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A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION INVOLVING THE GAMMA FUNCTION

Feng Qi and Bai-Ni Guo

Abstract. In this paper, sufficient conditions are found for a function involving the gamma function and its reciprocal to be logarithmically completely monotonic. Consequently, a decreasing monotonicity of the function is generalized and a known inequality is extended.

1. Introduction

A function f is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty \tag{1}$$

for $k \in \mathbb{N}$ on I. This terminology was first proposed in [2], but it seems to have been ignored until 2004 by the mathematical community. In early 2004, this notion was recovered in [16], the original version of the paper [14]. It was pointed out in [4] that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterized as the infinitely divisible completely monotonic functions studied in [8]. Furthermore, it was discovered in [4] that every Stieltjes transform is a logarithmically completely monotonic function on $(0, \infty)$, where a function f defined on $(0, \infty)$ is called a Stieltjes transform if it can be of the form

$$f(x) = a + \int_0^\infty \frac{1}{s+x} d\,\mu(s) \tag{2}$$

for some nonnegative number a and some nonnegative measure μ on $[0, \infty)$ satisfying $\int_0^\infty \frac{1}{1+s} d\,\mu(s) < \infty$. This demonstrates that the investigation of the logarithmically completely monotonic property of functions are naturally significant and meaningful.

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It is well-known that Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} d\,t. \tag{3}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

In [6, Theorem 2] and its preprint [20], the following decreasingly monotonic property was established: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{4}$$

is decreasing in $x \ge 1$ for fixed $y \ge 0$. Consequently, for positive real numbers $x \ge 1$ and $y \ge 0$, we have

$$\frac{x+y+1}{x+y+2} \le \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{5}$$

For more information on the history, background, motivation and generalizations of the function (4), please refer to [1, 3, 6, 7, 9, 10, 11, 17, 18, 19, 20, 21, 22] and a lot of related references therein.

The aim of this paper is to extend and generalize the above monotonicity result. Our main results can be stated as follows.

Theorem 1. *The function* (4) *is logarithmically completely monotonic with respect to* $x \in (0,\infty)$ *if* $y \ge 0$*, so is its reciprocal if* $-1 < y \le -\frac{1}{2}$*. Consequently, the inequality* (5) *is valid for* $(x,y) \in (0,\infty) \times [0,\infty)$ *and reversed for* $(x,y) \in (0,\infty) \times \left(-1, -\frac{1}{2}\right]$.

2. Proof of Theorem 1

For all $(x,y) \in (0,\infty) \times (-1,\infty)$, let

$$h(x,y) = \frac{\ln\Gamma(x+y+1) - \ln\Gamma(y+1)}{x} - \ln(x+y+1), \tag{6}$$

which is the logarithm of the function (4) clearly. Direct computation yields

$$\begin{aligned}\frac{\partial^k h(x,y)}{\partial x^k} &= \frac{k!}{x^{k+1}} \sum_{i=0}^{k} \frac{(-1)^{k-i} x^i \psi^{(i-1)}(x+y+1)}{i!} \\ &\quad - \frac{(-1)^k k! \ln\Gamma(y+1)}{x^{k+1}} - \frac{(-1)^{k-1}(k-1)!}{(x+y+1)^k}\end{aligned} \tag{7}$$

for $k \in \mathbb{N}$, where $\psi^{(-1)}(x+y+1)$ and $\psi^{(0)}(x+y+1)$ stand for $\ln\Gamma(x+y+1)$ and $\psi(x+y+1)$ respectively. Furthermore, a simple calculation gives

$$\frac{\partial}{\partial x}\left[x^{k+1}\frac{\partial^k h(x,y)}{\partial x^k}\right] = (-1)^{k-1}x^k\biggl[(-1)^{k-1}\psi^{(k)}(x+y+1) - \frac{(k-1)!}{(x+y+1)^k} - \frac{k!(y+1)}{(x+y+1)^{k+1}}\biggr]. \tag{8}$$

In [12, Lemma 1.3] and [13, Lemma 3], the function $\psi(x) - \ln x + \frac{\alpha}{x}$ was proved to be completely monotonic on $(0,\infty)$, i.e.,

$$(-1)^i\left[\psi(x) - \ln x + \frac{\alpha}{x}\right]^{(i)} \ge 0 \tag{9}$$

for $i \ge 0$, if and only if $\alpha \ge 1$, so is its negative, i.e., the inequality (9) is reversed, if and only if $\alpha \le \frac{1}{2}$. In [5], the function $\frac{e^x\Gamma(x)}{x^{x-\alpha}}$ was proved to be logarithmically completely monotonic on $(0,\infty)$, i.e.,

