral) generator \( A : D(A) \subset X \rightarrow X \) is a closed linear operator in \( X \) defined by \( D(A) = \{ x \mid x \in X \text{ and there exists a } y_x \in X \text{ such that } C(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)} Cx = \int_0^t \int_0^s C(r) y_x, dr ds \text{ for all } 0 \leq t < T_0 \} \) and \( Ax = y_x \) for all \( x \in D(A) \). In general, a local \( \alpha \)-times integrated \( C \)-cosine function on \( X \) is also called an \( \alpha \)-times integrated \( C \)-cosine function on \( X \) if \( T_0 = \infty \); or called a local \( \alpha \)-times integrated cosine function on \( X \) if \( C = I \) the identity operator on \( X \). The relation between the existence of an exponentially bounded \( \alpha \)-times integrated \( C \)-cosine function with generator \( A \) and the unique existence of strong solutions of \( \text{ACP}_2(A, f, x, y) \) have been considered as in [4, 5, 9, 12, 14, 15] if \( \alpha \in \mathbb{N} \cup \{0\} \). When \( \alpha = 0 \) and \( A \) is densely defined, some results concerning the relation between the existence of a \( C \)-cosine function with generator \( A \) and the unique existence of weak solutions of \( \text{ACP}_2(A, f, x, y) \) are also investigated in [9], and in [7] for the case \( C = I \). Just as in the case \( \alpha \in \mathbb{N} \cup \{0\} \), some equivalence relations between the existence of an \( \alpha \)-times integrated \( C \)-cosine function on \( X \) and the unique existence of strong solutions for \( \text{ACP}_2(A, f, x, y) \) are also obtained in [10, 11] for which the resolvent set \( \rho(A) \) of \( A \) may be nonempty and \( D(A) \) may be dense in \( X \). Several examples concerning \( \alpha \)-times integrated cosine functions with densely defined generators are given as in [8], and in [16] when integrated cosine functions are exponentially bounded. Unfortunately, the generator of a local \( C \)-cosine function or a local \( \alpha \)-times integrated cosine function may not be densely defined except for
the case $\alpha = 0$ and $C = I$. In this case, the adjoint of a closed linear operator $A : D(A) \subset X \to X$ is the multi-valued function $A^* : D(A^*) \subset X^* \to 2^{X^*}$ defined by $D(A^*) = \{ x^* \in X^* \mid$ there exists a $y_{x^*}^* \in X^*$ such that $\langle x^*, Ax \rangle = \langle y_{x^*}^*, x \rangle$ for all $x \in D(A) \}$ and $A^* x^* = \{ y_{x^*}^* \in X^* \mid \langle x^*, Ax \rangle = \langle y_{x^*}^*, x \rangle$ for all $x \in D(A) \}$. In particular, we write $A^* x^* = y_{x^*}^*$ for $x^* \in D(A^*)$ if $A$ is densely defined, where $2^{X^*}$ denotes the power set of $X^*$ and either $< x^*, x >$ or $< x^*, x^* >$ denotes the value of $x^*$ at $x$ for all $x \in X$ and $x^* \in X^*$. Moreover, a function $u$ is called a weak solution of $ACP_2(A, f, x, y)$, if $\langle u(\cdot), x^* \rangle \in W_{1,0}^{2,1}((0, T_0)), < u(t), x^* > \mid_{t=0} = \langle x, x^* >, \frac{d}{dt} < u(t), x^* > \mid_{t=0} = \langle y, x^* >$ and $\frac{d^2}{dt^2} < u(t), x^* > = \langle u(t), y^* > + \langle f(t), x^* >$ for a.e. $0 \leq t < T_0$ whenever $x^* \in D(A^*)$ and $y^* \in A^* x^*$. Here $W_{1,0}^{2,1}((0, T_0)) = \{ v \mid v : [0, T_0) \to F$ is continuously differentiable, $v'$ is differentiable a.e. on $[0, T_0)$ and $v''$ is locally Lebesgue integrable on $[0, T_0) \}$. The purpose of this paper is to obtain some generalization theorems concerning local $\alpha$-times integrated C-cosine functions for $\alpha \geq 0$ when their generators may not be densely defined. We first investigate an important result (see Lemma 2.1 below) which has been deduced by Ball in [3] when $A$ is densely defined. Under the assumption $C^{-1} AC = A$. We show that $A$ generates a nondegenerate local $(\alpha + 1)$-times (respectively, $\alpha$-times) integrated C-cosine function on $X$ if and only if $ACP_2(A, j_{\alpha -1}(\cdot)Cx + j_{\alpha -1} * Cg(\cdot), 0, 0)$ has a unique weak solution in $C([0, T_0), X)$(respectively, in $C^1([0, T_0), X)$) for all $x \in X$ and $g \in L_{1,0}^1([0, T_0), X)$ if and only if $ACP_2(A, j_{\alpha -1}(\cdot)Cx, 0, 0)$ has a unique weak solution in $C([0, T_0), X)$ (respectively, in $C^1([0, T_0), X)$) for all $x \in X$, where $\alpha > 0$ (see Theorems 2.4 and 2.5 below). Here $\beta(t) = \frac{t^\beta}{\Gamma(\beta + 1)}$ for $\beta > -1$ and $t > 0$. Applying these results, we then show that $A$ generates a nondegenerate local 1-times (respectively, 0-times) integrated C-cosine function on $X$ if and only if $ACP_2(A, Cg(\cdot), 0, Cx)$ has a unique weak solution in $C([0, T_0), X)$(respectively, in $C^1([0, T_0), X)$) for all $x \in X$ and $g \in L_{1,0}^1([0, T_0), X)$ if and only if $ACP_2(A, 0, 0, Cx)$ has a unique weak solution in $C([0, T_0), X)$(respectively, in $C^1([0, T_0), X)$) for all $x \in X$ (see Theorems 2.6 and 2.7 below), which can be applied to show that $A$ generates a nondegenerate local (0-times integrated) C-cosine function on $X$ if and only if $ACP_2(A, Cg(\cdot), Cx, Cy)$ has a unique weak solution in $C([0, T_0), X) \forall x, y \in X$ and $g \in L_{1,0}^1([0, T_0), X)$ if and only if $ACP_2(A, Cg(\cdot), Cx, Cy)$ has a unique weak solution in $C([0, T_0), X)$ if and only if $ACP_2(A, Cg(\cdot), Cx, Cy)$ has a unique weak solution in $C([0, T_0), X) \forall x, y \in X$ and if and only if $ACP_2(A, 0, Cx, Cy)$ has a unique weak solution in $C([0, T_0), X) \forall x, y \in X$ (see Theorem 2.8 below). Our results are still new even when $\alpha = 0$. An illustrative example concerning these theorems is also presented in the final part of this paper.

