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SOME WEIGHTED OPIAL-TYPE INEQUALITIES ON TIME SCALES

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Abstract. Motivated essentially by several recent investigations which claimed to have generalized, improved and extended such classical inequalities as the well-known Opial's inequality, here we establish some general weighted Opial-type inequalities on time scales. We also provide counterexamples, corrections and modifications of the aforementioned recent claims by Wong *et al.* [*Taiwanese J. Math.*, **12** (2008), 463-471].

1. Introduction

Almost five decades ago, Opial [12] established an integral inequality which we recall here as Theorem A below (see also a sequel by Olech [11] for a *simpler* proof under *weaker* conditions as well as for the explicit *extremal function*).

Theorem A. *Let* $f \in C^1\,[0,a]\ \ (a>0)$ *with*

$$f(0)=f(a)=0 \qquad \textit{and} \qquad f(x)>0 \qquad (0<x<a)\,.$$

Then

$$\int_0^a \left|f(x)f'(x)\right| dx \leqq \frac{a}{4}\int_0^a \left|f'(x)\right|^2 dx, \tag{1}$$

where the constant factor $\frac{a}{4}$ *is the best possible. Equality holds true in* (1) *if and only if*

$$f(x)=\begin{cases} cx & \left(0 \leqq x \leqq \dfrac{a}{2}\right) \\ c(a-x) & \left(\dfrac{a}{2} \leqq x \leqq a\right), \end{cases}$$

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where c *is a positive constant.*

The inequality (1) is well-known in the literature as Opial's inequality. For some recent results which generalize, improve and extend this classical inequality (1), see (for example) [1-5, 8-10, 13, 14, 17] and [18] (see also the edited volume [15]). In particular, Yang [18] established the following Opial-type inequalities.

Theorem B. *Let the function* $f(x)$ *be absolutely continuous on* $[0, a]\,(a > 0)$ *with* $f(0) = 0$. *Then*, *for*

$$\ell, m \in \mathbb{R} := (-\infty, \infty),$$

each of the following inequalities holds true under the additional conditions stated with it:

$$\int_0^a |f(x)|^\ell \cdot \left|f'(x)\right|^m \, dx \leqq \left(\frac{m}{\ell + m}\right) a^\ell \int_0^a \left|f'(x)\right|^{\ell+m} dx \tag{2}$$

$$\bigl(f(0) = 0; \;\; \ell \geqq 0; \;\; m \geqq 1\bigr),$$

$$\int_0^a |f(x)|^\ell \cdot \left|f'(x)\right|^m \, dx \leqq \left(\frac{m}{\ell + m}\right) a^\ell \int_0^a \left|f'(x)\right|^{\ell+m} dx \tag{3}$$

$$\bigl(f(a) = 0; \;\; \ell \geqq 0; \;\; m \geqq 1\bigr),$$

and

$$\int_0^a |f(x)|^\ell \cdot \left|f'(x)\right|^m \, dx \leqq \left(\frac{m}{\ell + m}\right) \left(\frac{a}{2}\right)^\ell \int_0^a \left|f'(x)\right|^{\ell+m} dx \tag{4}$$

$$\bigl(f(0) = f(a) = 0; \;\; \ell \geqq 0; \;\; m \geqq 1\bigr).$$

By applying the *Time Scales Theory* and the concept of *Delta Differentiability* (see Section 2 below for the details of the definitions and notations used here), Agarwal *et al.* [2] extended the Opial's inequality (1) to the following form.

Theorem C. *Let the function* $f(t)$ *given by*

$$f : [0, a] \cap \mathbb{T} \to \mathbb{R} \qquad (a > 0)$$

be delta differentiable on $[0, a]$. *Then*

$$\int_0^a \left|[f(t) + f^\sigma(t)]\, f^\Delta(t)\right| \Delta t \leqq a \int_0^a \left|f^\Delta(t)\right|^2 \, \Delta t. \tag{5}$$

Equality holds true in (5) *when* $f(t) = ct$ *for a constant* c.

In a more recent investigation, Wong *et al.* [17] presented several generalizations and variants of the inequality (5) for certain general cases involving time scales as asserted by Theorems D, E, F and G below.

