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THE ASYMPTOTIC TIAN-YAU-ZELDITCH EXPANSION ON RIEMANN SURFACES WITH CONSTANT CURVATURE

Chiung-ju Liu

Abstract. Let M be a regular Riemann surface with a metric which has constant scalar curvature ρ. We give the asymptotic expansion of the sum of the square norm of the sections of the pluricanonical bundles K_M^m. That is,

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1+\frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \cdots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for sufficiently large m.

1. Introduction

Let M be an n-dimensional compact complex Kähler manifold with an ample line bundle L over M. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$ for some positive Hermitian h metric on L. Such a Kähler metric g is called a polarized Kähler metric. The metric h induces a Hermitian metric h_m on L^m for all positive integers m. Let $\{S_0, \cdots, S_{d_m-1}\}$ be an orthonormal basis of the space $H^0(M, L^m)$ with respect to the inner product

$$(S,T) = \int_M \langle S(x), T(x)\rangle_{h_m} dV_g, \tag{1.1}$$

where $d_m = \dim H^0(M, L^m)$ and $dV_g = \frac{\omega_g^n}{n!}$ is the volume form of g. The quantity

$$\sum_{i=0}^{d_m-1} \|S_i(x)\|_{h_m}^2 \tag{1.2}$$

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is related to the existence of Kähler-Einstein metrics and stability of complex manifolds. A lot of work has been done for (1.2) on compact complex Kähler manifolds. Tian [6] applied Hömander's L^2-estimate to produce peak sections and proved the C^2 convergence of the Bergman metrics. Later, Ruan [5] proved the C^∞ convergence. About the same time, Zelditch [7] and Catlin [4] separately generalized the theorem of Tian by showing there is an asymptotic expansion

$$\sum_{i=0}^{d_m-1} \|S_i(x)\|^2_{h_m} \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \tag{1.3}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. In [10], Lu proved that each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives. In particular, $a_1 = \frac{\rho}{2}$, where ρ is the scalar curvature of M. These polynomials can be found by finitely many steps of algebraic operations. Recently, Song [3] generalized Zelditch's theorem on orbifolds of finite isolated singularities. The information on the singularities can be found in the expansion.

On the Riemann surfaces with bounded curvature, Lu [9] proved that there is a lower bound for (1.2). Later, the result of Lu and Tian [8] implies that on the Riemann surfaces with constant scalar curvature ρ, the asymptotic expansion (1.3) is given by

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1 + \frac{\rho}{2m}) + O\left(\frac{1}{m^p}\right)$$

for any $p > 0$. In the current paper, we obtain a more precise result for (1.3).

Theorem 1.1. *Let M be a regular compact Riemann surface and K_M be the canonical line bundle endowed with a Hermitian metric h such that the curvature $Ric(h)$ of h defines a Kähler metric g on M. Suppose that this metric g has constant scalar curvature ρ. Then there is a complete asymptotic expansion:*

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1 + \frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \cdots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for some $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$, where δ is the injective radius at x_0.

Our result indicates that the asymptotic expansion (1.3) is in exponential decay. Englis [2] has an asymptotically expansion for the Berezin transformation on any planar domain of hyperbolic type. He also showed that Berezin kernel [1] has

$$\tilde{B}(\eta, \eta) = m\left(1 + O(1)\rho_0(0)^{\frac{\pi m - 3}{2}}\right),$$

where $\rho_0(0)$ is a positive constant.

2. General Set Up

Let M be an n-dimensional compact complex Kähler manifold with a polarized line bundle $(L, h) \to M$. Choose the K-coordinates $(z_1, \cdots, z_n)$ on an open neighborhood U of a fixed point $x_0 \in M$. Then the Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

satisfies

$$g_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\bar{\beta}}, \qquad \frac{\partial^{p_1+\cdots+p_n} g_{\alpha\bar{\beta}}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0, \tag{2.1}$$

for $\alpha, \beta = 1, \cdots, n$ and any nonnegative integers $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n \neq 0$.

We also choose a local holomorphic frame e_L of the line bundle L at x_0 such that a is the local representation function of the Hermitian metric h. That is,

$$Ric(h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log a.$$

Under the K-coordinate, the function a has the properties

$$a(x_0) = 1, \qquad \frac{\partial^{p_1+\cdots+p_n}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(a)(x_0) = 0 \tag{2.2}$$

for any nonnegative integers $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n \neq 0$.

