TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 2, pp. 453-462, April 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

JENSEN'S FUNCTIONAL EQUATION IN MULTI-NORMED SPACES

M. S. Moslehian and H. M. Srivastava

Abstract. We investigate the Hyers-Ulam stability of the Jensen functional equation for mappings from linear spaces into multi-normed spaces. We then establish an asymptotic behavior of the Jensen equation in the framework of multi-normed spaces which are somewhat similar to the operator sequence spaces and have some connections with operator spaces and Banach lattices.

1. Introduction and Motivation

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940 Ulam [19] posed the first stability problem. In the following year, Hyers [7] gave a partial affirmative answer to the question of Ulam. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. The paper [15] of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. During the past decades, several stability problems for various functional equations have been investigated by a number of mathematicians; we refer the reader to [3, 8, 10, 16, 17] and also to the references cited therein.

The first result on the stability of the following classical Jensen functional equation:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

was given by Kominek [11]. The author who presumably investigated the stability problem on a restricted domain for the first time was Skof [18]. The stability of

Received June 10, 2008, accepted June 11, 2008.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: Primary 39B82; Secondary 39B52, 46B99, 47A99.
Key words and phrases: Hyers-Ulam stability, Multi-normed space, Jensen's functional equation, Asymptotic behavior.

the Jensen equation and of its generalizations were studied by numerous researchers (cf., e.g., [2, 4, 12, 13, 14]).

In this paper, using some ideas from the earlier works [5, 9], we investigate the Hyers-Ulam stability of the Jensen functional equation for mappings from linear spaces into multi-normed spaces. We then establish an asymptotic behavior of the Jensen equation in the framework of multi-normed spaces. Our results generalize those of Jung [9]. The theory of multi-normed spaces as well as the theory of multi-Banach algebras were originated in [6].

Let $(E, \|\cdot\|)$ be a complex linear space. Also let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \cdots, x_k)$, where $x_1, \cdots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, 3, \cdots, k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols.

2. Multi-normed Spaces and Multi-bounded Operators

We start this section by recalling the notion of a multi-normed space from [6]. Throughout this section, $(E, \|\cdot\|)$ denotes a complex normed space.

Definition 1. A *multi-norm* on $\{E^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geqq 2$:

(M1) $\left\|(x_{\sigma(1)}, \cdots, x_{\sigma(k)})\right\|_k = \|(x_1, \cdots, x_k)\|_k \quad (\sigma \in \mathfrak{S}_k;\ x_1, \cdots, x_k \in E)\,;$

(M2) $\|(\alpha_1 x_1, \cdots, \alpha_n x_k)\|_k \leqq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\, \|(x_1, \cdots, x_k)\|_k$

$(\alpha_1, \cdots, \alpha_k \in \mathbb{C};\ x_1, \cdots, x_k \in E)\,;$

(M3) $\|(x_1, \cdots, x_{k-1}, 0)\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in E)\,;$

(M4) $\|(x_1, \cdots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in E)\,.$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-normed space*.

The motivation for the study of multi-normed spaces (and multi-normed algebras) and many examples are detailed in the earlier investigation [6].

Suppose that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. The following properties are almost immediate consequences of the axioms.

(a) $\|(x, \cdots, x)\|_k = \|x\| \quad (x \in E)$;

(b) $\max_{i\in\mathbb{N}_k}\|x_i\|\leqq\|(x_1,\cdots,x_k)\|_k\leqq\sum_{i=1}^{k}\|x_i\|\leqq k\max_{i\in\mathbb{N}_k}\|x_i\|$ $(x_1,\cdots,x_k\in E)$.

It follows from the item (b) above that, if $(E,\|\cdot\|)$ is a Banach space, then $(E^k,\|\cdot\|_k)$ is a Banach space for each $k\in\mathbb{N}$; in this case, $((E^k,\|\cdot\|_k):k\in\mathbb{N})$ is a *multi-Banach space*.

Now we recall two important examples of multi-norms for an arbitrary normed space E (see, for details, [6]).