Theorem D. *Let the function* $f(t)$ *given by*

$$f : [a, b] \cap \mathbb{T} \to \mathbb{R} \qquad (b > a \geqq 0)$$

be delta differentiable on $[a, b] \cap \mathbb{T}$. *Suppose also that*

$$p \geqq 0, \quad q \geqq 1 \qquad \textit{and} \qquad h(t) \in C_{\mathrm{rd}}\left([a, b], [1, \infty)\right),$$

where $C_{\mathrm{rd}}([a, b], [1, \infty))$ *denotes the set of* rd*-continuous functions defined by*

$$C_{\mathrm{rd}}([a, b], [1, \infty)) := \{\mathfrak{f} \mid \mathfrak{f} : [a, b] \to [1, \infty) \quad \textit{and}$$
$$\mathfrak{f}(t) \ \textit{is an}\ \mathrm{rd}\textit{-continuous function}\}. \tag{6}$$

Then

$$\int_a^b h(t)\,|f(t)|^p \cdot \left|f^{\Delta}(t)\right|^q \Delta t$$
$$\leqq \left(\frac{q}{p+q}\right)(b-a)^p \int_a^b h(t)\left|f^{\Delta}(t)\right|^{p+q} \Delta t. \tag{7}$$

Theorem E. *Let the function* $f(t)$ *given by*

$$f : [a, b] \cap \mathbb{T} \to \mathbb{R} \qquad (b > a \geqq 0)$$

be delta differentiable n *times* $(n \in \mathbb{N})$ *on* $[a, b] \cap \mathbb{T}$, *where*

$$\mathbb{N} := \{1, 2, 3, \cdots\}.$$

Suppose also that

$$p \geqq 0, \quad q \geqq 1 \qquad \textit{and} \qquad h(t) \in C_{\mathrm{rd}}\left([a, b], [1, \infty)\right).$$

If

$$f(a) = f^{\Delta}(a) = \cdots = f^{\Delta^{n-1}}(a) = 0 \qquad (n \in \mathbb{N}), \tag{8}$$

then

$$\int_a^b h(t)\,|f(t)|^p \cdot \left|f^{\Delta^n}(t)\right|^q \Delta t$$
$$\leqq \left(\frac{q}{p+q}\right)[(b-a)^p]^n \int_a^b h(t)\left|f^{\Delta^n}(t)\right|^{p+q} \Delta t. \tag{9}$$

Theorem F. *Let each of the functions $f(t)$ and $g(t)$ given by*

$$f, g : [a, b] \cap \mathbb{T} \to \mathbb{R} \qquad (b > a \geqq 0)$$

be delta differentiable n times $(n \in \mathbb{N})$ on $[a, b] \cap \mathbb{T}$. Suppose also that

$$p \geqq 0, \quad q \geqq 1 \qquad \textit{and} \qquad h(t) \in C_{\mathrm{rd}}\left([a, b], [1, \infty)\right).$$

If the function $f(t)$ satisfies the conditions in (8) and the function $g(t)$ satisfies the following conditions:

$$g(a) = g^{\Delta}(a) = \cdots = g^{\Delta^{n-1}}(a) = 0 \qquad (n \in \mathbb{N}), \tag{10}$$

then

$$\begin{aligned} &\int_a^b h(t) \left\{ |f(t)|^p \cdot \left| g^{\Delta^n}(t) \right|^q + |g(t)|^p \cdot \left| f^{\Delta^n}(t) \right|^q \right\} \Delta t \\ &\leqq \left(\frac{2q}{p+q} \right) [(b-a)^p]^n \\ &\quad \cdot \int_a^b h(t) \left\{ \left| f^{\Delta^n}(t) \right|^{p+q} + \left| g^{\Delta^n}(t) \right|^{p+q} \right\} \Delta t. \end{aligned} \tag{11}$$

Theorem G. *Let each of the functions $f(t)$ and $g(t)$ given by*

$$f, g : [a, b] \cap \mathbb{T} \to \mathbb{R} \qquad (b > a \geqq 0)$$

be delta differentiable n times $(n \in \mathbb{N})$ on $[a, b] \cap \mathbb{T}$. Suppose also that

$$\frac{a+b}{2} \in [a, b], \quad p \geqq 0, \quad q \geqq 1 \qquad \textit{and} \qquad h(t) \in C_{\mathrm{rd}}\left([a, b], [1, \infty)\right).$$