2. Existence Theorems

In this section, we always assume that $C \in B(X)$ is an injection. We first inves-
tigate an important lemma which is used in the proofs of the following theorems, and has been obtained by Ball in [3] when \( A \) is densely defined.

**Lemma 2.1.** Let \( A : D(A) \subset X \rightarrow X \) be a closed linear operator. Assume that \( x_0, y_0 \in X \) and \( < y^*, x_0 > = < x^*, y_0 > \) for all \( x^* \in D(A^*) \) and \( y^* \in A^*x^* \). Then \( x_0 \in D(A) \) and \( Ax_0 = y_0 \).

**Proof.** If not, then there exist \( x^*, y^* \in X^* \) such that \( y^*(x_0) + x^*(y_0) \neq 0 \) and \( y^*(x) + x^*(Ax) = 0 \) for all \( x \in D(A) \), and so \( -y^*, x \geq x^*, Ax \) for all \( x \in D(A) \). Hence \( x^* \in D(A^*) \) and \( -y^* \in A^*x^* \). By hypothesis, we have \( -y^*, x_0 \geq x^*, y_0 \) or equivalently, \( y^*(x_0) + x^*(y_0) = 0 \). We obtain a contradiction. Consequently, \( x_0 \in D(A) \) and \( Ax_0 = y_0 \). \( \square \)

By slightly modifying the proofs of [11. Proposition 1.5] and [11. Lemma 1.6], the next proposition and lemma are also attained, and so their proofs are omitted.

**Proposition 2.2.** Let \( A \) be the generator of a nondegenerate local \( \alpha \)-times integrated C-cosine function \( C(\cdot) \) on \( X \). Then

\[
(2.1) \quad C(0) = C \quad \text{on} \quad X \quad \text{if} \quad \alpha = 0, \quad \text{and} \quad C(0) = 0 \quad \text{(the zero operator)} \quad \text{on} \quad X \quad \text{if} \quad \alpha > 0;
\]

\[
(2.2) \quad C \quad \text{is injective and} \quad C^{-1}AC = A;
\]

\[
(2.3) \quad C(t)x \in D(A) \quad \text{and} \quad AC(t)x = C(t)Ax \quad \text{for all} \quad x \in D(A) \quad \text{and} \quad 0 \leq t < T_0;
\]

\[
(2.4) \quad \int_0^t \int_0^s C(r)xdrds \in D(A) \quad \text{and} \quad A \int_0^t \int_0^s C(r)xdrds = C(t)x - j_\alpha(t)Cx
\]

\[
\quad \text{for all} \quad x \in X \quad \text{and} \quad 0 \leq t < T_0;
\]

\[
(2.5) \quad R(C(t)) \subset \overline{D(A)} \quad \text{for all} \quad 0 \leq t < T_0.
\]

**Lemma 2.3.** Let \( A \) be the generator of a nondegenerate local \( \alpha \)-times integrated C-cosine function \( C(\cdot) \) on \( X \), and let \( 0 < t_0 < T_0 \) be fixed. Assume that \( u \in C([0, t_0], X) \) satisfies \( u(t) = A \int_0^t (t - s)u(s)ds \) for all \( 0 \leq t < t_0 \). Then \( u \equiv 0 \) on \( [0, t_0] \).

**Theorem 2.4.** Let \( \alpha > 0 \), and \( A : D(A) \subset X \rightarrow X \) be a closed linear operator such that \( C^{-1}AC = A \). Then the following are equivalent:

(i) \( A \) generates a nondegenerate local \( (\alpha + 1) \)-times integrated C-cosine function \( S(\cdot) \) on \( X \);

(ii) For each \( x \in X \) and \( g \in L^1_{loc}([0, T_0], X) \), \( ACP_2(A, j_{\alpha-1}(\cdot)Cx + j_{\alpha-1} \ast Cg(\cdot), 0, 0) \) has a unique weak solution in \( C([0, T_0], X) \);
(iii) For each \( x \in X \), \( \text{ACP}_2(A, j_{\alpha-1} (\cdot) Cx, 0, 0) \) has a unique weak solution in \( C([0, T_0], X) \).

Here \( L_{loc}^1([0, T_0), X) \) denotes the set of all locally Bochner integrable functions from \([0, T_0) \) into \( X \) and \( j_\beta * g(t) = \int_0^t j_\beta(t - s) g(s) ds \) for all \( 0 \leq t < T_0 \) and \( g \in L_{loc}^1([0, T_0), X) \). Moreover,

(i) \( \| S(t) \| \leq Ke^{\omega t} \) for all \( t \geq 0 \) and for some \( K, \omega \geq 0 \) if and only if for each \( x \in X, \| u(t, Cx) \| \leq Ke^{\omega t} \| x \| \) for all \( t \geq 0 \);

(ii) \( \| S(t + h) - S(t) \| \leq Khe^{\omega(t+h)} \) for all \( t, h \geq 0 \) and for some \( K, \omega \geq 0 \) if and only if for each \( x \in X, \| u(t + h, Cx) - u(t, Cx) \| \leq Khe^{\omega(t+h)} \| x \| \) for all \( t, h \geq 0 \);

(iii) For each \( 0 < t_0 < T_0 \), \( \| S(t+h) - S(t) \| \leq K_{t_0} h \) for all \( 0 \leq t, h \leq t+h \leq t_0 \) and for some \( K_{t_0} > 0 \) if and only if for each \( x \in X \) and \( 0 < t_0 < T_0 \), \( \| u(t + h, Cx) - u(t, Cx) \| \leq K_{t_0} h \| x \| \) for all \( 0 \leq t, h \leq t+h \leq t_0 \) and for some \( K_{t_0} > 0 \).