Let $\{S_0, \cdots, S_{d_m-1}\}$ be a basis of $H^0(M, L^m)$. Assume that at the point $x_0 \in M$,

$$S_0(x_0) \neq 0, \qquad S_i(x_0) = 0, \qquad i = 1, \cdots, d_m - 1.$$

If the set $\{S_0, \cdots, S_{d_m-1}\}$ is not an orthonormal basis, we may do the following: Let the metric matrix

$$F_{ij} = (S_i, S_j), \qquad i, j = 0, \cdots, d_m - 1$$

with respect to the inner product (1.1). By definition, (F_{ij}) is a positive definite Hermitian matrix. We can find a $d_m \times d_m$ matrix G_{ij} such that

$$F_{ij} = \sum_{k=0}^{d_m-1} G_{ik} \overline{G_{jk}}.$$

Let (H_{ij}) be the inverse of (G_{ij}). Then $\{\sum_{j=0}^{d_m-1} H_{ij} S_j\}$ forms an orthonormal basis of $H^0(M, L^m)$. The left hand side of (1.2) is equal to

$$\sum_{i=0}^{d_m-1} \| \sum_{j=0}^{d_m-1} H_{ij} S_j(x_0) \|_{h^m}^2 = \sum_{i=0}^{d_m-1} |H_{i0}|^2 \|S_0(x_0)\|_{h^m}^2. \tag{2.3}$$

Let (I_{ij}) be the inverse matrix of (F_{ij}). Denote that

$$\sum_{i=0}^{d_m-1} |H_{i0}|^2 = I_{00}. \tag{2.4}$$

In order to compute (2.4), we need a suitable choice of the basis $\{S_0, \cdots, S_{d_m-1}\}$. We select some of Tian's peak sections in our basis. The following lemma is an improved version of Tian's result [6, Lemma 1.2], which is done by Lu and Tian.

Let $\mathbb{Z}_+^n$ be the set of n-tuple integers $P = (p_1, \cdots, p_n)$ such that each p_i is a nonnegative integer for $i = 1, \cdots, n$. For $P \in \mathbb{Z}_+^n$, we denote that $z^P = z_1^{p_1} \cdots z_n^{p_n}$ and $|P| = p_1 + \cdots p_n$.

Lemma 2.1. ([Tian]). *Suppose $Ric(g) \geq -K\omega_g$, where $K > 0$ is a constant. For $P \in \mathbb{Z}_+^n$ and an integer $p' > |P|$, let m be an integer such that*

$$m > \max\{e^{20\sqrt{n+2p'}} + 2K, e^{8(p'-1+n)}\}.$$

Then there is a holomorphic section $S_{P,m} \in H^0(M, L^m)$, satisfying

$$\left| \int_M \|S_{P,m}\|_{h_m}^2 dV_g - 1 \right| \leq Ce^{-\frac{1}{8}(\log m)^2}. \tag{2.5}$$

Moreover, $S_{P,m}$ can be decomposed as

$$S_{P,m} = \tilde{S}_{P,m} - u_{P,m}$$

such that

$$\tilde{S}_{P,m}(x) = \lambda_P \eta\left(\frac{m|z|^2}{(\log m)^2}\right) z^P e_L^m = \begin{cases} \lambda_P z^P e_L^m & x \in \{|z| \leq \dfrac{\log m}{\sqrt{2m}}\}, \\ 0 & x \in M \setminus \{|z| \leq \dfrac{\log m}{\sqrt{m}}\}, \end{cases} \tag{2.6}$$

and

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq Ce^{-\frac{1}{4}(\log m)^2}, \tag{2.7}$$

where η is a smoothly cut-off function

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{for } t \geq 1. \end{cases}$$

satisfying $0 \leq -\eta'(t) \leq 4$ and $|\eta''(t)| \leq 8$ and

$$\lambda_P^{-2} = \int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g. \tag{2.8}$$

Proof. Define the weight function

$$\Psi(z) = (n+2p')\eta\left(\frac{m|z|^2}{(\log m)^2}\right)\log\left(\frac{m|z|^2}{(\log m)^2}\right).$$

A straightforward computation gives

$$\sqrt{-1}\partial\bar{\partial}\Psi \geq -\frac{100m(n+2p')}{(\log m)^2}\omega_g. \tag{2.9}$$

By using (2.9), we can verify that

$$\langle \partial\bar{\partial}\Psi + \frac{2\pi}{\sqrt{-1}}(Ric(h^m) + Ric(g)), v \wedge \bar{v}\rangle_g \geq \frac{1}{4}m\|v\|_g^2.$$