If the function $f(t)$ satisfies the conditions in (8) as well as the following conditions:

$$f(b) = f^{\Delta}(b) = \cdots = f^{\Delta^{n-1}}(b) = 0 \qquad (n \in \mathbb{N}), \tag{12}$$

and if the function $g(t)$ satisfies the conditions in (10) as well as the following conditions:

$$g(b) = g^{\Delta}(b) = \cdots = g^{\Delta^{n-1}}(b) = 0 \qquad (n \in \mathbb{N}), \tag{13}$$

then

$$\begin{aligned} &\int_a^b h(t) \left\{ |f(t)|^p \cdot \left| g^{\Delta^n}(t) \right|^q + |g(t)|^p \cdot \left| f^{\Delta^n}(t) \right|^q \right\} \Delta t \\ &\leqq \left(\frac{2q}{p+q} \right) \left[\left(\frac{b-a}{2} \right)^p \right]^n \\ &\quad \cdot \int_a^b h(t) \left\{ \left| f^{\Delta^n}(t) \right|^{p+q} + \left| g^{\Delta^n}(t) \right|^{p+q} \right\} \Delta t. \end{aligned} \tag{14}$$

The object of this paper is to show that Theorems D, E, F and G above are not valid as asserted by Wong *et al.* [17, Section 2] by presenting some counterexamples for them. We also prove several general weighted Opial-type inequalities on time scales. Some of our results are intended to provide corrections and modifications of the aforementioned assertions by Wong *et al.* [17, Section 2].

2. A Set of Counterexamples

Since the derivations of Theorems E, F and G by Wong *et al.* [17, Section 2] are all based essentially upon Theorem D, it would suffice our purpose to give a counterexample for Theorem D only. Indeed, if (in Theorem D) we set

$$\mathbb{T}=\mathbb{R},\ p=q=1,\ a=0,\ b=r\ \ (r>0),\ f(t)=t \quad \text{and} \quad h(t)=t+1,$$

then the left-hand side of the inequality (7) becomes

$$\int_0^r (t+1)t\ dt = \frac{r^3}{3}+\frac{r^2}{2} \qquad (r>0), \tag{15}$$

whereas the right-hand side of the inequality (7) assumes the following form:

$$\frac{1}{2}\,r\int_0^r (t+1)dt = \frac{r^3}{4}+\frac{r^2}{2} \qquad (r>0). \tag{16}$$

Upon substituting from (15) and (16) into (7), we readily arrive at a contradiction in the inequality (7) for all $r>0$. Thus, obviously, Theorem D (and hence also Theorems E, F and G) do not hold true as asserted by Wong *et al.* [17, Section 2].

We remark in passing that a special case of Theorem E when $n=1$, a special case of Theorem F when $f(t)=g(t)$, and a special case of Theorem G when

$$n=1 \quad \text{and} \quad f(t)=g(t),$$

all correspond to the *erroneous* inequality (7) asserted by Theorem D.

3. Definitions, Notations and Preliminaries in Time Scales Theory and Delta Differentiability

In this section, we present some definitions, notations and preliminaries concerning the *Time Scales Theory* and the concept of *Delta Differentiability*. These concepts, together with the notion of rd-continuity, were used in Section 1 above.

Definition 1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, the two most popular examples being

$$\mathbb{T} = \mathbb{R} \qquad \text{and} \qquad \mathbb{T} = \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}.$$

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{\tau \mid \tau \in \mathbb{T} \quad \text{and} \quad \tau > t\} \qquad (t \in \mathbb{T}; \;\; t < \sup\{\mathbb{T}\})$$

and

$$\rho(t) := \sup\{\tau \mid \tau \in \mathbb{T} \quad \text{and} \quad s < t\} \qquad (t \in \mathbb{T}; \;\; t > \inf\{\mathbb{T}\}),$$

respectively, each of which is being supplemented by

$$\inf\{\varnothing\} = \sup\{\mathbb{T}\} \qquad \text{and} \qquad \sup\{\varnothing\} = \inf\{\mathbb{T}\}.$$

Furthermore, a point $t \in \mathbb{T}$ is called *right-scattered, right-dense, left-scattered* or *left-dense* if

$$\sigma(t) > t, \;\; \sigma(t) = t, \;\; \rho(t) < t \quad \text{or} \quad \rho(t) = t,$$

respectively.