Example 1. The sequence $(\|\cdot\|_k : k\in\mathbb{N})$ on $\{E^k : k\in\mathbb{N}\}$ defined by

$$\|(x_1,\cdots,x_k)\|_k := \max_{i\in\mathbb{N}_k}\|x_i\| \quad (x_1,\cdots,x_k\in E)$$

is a multi-norm called the *minimum multi-norm*. The terminology *minimum* is justified here by Property (c). ■

Example 2. Let

$$\{(\|\cdot\|_k^{\alpha} : k\in\mathbb{N}) \quad \text{and} \quad \alpha\in A\}$$

be the (non-empty) family of all multi-norms on $\{E^k : k\in\mathbb{N}\}$. For $k\in\mathbb{N}$, we set

$$|||(x_1,\cdots,x_k)|||_k := \sup_{\alpha\in A}\|(x_1,\cdots,x_k)\|_k^{\alpha} \quad (x_1,\cdots,x_k\in E).$$

Then $(|||\cdot|||_k : k\in\mathbb{N})$ is a multi-norm on $\{E^k : k\in\mathbb{N}\}$, which is called the *maximum multi-norm*. ■

We need the following observation which can be easily deduced from the triangle inequality for the norm $\|\cdot\|_k$ and the property (b) of multi-norms.

Lemma. *Suppose that* $k\in\mathbb{N}$ *and* $(x_1,\cdots,x_k)\in E^k$.
For each $j\in\{1,\cdots,k\}$, *let* $\{x_n^j\}_{n\in\mathbb{N}}$ *be a sequence in* E *such that*

$$\lim_{n\to\infty} x_n^j = x^j.$$

Then, for each $(y_1,\cdots,y_k)\in E_k$,

$$\lim_{n\to\infty}(x_n^1-y_1,\cdots,x_n^k-y_k)=(x_1-y_1,\cdots,x_k-y_k).$$

Definition 2. Let $((E^k,\|\cdot\|_k):k\in\mathbb{N})$ be a multi-normed space. A sequence (x_n) in E is a *multi-null sequence* if, for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\|(x_n,\cdots,x_{n+k-1})\|_k<\varepsilon \quad (n\geqq n_0).$$

Let $x\in E$. Then

$$\lim_{n\to\infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence; in this case, the sequence (x_n) is *multi-convergent* to x in E.

3. Hyers-Ulam Stability of the Jensen Equation

Theorem 1. *Let E be a linear space. Also let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that α is a nonnegative real number and $f : E \to F$ is a mapping satisfying $f(0) = 0$ and*

$$\sup_{k\in\mathbb{N}} \left\| \left(f\left(\frac{x_1+y_1}{2}\right) - \frac{f(x_1)+f(y_1)}{2}, \cdots, f\left(\frac{x_k+y_k}{2}\right) - \frac{f(x_k)+f(y_k)}{2} \right) \right\|_k \leqq \alpha \tag{3.1}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in E$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - T(x_1), \cdots, f(x_k) - T(x_k))\|_k \leqq 2\alpha \tag{3.2}$$

for all $x_1, \cdots, x_k \in E$.

Proof. Let $x_1, \cdots, x_k \in E$. Replacing $x_1, \cdots, x_k$ and $y_1, \cdots, y_k$ by $2x_1, \cdots, 2x_k$ and $0, \cdots, 0$ in (3.1) and multiplying the resulting inequality by 2, we obtain

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 2f(x_1), \cdots, f(2x_k) - 2f(x_k))\|_k \leqq 2\alpha\,. \tag{3.3}$$

By using (3.3) and the principle of mathematical induction, we can easily see that

$$\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x_1)}{2^n} - \frac{f(2^m x_1)}{2^m}, \cdots, \frac{f(2^n x_k)}{2^n} - \frac{f(2^m x_k)}{2^m} \right) \right\|_k \leqq 2\alpha \sum_{i=m}^{n-1} 2^{-i}\,. \tag{3.4}$$