Proof. (i) \( \Rightarrow \) (ii). Indeed, if \( A \) is the generator of a nondegenerate local \((\alpha + 1)\)-times integrated C-cosine function \( S(\cdot) \) on \( X \) and \( x \in X \) is given. Then for \( x^* \in D(A^*) \) and \( y^* \in A^* x^* \), we have \( < S(t)x, x^* > = \int_0^t \int_0^s < S(r)x, y^* > dr ds + j_{\alpha+1}(t) < Cx, x^* > \) for all \( 0 \leq t < T_0 \), and so

\[
\frac{d}{dt} < S(t)x, x^* > = \int_0^t < S(s)x, y^* > ds + j_{\alpha}(t) < Cx, x^* >
\]

for all \( 0 \leq t < T_0 \). Hence

\[
\frac{d^2}{dt^2} < S(t)x, x^* > = < S(t)x, y^* > + j_{\alpha-1}(t) < Cx, x^* >
\]

for all \( 0 < t < T_0 \). Now if \( g \in C([0, T_0), X) \) is given, then

\[
< \int_0^t S(t-s)g(s)ds, x^* > \\
= \int_0^t < S(t-s)g(s)ds, x^* > ds \\
= \int_0^t < S(t-s)g(s), y^* > ds + \int_0^t < j_{\alpha+1}(t-s)Cg(s), x^* > ds
\]

for all \( 0 \leq t < T_0 \). Here \( \tilde{S}(t)y = \int_0^t \int_0^s S(r)y dr ds \) for all \( 0 \leq t < T_0 \) and \( y \in X \). By differentiation, we have

\[
\frac{d}{dt} < \int_0^t S(t-s)g(s)ds, x^* > \\
= \int_0^t < \tilde{S}(t-s)g(s), y^* > ds + \int_0^t < j_{\alpha}(t-s)Cg(s), x^* > ds
\]
for all $0 \leq t < T_0$, and
\[
\frac{d^2}{dt^2} < \int_0^t S(t-s)g(s)ds, x^* > \\
= \int_0^t < S(t-s)g(s), y^* > ds + \int_0^t < j_{\alpha-1}(t-s)Cg(s), x^* > ds
\]
for all $t \in (0, T_0)$, where $\tilde{S}(t)y = \int_0^t S(r)ydr$ for all $0 \leq t < T_0$ and $y \in X$. Next we set $u_m(\cdot) = S(\cdot)x + S^*g(\cdot)$, then $u \in C([0, T_0], X)$, $u(0) = 0$ and $\frac{du}{dt} < u(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u(t), x^* >=< u(t), y^* > + < j_{\alpha-1}(t)Cx + j_{\alpha-1}^*Cg(t), x^* >$ for $t \in (0, T_0)$, which implies that $u \in C([0, T_0], X)$ is a weak solution of $ACP_2(A, j_{\alpha-1}^*C^x + j_{\alpha-1}^*Cg(\cdot), 0, 0)$ satisfying $u(0) = 0$. Finally, we turn to the case that $g$ is only an $L^1_{loc}([0, T_0], X)$ function and $\{g_m\}_{m=1}^\infty$ is a sequence in $C([0, T_0], X)$ such that $g_m \rightarrow g$ in $L^1([0, t_0], X)$ for all $0 < t_0 < T_0$. We define
\[
u(\cdot) = S(\cdot)x + S^*g(\cdot)
\]
and
\[
u_m(\cdot) = S(\cdot)x + S^*g_m(\cdot)
\]
for $m \in \mathbb{N}$, then $\|u_m(t) - u(t)\| \leq \int_{t_0}^{t} \sup_{\tau \in [t_0, t]} \|S(\tau)\|\|g_m(s) - g(s)\| ds$ for all $0 \leq t \leq t_0 < T_0$, and so $u_m(\cdot) \rightarrow u(\cdot)$ uniformly on compact subsets of $[0, T_0]$. Hence $u(\cdot)$ is continuous on $[0, T_0]$. The previous argument shows that $u_m(0) = 0$, $\frac{du}{dt} < u_m(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u_m(t), x^*> = < u_m(t), y^*> + < j_{\alpha-1}(t)Cx, x^*> + < j_{\alpha-1}^*Cg_m(t), x^*>$ for $t \in (0, T_0)$. By integration, we have
\[
\frac{d}{dt} < u_m(t), x^*> = \int_0^t < u_m(s), y^*> ds + j_{\alpha}(t)Cx, x^*> + < j_{\alpha}^*Cg_m(t), x^*> 
\]
and
\[
< u_m(t), x^*> = \int_0^t \int_0^s < u_m(r), y^*> dr ds + j_{\alpha+1}(t)Cx, x^*> + < j_{\alpha+1}^*Cg_m(t), x^*> 
\]
for all $0 \leq t < T_0$. Letting $m \rightarrow \infty$, we get that
\[
\int_0^t < u_m(s), y^*> ds + j_{\alpha}(\cdot)Cx, x^*> + < j_{\alpha}^*Cg_m(\cdot), x^*> \\
\rightarrow \int_0^t < u(s), y^*> ds + j_{\alpha}(\cdot)Cx, x^*> + < j_{\alpha}^*Cg(\cdot), x^*> 
\]
uniformly on compact subsets of \([0, T_0]\) and