Definition 2. Let the time scale $\mathbb{T}$ have a right-scattered minimum $\mathfrak{m}$. Then we define the set $\mathbb{T}^{\kappa}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus \{\mathfrak{m}\} & (\mathfrak{m} \quad \text{exists}) \\ \mathbb{T} & (\mathfrak{m} \quad \text{does not exist}). \end{cases} \tag{17}$$

On the other hand, if the time scale $\mathbb{T}$ has a left-scattered maximum $\mathfrak{M}$, then we define the set $\mathbb{T}^{\kappa}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus \{\mathfrak{M}\} & (\mathfrak{M} \quad \text{exists}) \\ \mathbb{T} & (\mathfrak{M} \quad \text{does not exist}). \end{cases} \tag{18}$$

Moreover, the *forward graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t \quad (t \in \mathbb{T}) \quad = \begin{cases} 0 & (\mathbb{T} = \mathbb{R}) \\ 1 & (\mathbb{T} = \mathbb{Z}) \end{cases}$$

and the *backward graininess* $\nu : \mathbb{T} \to [0, \infty)$ is defined by

$$\nu(t) := t - \rho(t) \quad (t \in \mathbb{T}) \quad = \begin{cases} 0 & (\mathbb{T} = \mathbb{R}) \\ 1 & (\mathbb{T} = \mathbb{Z}) \end{cases}$$

Definition 3. A mapping $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if

$$1 + \mu(t)\mathfrak{f}(t) \neq 0 \qquad (t \in \mathbb{T}).$$

Furthermore, if $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$, then the mapping $\mathfrak{f}^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$\mathfrak{f}^\sigma(t) := \mathfrak{f}\big(\sigma(t)\big) \qquad (t \in \mathbb{T}),$$

where $\sigma(t)$ is given in Definition 1 above.

Definition 4. A mapping $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ is said to be rd-*continuous* if it satisfies each of the following conditions:

(i) f is continuous at every right-dense point or maximal point of $\mathbb{T}$;

(ii) The left-sided limit:
$$\lim_{\tau \to t-} \mathfrak{f}(\tau) = \mathfrak{f}(t-)$$
exists at every left-dense point of $\mathbb{T}$.

Just as in Equation (6) above, the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is denoted as follows:

$$C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) := \{\mathfrak{f} \mid \mathfrak{f} : \mathbb{T} \to \mathbb{R} \ \text{ and } \ \mathfrak{f}(t) \ \text{ is an } \mathrm{rd}\text{-continuous function}\}.$$

Definition 5. Assume that $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$. Then we define $\mathfrak{f}^\Delta(t)$ to be the number (if it exists) with the property that, for any given $\epsilon > 0$, there is a neighborhood $\mathcal{N}$ of t such that

$$\left|\mathfrak{f}\big(\sigma(t)\big) - \mathfrak{f}(\tau) - \mathfrak{f}^\Delta(t)\,[\sigma(t) - \tau]\right| \leqq \epsilon\,|\sigma(t) - \tau| \qquad (\tau \in \mathcal{N}).$$

In this case, we say that $\mathfrak{f}^\Delta(t)$ is the *delta derivative* of $\mathfrak{f}(t)$ at the point $t \in \mathbb{T}^\kappa$. If $\mathfrak{f}$ is delta differentiable for every $t \in \mathbb{T}^\kappa$, then $\mathfrak{f}$ is delta differentiable on $\mathbb{T}$ and $\mathfrak{f}^\Delta(t)$ is a new function defined on $\mathbb{T}^\kappa$.

If $\mathfrak{f}$ is delta differentiable at $t \in \mathbb{T}^\kappa$, then it is easily seen that

$$\mathfrak{f}^\Delta(t) = \begin{cases} \displaystyle\lim_{\tau \to t\ (\tau \in \mathbb{T})} \frac{\mathfrak{f}(t) - \mathfrak{f}(\tau)}{t - \tau} & \big(\mu(t) = 0\big) \\[2ex] \dfrac{\mathfrak{f}\big(\sigma(t)\big) - \mathfrak{f}(t)}{\mu(t)} & \big(\mu(t) > 0\big). \end{cases} \tag{19}$$

Several useful delta derivative formulas are recorded here under Lemma 1 below.