For $P \in \mathbb{Z}_+^n$, consider the 1-form

$$w_P = \bar{\partial}\eta(\frac{m|z|^2}{(\log m)^2})z^P e_L^m.$$

Since $w_P \equiv 0$ in a neighborhood of x_0, we have

$$\int_M \|w_P\|_{h_m}^2 e^{-\Psi} dV_g < +\infty.$$

By [6, Prop. 2.1], there exists a smooth L^m-valued section u_P such that $\bar{\partial}u_P = w_P$ and

$$\int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{4}{m}\int_M \|w_P\|_{h_m}^2 e^{-\Psi} dV_g < \infty. \tag{2.10}$$

By direct computation, we get

$$\int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{C(\log m)^{2(p-1)}}{m^p}\int_{\frac{\log m}{\sqrt{2m}} \leq |z| \leq \frac{\log m}{\sqrt{m}}} a^m dV_0.$$

Under the K-coordinate, we have

$$a^m = e^{m\log a} = e^{m(-|z|^2 + O(|z|^4))}.$$

Hence we get

$$\int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{C_1(\log m)^{2(p-1+n)}}{m^{p+n}} e^{-\frac{1}{2}(\log m)^2}$$

for some constant C_1. Let $\tilde{S}_{P,m} = \lambda_P \eta(\frac{m|z|^2}{(\log m)^2})z^P e_L^m$ and $u_{P,m} = \lambda_P u_P$. Use a result in [10]

$$\lambda_P^2 \leq C_2 m^{n+|P|}$$

for some constant C_2. Then we have

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C(\log m)^{2(|P|-1+n)} e^{-\frac{1}{2}(\log m)^2}.$$

Choosing $m > e^{8(p'-1+n)}$, we obtain

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq Ce^{-\frac{1}{4}(\log m)^2}.$$ ■

3. Proof of Theorem 1.1

Proof. Let M be a smooth compact Riemann surface with a metric g that has constant scalar curvature. Let x_0 be a fixed point. Let

$$U = \{x : \operatorname{dist}(x, x_0) < \delta\},$$

where δ is the injective radius at x_0. It is well known that on a Riemann surface there is an isothermal coordinate at each point on U. We may assume that there is a holomorphic function z on U and it defines the conformal structure on U. That is,

$$ds^2 = gdzd\bar{z}$$

and $g > 0$. The metric g satisfies

$$\triangle \log g = -\rho, \quad g(x_0) = 1, \quad \text{and} \quad \frac{\partial g}{\partial z}(x_0) = 0, \tag{3.1}$$

where

$$\triangle = g^{-1}\frac{\partial^2}{\partial z \partial \bar{z}}$$

is the complex Laplace of M. Since the metric g has conformal structure, it is rotationally symmetric. We can write (3.1) in polar coordinates (r, θ):

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{1}{g}(\frac{\partial g}{\partial r})^2 = -4\rho g^2, \qquad g(0,\theta) = 1, \quad \frac{\partial g}{\partial r}(0,\theta) = 0, \tag{3.2}$$

where $z = re^{i\theta}$, and $|z|^2 = r^2$. There exists a solution

$$g = \frac{1}{(1 + \frac{\rho}{2}|z|^2)^2} \tag{3.3}$$

to (3.2) for $|z| < \sqrt{-\frac{2}{\rho}}$ if $\rho < 0$. Suppose that there exists another solution g_1 to (3.2). We have

$$\triangle \log (g_1/g) = 0 \quad \text{and} \quad g_1(x_0) = 1.$$

For $\rho < 0$, let $r_0 < \sqrt{-\frac{2}{\rho}}$. Since g and g_1 are rotationally symmetric, they remain constant on $|z| = r_0$. The harmonic function $\log(g/g_1)$ is a constant on $|z| \leq r_0$ by Maximum Principle. By definition, we have $g(x_0) = g_1(x_0) = 1$. Therefore, the solution in (3.3) is unique around x_0. By the same reason, $g = g_1$ on $\{\mathrm{dist}(x, x_0) \leq \delta_1\}$ for some $\delta_1 < \delta$ for $\rho \leq 0$.

Let a be the local representation of the metric h on K_M such that

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a = \omega_g.$$

If we normalize a and a satisfies

$$\triangle \log a = -1, \quad a(x_0) = 1, \quad \frac{\partial a}{\partial z}(x_0) = 0. \tag{3.4}$$

Since

$$-\frac{\partial^2}{\partial z\partial\bar{z}}\log a = g,$$

$\log a$ is also rotationally symmetric. Since

$$a = \begin{cases} \left(1 + \frac{\rho}{2}|z|^2\right)^{-\frac{2}{\rho}}, & \text{if } \rho \neq 0; \\ e^{-|z|^2}, & \text{if } \rho = 0. \end{cases} \tag{3.5}$$

satisfies (3.4), the local uniqueness is due to the same reason.