We now fix $x \in E$. We thus find that

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \cdots, \frac{f(2^{n+k-1} x)}{2^{n+k-1}} - \frac{f(2^{m+k-1} x)}{2^{m+k-1}} \right) \right\|_k \\
&\leqq \sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \cdots, \frac{1}{2^{k-1}} \left(\frac{f(2^n (2^{k-1} x))}{2^n} - \frac{f(2^m (2^{k-1} x))}{2^m} \right) \right) \right\|_k \\
&\leqq \sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \cdots, \frac{f(2^n (2^{k-1} x))}{2^n} - \frac{f(2^m (2^{k-1} x))}{2^m} \right) \right\|_k \\
&\leqq 2\alpha \sum_{i=m}^{n-1} 2^{-i}\,,
\end{aligned}$$

where we have used the axiom (M3) of Definition 1 and also replaced $x_1, x_2, \cdots, x_n$ by $x, 2x, \cdots, 2^{k-1}x$ in (3.4). Hence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence and so it is convergent in the multi-complete space F. Set

$$T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

Hence, for each $\varepsilon > 0$, there exists n_0 such that

$$\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x)}{2^n} - T(x), \cdots, \frac{f(2^{n+k-1}x)}{2^{n+k-1}} - T(x) \right) \right\|_k < \varepsilon$$

for all $n \geqq n_0$. In particular, by Property (b) of multi-norms, we have

$$\lim_{n\to\infty} \left\| \frac{f(2^n x)}{2^n} - T(x) \right\| = 0 \qquad (x \in E). \tag{3.5}$$

We next put $m = 0$ in (3.4) to get

$$\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(2^n x_1)}{2^n} - f(x_1), \cdots, \frac{f(2^n x_k)}{2^n} - f(x_k) \right) \right\|_k \leqq 2\alpha \sum_{i=0}^{n-1} 2^{-i}.$$

Letting $n \to \infty$ and utilizing the Lemma as well as (3.5), we obtain

$$\sup_{k\in\mathbb{N}} \|(T(x_1) - f(x_1), \cdots, T(x_k) - f(x_k))\|_k \leqq 2\alpha.$$

Let $x, y \in E$. Put

$$x_1 = \cdots = x_k = 2^n x \quad \text{and} \quad y_1 = \cdots = y_k = 2^n y$$

in (3.1) and divide both sides by 2^n. We thus obtain

$$\left\| 2^{-n} f\left(2^n \frac{(x+y)}{2}\right) - \frac{2^{-n}f(2^n x) + 2^{-n}f(2^n y)}{2} \right\| \leqq 2^{-n}\alpha,$$

which, upon taking the limit as $n \to \infty$, yields

$$T\left(\frac{x+y}{2}\right) - \frac{T(x) + T(y)}{2} = 0.$$

Hence T is Jensen and, using the fact that $T(0) = 0$, we conclude that T is also additive.

If T' is another additive mapping satisfying (3.2), then

$$\begin{aligned} \|T'(x) - T(x)\| &\leqq \frac{1}{2^n}\|T'(2^n x) - T(2^n x)\| \\ &\leqq \frac{1}{2^n}\|T'(2^n x) - f(2^n x)\| + \frac{1}{2^n}\|f(2^n x) - T(2^n x)\| \\ &\leqq \frac{1}{2^n}(2\alpha + 2\alpha), \end{aligned}$$

where we have combined (3.2) and Property (a) of multi-norms. Hence $T' = T$. This proves the uniqueness asserted by Theorem 1. This evidently completes the proof of Theorem 1. ■

Applying the method of proof of Theorem 3 of [9] *mutatis mutandis*, we get the following result.

Proposition. *Let E be a linear space, and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $\alpha, \beta \geqq 0$ and that $f : E \to F$ is a mapping satisfying $f(0) = 0$ and*

$$\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2}, \cdots, f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k \leqq \alpha \tag{3.6}$$

for all k and for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in E$ with

$$\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \geqq \beta.$$

Then there exists a unique additive mapping $T : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - T(x_1), \cdots, f(x_k) - T(x_k))\|_k \leqq 5\alpha$$

for all $x_1, \cdots, x_k \in E$.