Lemma 1. *The above-defined delta derivatives satisfy each of the following properties*:

$$f\big(\sigma(t)\big) = f(t) + \mu(t) f^\Delta(t), \tag{20}$$

$$\big(f(t)g(t)\big)^{\Delta} = f^{\Delta}(t)g(t) + f\big(\sigma(t)\big)g^{\Delta}(t) \tag{21}$$

and

$$\left(\frac{f(t)}{g(t)}\right)^{\Delta} = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g\big(\sigma(t)\big)}. \tag{22}$$

Lemma 2 below is an easy consequence of the property (20) asserted by Lemma 1.

Lemma 2. *If* $f : \mathbb{T} \to \mathbb{R}$ *is* rd*-continuous at* $t \in \mathbb{T}$ *and* t *is right-scattered*, *then*

$$f^{\Delta}(t) = \frac{f\big(\sigma(t)\big) - f(t)}{\sigma(t) - t}. \tag{23}$$

Definition 6. A function $\mathfrak{F} : \mathbb{T} \to \mathbb{R}$ is said to be an *antiderivative* of $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ if

$$\mathfrak{F}^{\Delta}(t) = \mathfrak{f}(t) \qquad (t \in \mathbb{T}^{\kappa}).$$

In this case, we define the integral of $\mathfrak{f}$ by

$$\int_s^t \mathfrak{f}(\tau)\Delta\tau = \mathfrak{F}(t) - \mathfrak{F}(s) \qquad (s, t \in \mathbb{T}) \tag{24}$$

and we say that $\mathfrak{f}$ is integrable on $\mathbb{T}$.

The results asserted by Lemma 3 below are rather immediate consequences of (21) and (24).

Lemma 3. *Each of the following integral formulas holds true*:

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\Delta} = f(t) \tag{25}$$

and

$$\int_a^t f(\tau)g^{\Delta}(\tau)\Delta\tau = f(\tau)g(\tau)\Big|_{\tau=a}^{t} - \int_a^t f^{\Delta}(\tau)g\big(\sigma(\tau)\big)\Delta\tau \tag{26}$$

for any constant $a \in \mathbb{T}$.

2. The Main Opial-type Inequalities on Time Scales

We begin this section by proving our first Opial-type inequality on time scales.

Theorem 1. *Let $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$ be defined as in Theorem* D. *Also let*

$$p \geqq 0 \qquad \textit{and} \qquad q \geqq 1,$$

and suppose that $h(t)$ is a right-continuous, positive and non-increasing function on $[a,\tau] \cap \mathbb{T}$. Then the Opial-type inequality (7) holds true.

Proof. Suppose that the function $g(t)$ is given by

$$g(t) = \int_a^t [h(s)]^{\frac{q}{p+q}} \left|f^{\Delta}(s)\right|^q \Delta s \qquad\qquad (t \in [a,\tau] \cap \mathbb{T}),$$

so that

(27) $$g(a) = 0 \qquad \text{and} \qquad g^{\Delta}(t) = [h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q.$$

In the case when $q > 1$, by using Hölder's inequality with indices

$$q \qquad \text{and} \qquad \frac{q}{q-1},$$

we have

$$\begin{aligned}
|f(t)| &\leqq \int_a^t \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_a^t [h(s)]^{-\frac{1}{p+q}} [h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right| \Delta s \\
&\leqq \left[\int_a^t \left([h(s)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \Delta s\right]^{\frac{q-1}{q}} \left[\int_a^t \left([h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right|\right)^q \Delta s\right]^{\frac{1}{q}} \\
&\leqq \left[\left([h(t)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \left(\int_a^t 1 \cdot \Delta s\right)\right]^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}} \\
&= [h(t)]^{-\frac{1}{p+q}} (t-a)^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}},
\end{aligned}$$

which readily yields

(28) $$[h(t)]^{\frac{p}{p+q}} |f(t)|^p \leqq (t-a)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}}.$$