\[
<u(t), x^*> = \int_0^t \int_0^s <u(r), y^*> dr ds + <j_{\alpha+1}(t)Cx, x^*> \\
+ <j_{\alpha+1} * Cg(t), x^*> 
\]

for all \(0 \leq t < T_0\). In particular, \(u(0) = 0\), \(\frac{d}{dt} <u(t), x^*> |_{t=0} = 0\) and \(\frac{d^2}{dt^2} <u(t), x^*> \leq <j_{\alpha-1}(t)Cx, x^*> + <j_{\alpha-1} * Cg(t), x^*>\) for \(t \in (0, T_0)\), which implies that \(u \in C([0, T_0], X)\) is a weak solution of \(ACP_2(\alpha, j_{\alpha-1} \cdot Cx + j_{\alpha-1} * Cg(\cdot), 0, 0)\) satisfying \(u(0) = 0\). To prove the uniqueness, let \(v\) be another weak solution of \(ACP_2(\alpha, j_{\alpha-1} \cdot Cx + j_{\alpha-1} * Cg(\cdot), 0, 0)\) in \(C([0, T_0], X)\) and \(w(\cdot) = u(\cdot) - v(\cdot)\) on \([0, T_0]\). Applying the continuity of \(w\), we get that

\[
<w(t), x^*> = \left(\int_0^t \int_0^s w(r) dr ds, y^*> \right) \\
\quad \text{for all } 0 \leq t < T_0, \quad x^* \in D(A^*) \text{ and } y^* \in A^* x^*,
\]

which together with Lemma 2.1 implies that \(\int_0^t \int_0^s w(r) dr ds \in D(A)\) and \(A \int_0^t \int_0^s w(r) dr ds = w(t)\) for all \(0 \leq t < T_0\). It follows from Lemma 2.3 that we have \(w = 0\) on \([0, T_0]\) or equivalently, \(u = v\) on \([0, T_0]\).

(iii) \(\Rightarrow\) (i). Indeed, if the unique weak solution of \(ACP_2(\alpha, j_{\alpha-1} \cdot Cx, 0, 0)\) in \(C([0, T_0], X)\) is denoted by \(w(\cdot)\) for all \(x \in X\). We define the map \(S(t) : X \rightarrow X\) by \(S(t)x = w(t, Cx)\) for all \(x \in X\) and \(0 \leq t < T_0\). Clearly, \(S(\cdot) x : [0, T_0] \rightarrow X\) is continuous for all \(x \in X\). It follows from the uniqueness of weak solutions of \(ACP_2(\alpha, j_{\alpha-1} \cdot Cx, 0, 0)\) in \(C([0, T_0], X)\) and Lemma 2.1 that \(S(t)\) is linear for all \(0 \leq t < T_0\), \(S(\cdot) (\{S(t) | 0 \leq t < T_0\})\) commutes with \(C\) and is nondegenerate. Next we shall show that \(S(\cdot) \subset B(X)\). By the closed graph theorem, we need only to show that the linear map \(\eta : X \rightarrow C([0, T_0], X)\) defined by \(\eta(x) = S(\cdot) x\) for \(x \in X\), is a continuous function from the Banach space \(X\) into the Frechet space \(C([0, T_0], X)\) with the quasi-norm \(|\cdot|\) defined by \(|v| = \sum_{k=1}^{\infty} \frac{\|v\|_k}{k!(1+\|v\|_k)}\) for \(v \in C([0, T_0], X)\), where \(\|v\|_k = \max_{t \in [0,k]} \|v(t)\|\). Indeed, if \(\{x_m\}_{m=1}^{\infty}\) is a sequence in \(X\) such that \(x_m \rightarrow x\) in \(X\) and \(\eta(x_m) \rightarrow u(\cdot)\) in \(C([0, T_0], X)\) as \(m \rightarrow \infty\). Then for \(x^* \in D(A^*)\) and \(y^* \in A^* x^*\), we have

\[
\int_0^t \int_0^s <S(r)x_m, y^*> dr ds \\
= \int_0^t \int_0^s \frac{d^2}{dt^2} <S(r)x_m, x^*> dr ds - \int_0^t \int_0^s <j_{\alpha-1}(r)Cx_m, x^*> dr ds \\
= <S(t)x_m, x^*> - j_{\alpha+1}(t) <Cx_m, x^*>
\]
for all $0 \leq t < T_0$, and so

$$< u(t), x^* > = j_{\alpha+1}(t) < Cx, x^* > + \int_0^t \int_0^s < u(r), y^* > dr ds$$

for all $0 \leq t < T_0$. Hence $< u(\cdot), x^* > \in W_{loc}^{2,1}([0, T_0]), < u(t), x^* > |_{t=0} = \frac{d}{dt} < u(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u(t), x^* > = j_{\alpha-1}(t) < Cx, x^* > + < u(t), y^* >$ for a.e. $0 \leq t < T_0$, which implies that $u$ is a weak solution of $ACP_2(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$ in $C([0, T_0], X)$. The uniqueness of weak solutions in $C([0, T_0], X)$ implies that $u(\cdot) = S(\cdot)x = \eta(x)$. Consequently, $\eta$ is closed. In order, we shall show that $S(\cdot)$ is a local $(\alpha + 1)$-times integrated C-cosine function on $X$. Indeed, if $x \in X$ and $0 \leq s < T_0$ are given. We first assume that $\alpha \geq 1$ and define

$$v_s(t) = \frac{1}{\Gamma(\alpha + 1)} \{ \int_0^{t+s} - \int_0^t - \int_0^s (t + s - r)^\alpha S(r)Cx dr$$