We need to choose sufficient large m such that $\frac{\log m}{\sqrt{m}} < \min\{\delta, \sqrt{\frac{2}{|\rho|}}\}$. With these particular solutions of g and a, we further compute

$$\begin{aligned} \lambda_0^{-2} &= \int_{|z|\leq\frac{\log m}{\sqrt{m}}} a^m g\frac{\sqrt{-1}}{2\pi}dz\wedge d\bar{z} \\ &= 2\int_o^{\frac{\log m}{\sqrt{m}}}(1+\frac{\rho}{2}r^2)^{-\frac{2m}{\rho}-2}rdr \\ &= \frac{1}{m+\frac{\rho}{2}}\left(1-\left(1+\frac{\rho}{2}\frac{(\log m)^2}{m}\right)^{-1-\frac{2m}{\rho}}\right) \qquad \text{for } \rho \neq 0. \end{aligned} \tag{3.6}$$

For $m > \max\{|\rho|^{4/3}, 10\}$, we have $\left|\frac{\rho}{2}\frac{(\log m)^2}{m}\right| < 1/2$. For $\rho \neq 0$, this gives

$$\left(1+\frac{\rho}{2}\frac{(\log m)^2}{m}\right)^{-1-\frac{2m}{\rho}} \leq 2e^{-\frac{2m}{\rho}\left(\frac{\rho}{2}\frac{(\log m)^2}{m}-\frac{1}{2}(\frac{\rho}{2}\frac{(\log m)^2}{m})^2+\cdots\right)} \leq Ce^{-(\log m)^2}.$$

For $\rho = 0$, we have

$$\lambda_0^{-2} = \int_{|z|\leq\frac{\log m}{\sqrt{m}}} e^{-m|z|^2}\frac{\sqrt{-1}}{2\pi}dz\wedge d\bar{z} = \frac{1}{m}(1+O(e^{-(\log m)^2})).$$

Hence we obtain

$$\lambda_0^{-2} = \frac{1}{m+\frac{\rho}{2}}\left(1+O\left(e^{-(\log m)^2}\right)\right). \tag{3.7}$$

From the properties of g and a, the isothermal coordinate (U, z) is a K-coordinate. According to Lemma 2.1, we may choose two peak sections

$$\begin{aligned} S_{0,m} &= \lambda_0(\eta(\frac{m|z|^2}{(\log m)^2})(dz)^m - u_0) \\ S_{1,m} &= \lambda_1(\eta(\frac{m|z|^2}{(\log m)^2})z(dz)^m - u_1) \end{aligned}$$

in $H^0(M, K_M^m)$ for some $m > e^{20\sqrt{1+4}} + 2|\rho|$. Obviously, we have $S_{0,m}(x_0) \neq 0$ and $S_{1,m}(x_0) = 0$. Let the subspace

$$V = \{S \in H^0(M, K_M^m) | S(x_0) = 0, DS(x_0) = 0\},$$

where D is a covariant derivative on K_M^m. Let $T_1, \cdots, T_{d_m-2}$ be an orthonormal basis of V. Let

$$S_i = \begin{cases} S_{i,m} & \text{if } i = 0, 1 \\ T_{i-1} & \text{if } 2 \leq i \leq d_m - 1 \end{cases}. \tag{3.8}$$

Then $\{S_i\}_{i=0}^{d_m-1}$ forms a basis for $H^0(M, K_M^m)$. Locally, each T_i has the form $f_i(dz)^m$ for some holomorphic function f_i defined in U. The holomorphic function f_i has Taylor expansion as $f_i = \sum_{\alpha=2}^{\infty} b_{i\alpha} z^\alpha$ in U, since $T_i(x_0) = 0$ and $DT_i(x_0) = 0$ for $1 \leq i \leq d_m - 2$

Lemma 3.2. *Let $\{S_i\}_{i=0}^{d_m-1}$ be the basis of $H^0(M, K_M^m)$, defined in* (3.8). *For $m > e^{20\sqrt{5}} + 2|\rho|$, the Hermitian matrix*

$$(S_i, S_j) = \int_M \langle S_i(x), S_j(x) \rangle_{h_m} dV_g$$

is given by

$$\begin{aligned} (S_0, S_0) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_0, S_1) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_1, S_1) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_0, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \end{aligned}$$

$$
\begin{aligned}
(S_1, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_i, S_j) &= \delta_{ij}
\end{aligned}
$$

for $i, j = 2, \cdots, d_m - 1$.