In the case when $q = 1$, we find that

$$\begin{aligned}
|f(t)| &\leqq \int_a^t \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_a^t [h(s)]^{-\frac{1}{p+1}} [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&\leqq [h(t)]^{-\frac{1}{p+1}} \int_a^t [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} g(t),
\end{aligned}$$

which shows that the inequality (28) holds true also when $q = 1$. Thus, by means of (27) and (28), we observe that

$$\begin{aligned}
&\int_a^\tau h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \\
&= \int_a^\tau [h(t)]^{\frac{p}{p+q}}\,|f(t)|^p\,[h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \int_{t=a}^\tau (t-a)^{\frac{p(q-1)}{q}}\,[g(t)]^{\frac{p}{q}}\, g^\Delta(t)\Delta t \\
&\leqq (\tau-a)^{\frac{p(q-1)}{q}} \int_a^\tau [g(t)]^{\frac{p}{q}}\,\Delta g(t) \\
&\leqq (\tau-a)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right)\,[g(\tau)]^{\frac{p+q}{q}},
\end{aligned}$$

since (by definition) $g(a) = 0$ as given by (27).

On the other hand, from Hölder's inequality with indices

$$\frac{p+q}{p} \qquad \text{and} \qquad \frac{p+q}{q},$$

we obtain

$$\begin{aligned}
g(\tau) &= \int_a^\tau [h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \left(\int_a^\tau 1^{\frac{p+q}{p}} \cdot \Delta t\right)^{\frac{p}{p+q}} \left[\int_a^\tau \left([h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q\right)^{\frac{p+q}{q}} \Delta t\right]^{\frac{q}{p+q}} \\
&\leqq (\tau-a)^{\frac{p}{p+q}} \left(\int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t\right)^{\frac{q}{p+q}}.
\end{aligned}$$

Therefore, we finally have

$$\begin{aligned}
&\int_a^\tau h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \left(\frac{q}{p+q}\right) (\tau-a)^p \int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t,
\end{aligned}$$

which evidently completes the proof of Theorem 1.

Theorem 2. *Let the function* $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$ *be delta differentiable with* $f(b) = 0$. *Also let*

$$p \geqq 0 \qquad \textit{and} \qquad q \geqq 1,$$

and suppose that $h(t)$ is a right-continuous, positive and non-increasing function on $[\tau, b] \cap \mathbb{T}$. Then

$$\int_\tau^b h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \leqq \left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t. \tag{29}$$

Proof. We consider a function $g(t)$ given by

$$g(t) = \int_t^b [h(s)]^{\frac{q}{p+q}} \left|f^\Delta(s)\right|^q \Delta s \qquad (t \in [\tau, b] \cap \mathbb{T}),$$

so that

$$g(b) = 0 \qquad \text{and} \qquad g^\Delta(t) = -[h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q. \tag{30}$$

In the case when $q > 1$, by using Hölder's inequality with indices

$$q \qquad \text{and} \qquad \frac{q}{q-1},$$

we get

$$\begin{aligned}
|f(t)| &\leqq \int_t^b \left|f^\Delta(s)\right| \Delta s \\
&= \int_t^b [h(s)]^{-\frac{1}{p+q}} [h(s)]^{\frac{1}{p+q}} \left|f^\Delta(s)\right| \Delta s \\
&\leqq \left[\int_t^b \left([h(s)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \Delta s\right]^{\frac{q-1}{q}} \left[\int_t^b \left([h(s)]^{\frac{1}{p+q}} \left|f^\Delta(s)\right|\right)^q \Delta s\right]^{\frac{1}{q}} \\
&\leqq \left[\left([h(t)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \left(\int_t^b 1 \cdot \Delta s\right)\right]^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}} \\
&= [h(t)]^{-\frac{1}{p+q}} (b-t)^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}},
\end{aligned}$$

which implies the following inequality:

$$[h(t)]^{\frac{p}{p+q}} |f(t)|^p \leqq (b-t)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}}. \tag{31}$$