$$+ \int_{|t-s|}^t (s - t + r)^\alpha S(r)Cx dr + \int_{|t-s|}^s (t - s + r)^\alpha S(r)Cx dr$$

$$+ \int_0^{t-s} (|t - s| + r)^\alpha S(r)Cx dr \}$$

for all $0 \leq t \leq t + s < T_0$. Then for $x^* \in D(A^*)$ and $y^* \in A^*x^*$, we obtain from [11, Lemma 2.1] that

$$< v_s(t), y^* > = \frac{1}{\Gamma(\alpha + 1)} \{ \int_0^{t+s} - \int_0^t - \int_0^s (t + s - r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > - < j_{\alpha-1}(r)C^2x, x^* > | dr$$

$$+ \int_{|t-s|}^t (s - t + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > - < j_{\alpha-1}(r)C^2x, x^* > | dr$$

$$+ \int_{|t-s|}^s (t - s + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > - < j_{\alpha-1}(r)C^2x, x^* > | dr$$

$$+ \int_0^{t-s} (|t - s| + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > - < j_{\alpha-1}(r)C^2x, x^* > | dr \}$$

$$= \frac{1}{\Gamma(\alpha + 1)} \{ \int_0^{t+s} - \int_0^t - \int_0^s (t + s - r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > | dr$$

$$+ \int_{|t-s|}^t (s - t + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > | dr$$

$$+ \int_{|t-s|}^s (t - s + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > | dr$$

$$+ \int_0^{t-s} (|t - s| + r)^\alpha \frac{d^2}{dr^2} < S(r)Cx, x^* > | dr \}$$
for all $0 \leq t \leq t + s < T_0$. Using integration by parts twice, we also have, for $0 \leq t < t + s < T_0$, 

$$
\begin{align*}
&< v_s(t), y^*> = \frac{1}{\Gamma(\alpha - 1)} \{ \int_0^{t+s} (t + s - r)^{\alpha-2} < S(r)Cx, x^*> dr \\
&\quad + \int_0^t (s-t+r)^{\alpha-2} < S(r)Cx, x^*> dr + \int_{s}^{t+s} (t-s+r)^{\alpha-2} < S(r)Cx, x^*> dr \\
&\quad + \int_0^{|t-s|} (|t-s|+r)^{\alpha-2} < S(r)Cx, x^*> dr \} - 2j_{\alpha-1}(s) < S(t)Cx, x^*> \\
&\frac{d}{dt} < v_s(t), x^*> |_{t=0} = 0 = < v_s(t), x^*> |_{t=0} \end{align*}
$$

when $\alpha > 1$. Similarly, we can show that for $0 \leq t \leq t + s < T_0$

$$
\begin{align*}
&< v_s(t), y^*> = < S(t+s)Cx, x^*> + < S(|t-s|)Cx, x^*> \\
&\quad - 2 < S(s)Cx, x^*> - 2 < S(t)Cx, x^*> ,
\end{align*}
$$

and

$$
\begin{align*}
&\frac{d^2}{dt^2} < v_s(t), x^*> |_{t=0} = 0 = < v_s(t), x^*> |_{t=0} \quad \text{and} \\
&\frac{d^2}{dt^2} < v_s(t), x^*> = < S(t+s)Cx, x^*> + < S(|t-s|)Cx, x^*> \\
&\quad - 2 < S(t)Cx, x^*> \\
&\quad + < v_s(t), y^*> + 2 < j_{\alpha-1}(t)CS(s)x, x^*> 
\end{align*}
$$
when $\alpha = 1$. Applying the uniqueness of weak solutions of $ACP_2(A, 2j_{\alpha-1}(\cdot)CS(s)x, 0, 0)$ in $C([0, T_0], X)$, we get that $v_t(t) = 2S(t)sx$ for all $0 \leq t \leq t + s < T_0$. Consequently, $S(\cdot)$ is a nondegenerate local $(\alpha + 1)$-times integrated C-cosine function on $X$ when $\alpha \geq 1$. We now turn to the case $0 < \alpha < 1$. By hypothesis, $\int_0^t w(s, Cx)ds$ is a unique weak solution of $ACP_2(A, j_{\alpha}(\cdot)Cx, 0, 0)$ in $C^1([0, T_0], X)$ for all $x \in X$. Just as in the proof of the case $\alpha > 1$, we can show that $\tilde{S}(\cdot)$ is a nondegenerate local $(\alpha + 2)$-times integrated C-cosine function on $X$. Here $S(t)x = \int_0^t S(s)xds$ for all $0 \leq t < T_0$ and $x \in X$. An easy computation shows that $S(\cdot)$ is a nondegenerate local $(\alpha + 1)$-times integrated C-cosine function on $X$. Finally, we shall show that $A$ is its generator. Indeed, if $B$ denotes the generator of $S(\cdot)$ and $x \in D(B)$ is given. Then for $x^* \in D(A^*)$ and $y^* \in A^*x^*$, we have

$$<S(t)x, y^*> = \frac{d^2}{dt^2} <S(t)x, x^*> - <j_{\alpha-1}(t)Cx, x^*>$$

$$= <S(t)Bx, x^*>$$

for a.e. $0 \leq t < T_0$ because $S(\cdot)x$ is a weak solution of $ACP_2(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$. The strong continuity of $S(\cdot)$ implies that $<S(t)x, y^*> = <S(t)Bx, x^*>$ for all $0 \leq t < T_0$. Applying Lemma 2.1, we get that $S(t)x \in D(A)$ and $AS(t)x = S(t)Bx$ for all $0 \leq t < T_0$. Since $S(\cdot)Bx$ is a weak solution of $ACP_2(A, j_{\alpha-1}(\cdot)CBx, 0, 0)$, we also have