Proof. Let us fix $k \in \mathbb{N}$ and set

$$\mathbf{x} = (x_1, \cdots, x_k) \quad \text{and} \quad \mathbf{y} = (y_1, \cdots, y_k).$$

Assume that

$$\|\mathbf{x}\|_k + \|\mathbf{y}\|_k < \beta.$$

Suppose that, for $\mathbf{x} = \mathbf{y} = 0$, $\mathbf{z} = (z_1, \cdots, z_k) \in E^k$ is an element of E with

$$\|\mathbf{z}\|_k = \beta.$$

Furthermore, for $\mathbf{x} \neq 0$ or $\mathbf{y} \neq 0$, let

$$\mathbf{z} := \begin{cases} \mathbf{x} + \dfrac{\beta\mathbf{x}}{\|\mathbf{x}\|_k} & (\|\mathbf{x}\|_k \geqq \|\mathbf{y}\|_k) \\[2ex] \mathbf{y} + \dfrac{\beta\mathbf{y}}{\|\mathbf{y}\|_k} & (\|\mathbf{x}\|_k < \|\mathbf{y}\|_k). \end{cases}$$

Then

$$
\begin{aligned}
\|\mathbf{x}\text{-}\mathbf{z}\|_k + \|\mathbf{y}\text{+}\mathbf{z}\|_k &\geqq \beta \\
\|2\mathbf{z}\|_k + \|\mathbf{x}\text{-}\mathbf{z}\|_k &\geqq \beta \\
\|\mathbf{y}\|_k + \|2\mathbf{z}\|_k &\geqq \beta \qquad (3.7)\\
\|\mathbf{y}\text{+}\mathbf{z}\|_k + \|\mathbf{z}\|_k &\geqq \beta \\
\|\mathbf{x}\|_k + \|\mathbf{z}\|_k &\geqq \beta\,.
\end{aligned}
$$

It follows from (3.6) and (3.7) that

$$
\begin{aligned}
&\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k\\
\leqq&\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1-z_1)+f(y_1+z_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k-z_k)+f(y_k+z_k)}{2}\right)\right\|_k\\
+&\left\|\left(f\left(\frac{x_1+z_1}{2}\right)-\frac{f(2z_1)+f(x_1-z_1)}{2},\cdots,f\left(\frac{x_k+z_k}{2}\right)-\frac{f(2z_k)+f(x_k-z_k)}{2}\right)\right\|_k\\
+&\left\|\left(f\left(\frac{y_1+2z_1}{2}\right)-\frac{f(y_1)+f(2z_1)}{2},\cdots,f\left(\frac{y_k+2z_k}{2}\right)-\frac{f(y_k)+f(2z_k)}{2}\right)\right\|_k\\
+&\left\|\left(f\left(\frac{y_1+2z_1}{2}\right)-\frac{f(y_1+z_1)+f(z_1)}{2},\cdots,f\left(\frac{y_k+2z_k}{2}\right)-\frac{f(y_k+z_k)+f(z_k)}{2}\right)\right\|_k\\
+&\left\|\left(f\left(\frac{x_1+z_1}{2}\right)-\frac{f(x_1)+f(z_1)}{2},\cdots,f\left(\frac{x_k+z_k}{2}\right)-\frac{f(x_k)+f(z_k)}{2}\right)\right\|_k.
\end{aligned}
$$

We thus obtain

$$\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k \leqq 5\alpha$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in E$. Now the result asserted by the above Proposition can be deduced fairly easily from Theorem 1. ∎

If

$$D=\{(\mathbf{x},\mathbf{y})\in E^k\times E^k : \|\mathbf{x}\|_k<\beta, \|\mathbf{y}\|_k<\beta\}$$

for some $\beta>0$, then

$$\{(\mathbf{x},\mathbf{y})\in E^k\times E^k : \|\mathbf{x}\|_k+\|\mathbf{y}\|_k \geqq 2\beta\}\subseteq (E^k\times E^k)\backslash D.$$

Hence we have the result asserted by the Corollary below.

Corollary. *Let E be a linear space. Also let $((F^n,\|\cdot\|_n):n\in\mathbb{N})$ be a multi-Banach space. Suppose that $\alpha,\beta\geqq 0$ and that $f:E\to F$ is a mapping satisfying $f(0)=0$ and*

$$\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k \leqq \alpha$$

for all k and for all $(x_1, \cdots, x_k), (y_1, \cdots, y_k) \in (E^k \times E^k) \setminus D$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - T(x_1), \cdots, f(x_k) - T(x_k))\|_k \leqq 5\alpha\,,$$

for all $x_1, \cdots, x_k \in E$.