Proof. By definition, we have $(S_i, S_j) = \delta_{ij}$ for $2 \leq i, j \leq d_m - 1$. The inner product of (S_i, S_i) for $0 \leq i \leq 1$ is directly from Lemma 2.1. Since $a^m g$ is rotationally symmetric, we have

$$
\int_{|z| \leq \frac{\log m}{\sqrt{m}}} \bar{z}^\alpha a^m g dV_0 = 0 \qquad \text{for arbitrary positive integer } \alpha.
$$

Then we get

$$
\begin{aligned}
(S_0, S_1) &= (\tilde{S}_0, \tilde{S}_1) + (\lambda_0 u_0, \tilde{S}_1) + (\tilde{S}_0, \lambda_1 u_1) + (u_0, u_1) \\
&= O\left(e^{-\frac{(\log m)^2}{8}}\right).
\end{aligned}
$$

Consider

$$
\begin{aligned}
(S_0, S_i) &= \int_M \langle \lambda_0 (\eta(\frac{m|z|^2}{(\log m)^2})(dz)^m - u_0), f_{i-1}(dz)^m \rangle_{h_m} dV_g \\
&\leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} \bar{z}^\alpha a^m g dV_0 + \lambda_0 \|u_0\| \cdot \|S_i\|.
\end{aligned}
$$

Thus we have

$$
(S_0, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right) \qquad \text{for } 2 \leq i \leq d_m - 1.
$$

Similarly, consider

$$
(S_1, S_j) \leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} z \bar{z}^\alpha a^m g dV_0 + \lambda_1 \|u_1\| \cdot \|S_i\| \qquad \text{for } 2 \leq i \leq d_m - 1.
$$

Since $a^m g$ is rotationally symmetric, $\int_{|z| \leq \frac{\log m}{\sqrt{m}}} z \bar{z}^\alpha a^m g dV_0 = 0$ for $\alpha \geq 2$. Hence we obtain

$$
(S_1, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right). \qquad \blacksquare
$$

According to [10, Definition 3.1], the metric matrix (F_{ij}) can be represented by the block matrices

$$(F_{ij}) = \begin{pmatrix} (S_0, S_0) & (S_0, S_1) & M_{13} \\ (S_1, S_0) & (S_1, S_1) & M_{23} \\ M_{31} & M_{32} & E \end{pmatrix}, \tag{3.9}$$

where $M_{13} = ((S_0, S_2), \cdots, (S_0, S_{d_m-1}))$, $M_{23} = ((S_1, S_2), \cdots, (S_1, S_{d_m-1}))$, $M_{31} = \overline{M_{13}^T}$, $M_{32} = \overline{M_{23}^T}$, and E is a $(d_m - 2) \times (d_m - 2)$ identity matrix. By using [10, Lemma 3.1], we obtain

$$I_{00} = \frac{1}{(S_0, S_0)} + (\frac{1}{(S_0, S_0)})^2 \begin{pmatrix} (S_0, S_1) & M_{13} \end{pmatrix} \tilde{M}^{-1} \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix}, \tag{3.10}$$

where

$$\tilde{M} = \begin{pmatrix} (S_1, S_1) & M_{23} \\ M_{32} & E \end{pmatrix} - \frac{1}{(S_0, S_0)} \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix} \begin{pmatrix} (S_0, S_1) & M_{13} \end{pmatrix}.$$

Applying Lemma 3.2 in (3.10), we get

$$I_{00} = 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right). \tag{3.11}$$

In order to evaluate the expansion of (2.3), we are left to find $\|S_0(x_0)\|^2_{h_m} = \lambda_0^2$. From (3.7), we have

$$\lambda_0^2 = m(1 + \frac{\rho}{2m}) \left(1 + O\big(e^{-(\log m)^2}\big)\right).$$

Therefore, the Tian-Yau-Zelditch expansion according to (2.3) on a Riemann surface with constant scalar curvature ρ is

$$\begin{aligned} I_{00}\lambda_0^2 &= (1 + O\left(e^{-\frac{(\log m)^2}{8}}\right))m(1 + \frac{\rho}{2m}) \left(1 + O\big(e^{-(\log m)^2}\big)\right) \\ &= m(1 + \frac{\rho}{2m}) + O\left(e^{-(\frac{(\log m)^2}{8})}\right) \end{aligned}$$

for $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$. ∎

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Chiung-ju Liu
Taida Institute for Mathematical Sciences,
National Taiwan University,
Taipei 106, Taiwan
E-mail: cjliu4@ntu.edu.tw