$$<\int_0^t \int_0^s S(r)Bxdrds, y^*>$$

$$= \int_0^t \int_0^s <S(r)Bx, y^*> drds$$

$$= \int_0^t \int_0^s \frac{d^2}{dt^2} <S(r)Bx, x^*> - <j_{\alpha-1}(r)CBx, x^*> drds$$

$$= <S(t)Bx, x^*> - <j_{\alpha+1}(t)CBx, x^*>$$

for all $x^* \in D(A^*)$, $y^* \in A^*x^*$ and $0 \leq t < T_0$. Applying Lemma 2.1 again, we get that

$$\int_0^t \int_0^s S(r)Bxdrds \in D(A)$$

and

$$A \int_0^t \int_0^s S(r)Bxdrds = S(t)Bx - j_{\alpha+1}(t)CBx$$

for all $0 \leq t < T_0$, and so

$$-j_{\alpha+1}(t)ACx = A \int_0^t \int_0^s S(r)Bxdrds - AS(t)x$$

$$= [S(t)Bx - j_{\alpha+1}(t)CBx] - S(t)Bx$$

$$= -j_{\alpha+1}(t)CBx$$
for all $0 \leq t < T_0$. Hence $x \in D(C^{-1}AC)$ and $C^{-1}ACx = Bx$. Having shown that $B \subset C^{-1}AC$. We next show that $A \subset B$. Indeed, if $x \in D(A)$ is given, then $\int_0^t \int_0^s S(r)xdrds + \int_0^t \int_0^s S(r)A xdrds \in D(A)$.

\begin{equation}
S(t)x = j_{\alpha+1}(t)C x + A \int_0^t \int_0^s S(r)xdrds
\end{equation}
and

\begin{equation}
S(t)A x = j_{\alpha+1}(t)(C A x) + A \int_0^t \int_0^s S(r)A xdrds
\end{equation}
for all $0 \leq t < T_0$. It is easy to see from (2.6) and (2.7) that the function $t \mapsto \int_0^t \int_0^s S(r)xdrds - A \int_0^t \int_0^s S(r)A xdrds$ is a weak solution of $ACP_2(A,0,0,0)$ in $C([0,T_0],X)$, and hence it must be the zero function on $[0,T_0]$ or equivalently, $\int_0^t \int_0^s S(r)xdrds - A \int_0^t \int_0^s S(r)A xdrds$ for all $0 \leq t < T_0$, which together with (2.6) implies that $x \in D(B)$ and $Bx = Ax$. Consequently, $A = B$.

**Theorem 2.5.** Let $\alpha > 0$, and $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent:

(i) For each $x \in X$ and $g \in L^1_{loc}([0,T_0),X)$, $ACP_2(A,j_{\alpha}(-)Cx + j_{\alpha} * Cg(-),0,0)$ has a unique strong solution in $C^2([0,T_0),X) \cap C([0,T_0],[D(A)])$;

(ii) For each $x \in X$ $ACP_2(A,j_{\alpha}(-)Cx,0,0)$ has a unique strong solution in $C^2([0,T_0),X) \cap C([0,T_0],[D(A)])$;

(iii) $A$ generates a nondegenerate local $\alpha$-times integrated C-cosine function $C(\cdot)$ on $X$;

(iv) For each $x \in X$ and $g \in L^1_{loc}([0,T_0),X)$, $ACP_2(A,j_{\alpha-1}(-)Cx + j_{\alpha-1} * Cg(-),0,0)$ has a unique weak solution in $C^1([0,T_0),X)$;

(v) For each $x \in X ACP_2(A,j_{\alpha-1}(-)Cx,0,0)$ has a unique weak solution in $C^1([0,T_0),X)$.

Moreover, $\|C(t)\| \leq K e^{\omega t}$ for all $t \geq 0$ and for some $K, \omega \geq 0$ if and only if for each $x \in X$, the unique weak solution $u(\cdot,Cx)$ of $ACP_2(A,j_{\alpha-1}(-)Cx,0,0)$ satisfies $\|u(t+h,Cx) - u(t,Cx)\| \leq K e^{\omega(t+h)}\|x\|$ for all $t, h \geq 0$.

**Proof.** The equivalence relations (i)-(iii) follow from [11, Theorem 2.3]. To show that (iii)$\Rightarrow$(iv). Indeed, if $C(\cdot)$ is a nondegenerate local $\alpha$-times integrated C-cosine function on $X$ with generator $A$, then $S(\cdot)$ is a nondegenerate local $(\alpha+1)$-times integrated C-cosine function on $X$ with generator $A$ and satisfies $S(\cdot)x \in C^1([0,T_0),X)$ for all $x \in X$, where $S(t)x = \int_0^t C(r)xdr$. It follows from Theorem 2.4 that $S(\cdot)x + S * g(\cdot)$ is the unique weak solution of $ACP_2(A,j_{\alpha-1}(-)Cx +
has a unique weak solution $\ACP$ of $S$ on $X$ with generator $A$. Applying Theorem 2.4, we get that $S(\cdot)$ is a nondegenerate local $(\alpha + 1)$-times integrated C-cosine function on $X$ with generator $A$, which implies that $C(\cdot)$ is a nondegenerate local $\alpha$-times integrated C-cosine function on $X$ with generator $A$, where $C(t)x = \frac{d}{dt}S(t)x$ for all $0 \leq t < T_0$ and $x \in X$.

Applying Theorem 2.5, the next theorem concerning local 1-times integrated C-cosine functions is also obtained.

**Theorem 2.6.** Let $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent:

(i) $A$ generates a nondegenerate local 1-times integrated C-cosine function $C(\cdot)$ on $X$;

(ii) For each $x \in X$ and $g \in L^1_{\text{loc}}([0, T_0), X)$, $\ACP(A, Cg(\cdot), 0, Cx)$ has a unique weak solution in $C([0, T_0), X)$;

(iii) For each $x \in X$, $\ACP(A, 0, 0, Cx)$ has a unique weak solution $u(\cdot, Cx)$ in $C([0, T_0), X)$.