Theorem 2. *Let E be a linear space. Also let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : E \to F$ is a mapping satisfying $f(0) = 0$. Then f is additive if and only if*

$$\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k\to 0 \tag{3.8}$$

uniformly on $k \in \mathbb{N}$ as

$$\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \to \infty.$$

Proof. If f is additive, then (3.8) evidently holds true. Conversely, we use the uniform limit (3.8) to get a sequence $\{\beta_n\}$ such that, for each k,

$$\left\|\left(f\left(\frac{x_1+y_1}{2}\right)-\frac{f(x_1)+f(y_1)}{2},\cdots,f\left(\frac{x_k+y_k}{2}\right)-\frac{f(x_k)+f(y_k)}{2}\right)\right\|_k\leqq\frac{1}{n} \tag{3.9}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in E$ with

$$\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \geqq \beta_n\,.$$

Next, by using the above Proposition, we see that there exists a unique additive mapping T_n such that

$$\|f(x) - T_n(x)\| \leqq \frac{5}{n} \tag{3.10}$$

for all $x \in E$, so that

$$\|f(x) - T_1(x)\| \leqq 5$$

and

$$\|f(x) - T_n(x)\| \leqq 5$$

for each n. By the uniqueness of T_1, we conclude that $T_n = T_1$ for all n. Now, letting $n \to \infty$ in (3.10), we deduce that $f = T_1$ is additive. ■

ACKNOWLEDGMENTS

The work of the second-named author was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

REFERENCES

1. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950) 64-66.
2. D.-H. Boo, S.-Q. Oh, C.-G. Park and J.-M. Park, Generalized Jensen's equations in Banach modules over a C^*-algebra and its unitary group, *Taiwanese J. Math.*, **7** (2003), 641-655.
3. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2002.
4. V. Faziev and P. K. Sahoo, On the stability of Jensen's functional equation on groups, *Proc. Indian Acad. Sci. Math. Sci.*, **117** (2007), 31-48.
5. H. G. Dales and M. S. Moslehian, Stability of mappings on multi-normed spaces, *Glasgow Math. J.*, **49** (2007), 321-332.
6. H. G. Dales and M. E. Polyakov, *Multi-normed spaces and multi-Banach algebras*, Preprint 2008.
7. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.*, **27** (1941), 222-224.
8. D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
9. S.-M. Jung, Hyers-Ulam-Rassias stability of Jensen's equation and its application, *Proc. Amer. Math. Soc.*, **126** (1998), 3137-3143.
10. S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Florida, 2001.
11. Z. Kominek, On a local stability of the Jensen functional equation, *Demonstratio Math.*, **22** (1989), 499-507.
12. L. Li, J. Chung and D. Kim, Stability of Jensen equations in the space of generalized functions, *J. Math. Anal. Appl.*, **299** (2004), 578-586.
13. Y.-H. Lee and K.-W. Jun, A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation, *J. Math. Anal. Appl.*, **238** (1999), 305-315.
14. M. S. Moslehian and L. Székelyhidi, Stability of ternary homomorphisms via generalized Jensen equation, *Results Math.*, **49** (2006), 289-300.
15. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

16. Th. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23-130.

17. Th. M. Rassias (Editor), *Functional Equations*, *Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

18. F. Skof, Sulle approssimazione delle applicazioni localmente δ-additive, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **117** (1983), 377-389.

19. S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science Editions, John Wiley and Sons, New York, 1964.

M. S. Moslehian
Department of Mathematics,
Ferdowsi University of Mashhad,
and
Centre of Excellence in Analysis on Algebraic Structures (CEAAS),
P. O. Box 1159,
Mashhad 91775-1159, Iran
and
Banach Mathematical Research Group (BMRG),
Mashhad 91775-1159, Iran
E-mail: moslehian@ferdowsi.um.ac.ir
moslehian@ams.org

H. M. Srivastava
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia V8W 3R4,
Canada
E-mail: harimsri@math.uvic.ca