$$(-1)^k\left[\ln\frac{e^x\Gamma(x)}{x^{x-\alpha}}\right]^{(k)} \ge 0 \tag{10}$$

for $k \in \mathbb{N}$, if and only if $\alpha \ge 1$, so is its reciprocal, i.e., the inequality (10) is reversed, if and only if $\alpha \le \frac{1}{2}$. As straightforward consequences of any one of these two conclusions (9) and (10), the following double inequalities are derived readily:

$$\ln x - \frac{1}{x} \le \psi(x) \le \ln x - \frac{1}{2x}$$

and

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \le (-1)^{k+1}\psi^{(k)}(x) \le \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{11}$$

hold on $(0,\infty)$ for $k \in \mathbb{N}$. See also [15, Lemma 3]. Utilization of (11) in (8) leads to

$$-\frac{k!(y+1/2)}{(x+y+1)^{k+1}} \le \frac{(-1)^{k-1}}{x^k}\frac{\partial}{\partial x}\left[x^{k+1}\frac{\partial^k h(x,y)}{\partial x^k}\right] \le -\frac{k!y}{(x+y+1)^{k+1}}$$

for $k \in \mathbb{N}$ and $(x,y) \in (0,\infty)\times(-1,\infty)$. Therefore,

$$\frac{(-1)^{k-1}}{x^k}\frac{\partial}{\partial x}\left[x^{k+1}\frac{\partial^k h(x,y)}{\partial x^k}\right]\begin{cases}\le 0, & y \ge 0;\\ \ge 0, & -1 < y \le -\dfrac{1}{2}.\end{cases}$$

This means that

$$\frac{\partial}{\partial x}\left[x^{2k}\frac{\partial^{2k-1}h(x,y)}{\partial x^{2k-1}}\right]\begin{cases}\le 0, & y\ge 0\\ \ge 0, & -1<y\le -\frac12\end{cases}$$

and

$$\frac{\partial}{\partial x}\left[x^{2k+1}\frac{\partial^{2k}h(x,y)}{\partial x^{2k}}\right]\begin{cases}\ge 0, & y\ge 0\\ \le 0, & -1<y\le -\frac12\end{cases}$$

for $k\in\mathbb{N}$ and $x\in(0,\infty)$. In other words, the functions

$$x^{2k}\frac{\partial^{2k-1}h(x,y)}{\partial x^{2k-1}}\quad\text{and}\quad -x^{2k+1}\frac{\partial^{2k}h(x,y)}{\partial x^{2k}}$$

are decreasing if $y\ge 0$ or increasing if $-1<y\le -\frac12$ with respect to $x\in(0,\infty)$. From (7), it is easy to see that

$$\lim_{x\to0^+}\left[x^{k+1}\frac{\partial^k h(x,y)}{\partial x^k}\right]=0$$

for $k\in\mathbb{N}$ and any given $y>-1$. Since $x^{k+1}\frac{\partial^k h(x,y)}{\partial x^k}$ is not constant for x near 0, we must have

$$x^{2k}\frac{\partial^{2k-1}h(x,y)}{\partial x^{2k-1}}\begin{cases}<0, & y\ge 0\\ >0, & -1<y\le -\frac12\end{cases}$$

and

$$-x^{2k+1}\frac{\partial^{2k}h(x,y)}{\partial x^{2k}}\begin{cases}<0, & y\ge 0\\ >0, & -1<y\le -\frac12\end{cases}$$

for $k\in\mathbb{N}$ and $x\in(0,\infty)$, which are equivalent to

$$\frac{\partial^{2k-1}h(x,y)}{\partial x^{2k-1}}\begin{cases}<0, & y\ge 0\\ >0, & -1<y\le -\frac12\end{cases}$$

and

$$\frac{\partial^{2k}h(x,y)}{\partial x^{2k}}\begin{cases}>0, & y\ge 0\\ <0, & -1<y\le -\frac12\end{cases}$$

for $k\in\mathbb{N}$ and $x\in(0,\infty)$. In conclusion,

$$(-1)^k\frac{\partial^k h(x,y)}{\partial x^k}\begin{cases}>0, & y\ge 0\\ <0, & -1<y\le -\frac12\end{cases}$$

for $k \in \mathbb{N}$ and $x \in (0, \infty)$. Hence, the function (4) is logarithmically completely monotonic with respect to x on $(0, \infty)$ if $y \geq 0$, so is the reciprocal of the function (4) if $-1 < y \leq -\frac{1}{2}$. The proof of Theorem 1 is complete.

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Feng Qi
College of Mathematics and Information Science,
Henan Normal University,
Xinxiang City, Henan Province 453007,
P. R. China
E-mail: qifeng618@gmail.com
qifeng618@hotmail.com
qifeng618@qq.com

Bai-Ni Guo
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City, Henan Province 454010,
P. R. China
E-mail: bai.ni.guo@gmail.com