Moreover,

(i) \[ \|C(t)\| \leq Ke^{\omega t} \text{ for all } t \geq 0 \text{ and for some } K, \omega \geq 0 \text{ if and only if for each } x \in X, \|u(t, Cx)\| \leq Ke^{\omega t}\|x\| \text{ for all } t \geq 0; \]

(ii) \[ \|C(t + h) - C(t)\| \leq Khe^{\omega(t + h)} \text{ for all } t, h \geq 0 \text{ and for some } K, \omega \geq 0 \text{ if and only if for each } x \in X, \|u(t + h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t + h)}\|x\| \text{ for all } t, h \geq 0; \]

(iii) For each $0 < t_0 < T_0$, \[ \|C(t + h) - C(t)\| \leq K_{t_0}h \text{ for all } 0 \leq t, h \leq t + h \leq t_0 \text{ and for some } K_{t_0} > 0 \text{ if and only if for each } x \in X \text{ and } 0 < t_0 < T_0, \]
\[ \|u(t + h, Cx) - u(t, Cx)\| \leq K_{t_0}h\|x\| \text{ for all } 0 \leq t, h \leq t + h \leq t_0 \text{ and for some } K_{t_0} > 0. \]

**Proof.** We first show that (i)⇒(ii). Indeed, if $A$ generates a nondegenerate local 1-times integrated C-cosine function on $X$. Then for each $x \in X$ and $g \in L^1_{\text{loc}}([0, T_0), X)$, we obtain from Theorem 2.5 that $\ACP(A, Cx + j_0 \ast Cg(\cdot), 0, 0)$ has a unique weak solution $u$ in $C^1([0, T_0), X)$ which satisfies $u(0) = 0$, so that for each $x^* \in D(A^*)$ and $y^* \in A^*x^*$, we have $\langle u'(t), x^* \rangle = \frac{d}{dt} \langle u(t), x^* \rangle |_{t=0} = 0$. The result follows from the above.

\[ \langle u'(t), x^* \rangle = \frac{d}{dt} \langle u(t), x^* \rangle |_{t=0} = 0. \]
\[
\frac{d^2}{dt^2} < u'(t), x^* > = \frac{d^3}{dt^3} < u(t), x^* > \\
= \frac{d}{dt}[< u(t), y^* > + < Cx + j_0 * Cg(t), x^* >] \\
=< u'(t), y^* > + < Cg(t), x^* >
\]

for a.e. \(0 \leq t < T_0\). Clearly, \(\frac{d}{dt} < u'(t), x^* > = \frac{d^2}{dt^2} < u(t), x^* > = < u(t), y^* > + < Cx + j_0 * Cg(t), x^* > \) for all \(0 \leq t < T_0\). In particular, \(\frac{d}{dt} < u'(t), x^* > \mid_{t=0} = < Cx, x^* >\). It follows that \(u'\) is a weak solution of \(ACP_2(A, Cg, 0, Cx)\) in \(C([0, T_0]), X\). The uniqueness of weak solutions of \(ACP_2(A, Cg, 0, Cx)\) in \(C([0, T_0]), X\) follows from the uniqueness of weak solutions of \(ACP_2(A, 0, 0, 0)\) in \(C([0, T_0]), X\). In order, we show that (iii) \(\Rightarrow\) (i). Indeed, if \(u(\cdot, x)\) denotes the unique weak solution of \(ACP_2(A, 0, 0, Cx)\) in \(C([0, T_0]), X\) for all \(x \in X\), then \(v = j_0 * u\) is the unique weak solution of \(ACP_2(A, Cx, 0, 0)\) in \(C^1([0, T_0]), X)\). Applying Theorem 2.5, we get that \(A\) generates a nondegenerate local \(1\)-times integrated \(C\)-cosine function \(C(\cdot)\) on \(X\) which is defined by \(C(t)x = u(t, x)\) for all \(0 \leq t < T_0\) and \(x \in X\).

By slightly modifying the proof of Theorem 2.5, we can apply Theorem 2.6 to prove the next theorem concerning local \((0\)-times integrated) \(C\)-cosine functions.

**Theorem 2.7.** Let \(A : D(A) \subset X \rightarrow X\) be a closed linear operator such that \(C^{-1}AC = A\). Then the following are equivalent:

(i) For each \(x \in X\) and \(g \in L^1_{\text{loc}}([0, T_0], X)\), \(ACP_2(A, Cx + j_0 * Cg(\cdot), 0, 0)\) has a unique strong solution in \(C^2([0, T_0], X) \cap C([0, T_0], [D(A)])\);

(ii) For each \(x \in X\), \(ACP_2(A, Cx, 0, 0)\) has a unique strong solution in \(C^2([0, T_0], X) \cap C([0, T_0], [D(A)])\);

(iii) \(A\) generates a nondegenerate local \((0\)-times integrated) \(C\)-cosine function \(C(\cdot)\) on \(X\);

(iv) For each \(x \in X\) and \(g \in L^1_{\text{loc}}([0, T_0], X)\), \(ACP_2(A, Cg(\cdot), 0, Cx)\) has a unique weak solution in \(C^1([0, T_0], X)\);

(v) For each \(x \in X\), \(ACP_2(A, 0, 0, Cx)\) has a unique weak solution \(u(\cdot, Cx)\) in \(C^1([0, T_0], X)\).

Moreover, \(\|C(t)\| \leq K e^{\omega t}\) for all \(t \geq 0\) and for some \(K, \omega \geq 0\) if and only if for each \(x \in X\), \(\|u(t + h, Cx) - u(t, Cx)\| \leq K e^{\omega(t+h)} \|x\|\) for all \(t, h \geq 0\).