In the case when $q = 1$, we have

$$\begin{aligned}
|f(t)| &\leqq \int_t^b \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_t^b [h(s)]^{-\frac{1}{p+1}} [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} \int_t^b [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} g(t),
\end{aligned}$$

which shows that the inequality (31) holds true also when $q = 1$. Thus, by (30) and (31), we find that

$$\begin{aligned}
\int_\tau^b h(t) \left|f(t)\right|^p \left|f^{\Delta}(t)\right|^q \Delta t &= \int_\tau^b [h(t)]^{\frac{p}{p+q}} \left|f(t)\right|^p [h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q \Delta t \\
&\leqq \int_\tau^b (b-t)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}} \left[-g^{\Delta}(t)\right] \Delta t \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \int_{t=\tau}^b \left(-[g(t)]^{\frac{p}{q}}\right) \Delta g(t) \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right) \left(-[g(t)]^{\frac{p+q}{q}}\right) \Big|_{t=\tau}^b \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right) [g(\tau)]^{\frac{p+q}{q}},
\end{aligned}$$

since (by definition) $g(b) = 0$ as given by (30).

On the other hand, in view of Hölder's inequality with indices

$$\frac{p+q}{p} \qquad \text{and} \qquad \frac{p+q}{q},$$

we have

$$\begin{aligned}
g(\tau) &= \int_\tau^b [h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q \Delta t \\
&\leqq \left(\int_\tau^b 1^{\frac{p+q}{p}} \cdot \Delta t\right)^{\frac{p}{p+q}} \left[\int_\tau^b \left([h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q\right)^{\frac{p+q}{q}} \Delta t\right]^{\frac{q}{p+q}} \\
&\leqq (b-\tau)^{\frac{p}{p+q}} \left(\int_\tau^b h(t) \left|f^{\Delta}(t)\right|^{p+q} \Delta t\right)^{\frac{q}{p+q}}.
\end{aligned}$$

Therefore, we finally obtain

$$\int_\tau^b h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \leqq \left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t,$$

which is precisely the inequality (29) asserted by Theorem 2.

Theorem 3. *Let the function* $f : [a, b] \cap \mathbb{T} \to \mathbb{R}$ *be delta differentiable with*

$$f(a) = f(b) = 0.$$

Also let

$$p \geqq 0, \qquad q \geqq 1 \qquad \text{and} \qquad \tau \in [a, b] \cap \mathbb{T}.$$

Furthermore, suppose that the function $h(t)$ *is positive and non-increasing on* $[a, \tau] \cap \mathbb{T}$ *and non-increasing on* $[\tau, b] \cap \mathbb{T}$. *Then*

$$\begin{aligned}\int_a^b h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t &\leqq \left(\frac{q}{p+q}\right)(\tau-a)^p \int_a^\tau h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t \\ &\quad + \left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t. \end{aligned} \tag{32}$$

Proof. By Theorem 1 and 2, we get

$$\begin{aligned}\int_a^b h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t &= \int_a^\tau h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \\ &\quad + \int_\tau^b h(t)\,|f(t)|^p \left|f^\Delta(t)\right|^q \Delta t \\ &\leqq \left(\frac{q}{p+q}\right)(\tau-a)^p \int_a^\tau h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t \\ &\quad + \left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t)\left|f^\Delta(t)\right|^{p+q} \Delta t,\end{aligned}$$

which obviously completes the proof of Theorem 3.

By setting

$$\tau = \frac{a+b}{2}$$

in Theorem 3, we arrive at an interesting special case as given below.

Corollary. *The following weighted Opial-type inequality holds true on time scales*:

$$\int_a^b h(t)\,|f(t)|^p\,\left|f^{\Delta}(t)\right|^q\,\Delta t$$

$$\leqq \left(\frac{q}{p+q}\right)\left(\frac{b-a}{2}\right)^p \int_a^b h(t)\,\left|f^{\Delta}(t)\right|^{p+q}\,\Delta t. \tag{33}$$

Theorems 4, 5 and 6 below can be proven by employing the same methods as those used in the proofs of Theorems E, F and G, respectively (see, for details, [17]).

Theorem 4. *Let the function f be defined as in Theorem* E. *Also let p, q and the function $h(t)$ be defined as in Theorem* 1. *Then the inequality* (9) *holds true.*

Theorem 5. *Let the functions f and g be defined as in Theorem* F. *Also let p, q and the function $h(t)$ be defined as in Theorem* 1. *Then the inequality* (11) *holds true.*

Theorem 6. *Let the functions f and g be defined as in Theorem* G. *Also let p, q and the function $h(t)$ be defined as in Theorem* 1. *Then the inequality* (14) *holds true.*

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