Similarly, we can apply Theorem 2.7 to prove the next theorem concerning local \((0\)-times integrated) \(C\)-cosine functions which has been obtained in [9] when \(A\) is densely defined.
Theorem 2.8. Let \( A : D(A) \subset X \to X \) be a closed linear operator such that \( C^{-1}AC = A \). Then the following are equivalent:

(i) For each \( x, y \in X \) and \( g \in L^1_{\text{loc}}([0, T_0), X) \), \( \text{ACP}_2(A, Cx + j_1(\cdot)Cy + j_0 * Cg(\cdot), 0, 0) \) has a unique strong solution in \( C^2([0, T_0), X) \cap C([0, T_0), [D(A)]) \);

(ii) For each \( x \in X \) and \( g \in L^1_{\text{loc}}([0, T_0), X) \), \( \text{ACP}_2(A, Cx + j_0 * Cg(\cdot), 0, 0) \) has a unique strong solution in \( C^2([0, T_0), X) \cap C([0, T_0), [D(A)]) \);

(iii) For each \( x, y \in X \), \( \text{ACP}_2(A, Cx + j_1(\cdot)Cy, 0, 0) \) has a unique strong solution in \( C^2([0, T_0), X) \cap C([0, T_0), [D(A)]) \);

(iv) \( A \) generates a nondegenerate local \( (0\text{-times integrated}) C\)-cosine function \( C(\cdot) \) on \( X \);

(v) For each \( x, y \in X \) and \( g \in L^1_{\text{loc}}([0, T_0), X) \), \( \text{ACP}_2(A, Cg(\cdot), Cx, Cy) \) has a unique weak solution in \( C([0, T_0), X) \);

(vi) For each \( x \in X \) and \( g \in L^1_{\text{loc}}([0, T_0), X) \), \( \text{ACP}_2(A, Cg(\cdot), Cx, 0) \) has a unique weak solution in \( C([0, T_0), X) \);

(vii) For each \( x, y \in X \), \( \text{ACP}_2(A, 0, Cx, Cy) \) has a unique weak solution in \( C([0, T_0), X) \);

(viii) For each \( x \in X \), \( \text{ACP}_2(A, 0, Cx, 0) \) has a unique weak solution \( u(\cdot, Cx) \) in \( C([0, T_0), X) \).

Moreover,

(i) \( \|C(t)\| \leq Ke^{\omega t} \) for all \( t \geq 0 \) and for some \( K, \omega \geq 0 \) if and only if for each \( x \in X \), \( \|u(t, Cx)\| \leq Ke^{\omega t}\|x\| \) for all \( t \geq 0 \);

(ii) \( \|C(t + h) - C(t)\| \leq Khe^{\omega(t+h)} \) for all \( t, h \geq 0 \) and for some \( K, \omega \geq 0 \) if and only if for each \( x \in X \), \( \|u(t + h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t+h)}\|x\| \) for all \( t, h \geq 0 \);

(iii) For each \( 0 < t_0 < T_0 \), \( \|C(t+h) - C(t)\| \leq K_{t_0}h \) for all \( 0 \leq t, h \leq t + h \leq t_0 \) and for some \( K_{t_0} > 0 \) if and only if for each \( x \in X \) and \( 0 < t_0 < T_0 \), \( \|u(t + h, Cx) - u(t, Cx)\| \leq K_{t_0}h\|x\| \) for all \( 0 \leq t, h \leq t + h \leq t_0 \) and for some \( K_{t_0} > 0 \).

We end this paper with a simple illustrative example. Let \( X = C_b(\mathbb{R})( \text{ or } L^\infty(\mathbb{R})) \), and \( A \) be the maximal differential operator in \( X \) defined by \( A = \sum_{j=0}^k a_j D^j u \) on \( \mathbb{R} \) for all \( u \in D(A) \), then \( UC_b(\mathbb{R})( \text{ or } C_0(\mathbb{R})) = \overline{D(A)} \). Here \( a_0, a_1, \cdots, a_k \in \mathbb{C} \) and \( D^j u(x) = u^{(j)}(x) \) for all \( x \in \mathbb{R} \). It is shown in [2, Theorem 6.7] that \( A \) generates an exponentially bounded, norm continuous \( 1\)-times integrated cosine function \( C(\cdot) \) on \( X \) which is defined by \( (C(t)f)(x) = \frac{1}{\sqrt{2\pi}}(\tilde{g}_t * f)(x) \)
for all $f \in X$ and $t \geq 0$ if the real-valued polynomial $p(x) = \sum_{j=0}^{k} a_j (ix)^j$ satisfies $\sup_{x \in \mathbb{R}} p(x) < \infty$. Here $\tilde{\phi}_t$ denotes the inverse Fourier transform of $\phi_t$ with $\phi_t(x) = \int_0^t \cosh(\sqrt{p(x)}s)ds$. Applying Theorem 2.6, we get that for each $f \in X$ and continuous function $g$ on $[0, T_0) \times \mathbb{R}$ with $\int_0^t \sup_{x \in \mathbb{R}} |g(s, x)|ds < \infty$ for all $0 \leq t < T_0$, the function $u$ on $[0, T_0) \times \mathbb{R}$ defined by $u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}_t(x-y)f(y)dy + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \tilde{\phi}_t-s(x-y)g(s, y)dyds$ for all $0 \leq t < T_0$ and $x \in \mathbb{R}$, is the unique weak solution of

$$\begin{cases} 
\frac{\partial^2 u(t, x)}{\partial t^2} = \sum_{j=0}^{k} a_j \frac{\partial}{\partial x}^j u(t, x) + g(t, x) \text{ for } t \in (0, T_0) \text{ and a.e. } x \in \mathbb{R}, \\
u(0, x) = 0 \text{ and } \frac{\partial u}{\partial t}(0, x) = f(x) \text{ for a.e. } x \in \mathbb{R} 
\end{cases}$$

in $C([0, T_0), X)$.

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