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SOME NEW EXISTENCE THEOREMS OF GENERALIZED ABSTRACT FUZZY ECONOMIES WITH APPLICATIONS

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Abstract. In this paper, some new existence theorems of equilibrium and maximal element for generalized abstract fuzzy economies with uncountable number of agents and qualitative fuzzy games are proved in G-convex spaces, respectively. As applications, some existence theorems of equilibria for abstract economies are given in G-convex spaces. The results presented in this paper generalize some known results in the literature.

1. INTRODUCTION

Since the classical Arrow and Debrew [2] existence theorem of Walrasian equilibria was proved, the result was generalized in many directions. Mas-colell [18] was first to show that the existence of equilibrium can be established without assuming preferences to be total and transitive. On the other hand, Borgin and Keiding [5] proved a new existence theorem for a compact abstract economy with KF-majorized preference correspondences. Following their ideas, many authors studied the existence of equilibria for generalized games (see [4, 6-9, 12-17, 22-27] and the references therein).

Billot [4] studied the equilibrium points of fuzzy games and fuzzy economic equilibrium, and proved the existence of a fuzzy general equilibrium. Huang [13, 14] first introduced the concepts of abstract fuzzy economies and generalized abstract

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fuzzy economies, and proved some existence theorems of equilibrium and maximal element for the abstract fuzzy economies and qualitative fuzzy games, respectively.

In this paper, we introduce a new class of generalized abstract fuzzy economies with an uncountable number of agents with fuzzy constraint correspondences and fuzzy preference correspondence in G-convex spaces. We prove some new existence theorems of equilibrium and maximal element for generalized abstract fuzzy economies with uncountable number of agents and qualitative fuzzy games in G-convex spaces, respectively. As applications, we give some existence theorems of equilibria for abstract economies in G-convex spaces. The results presented in this paper generalize some known results in [13-16].

2. Preliminaries

For a set X, we shall denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subset of X respectively. For $A \in \langle X\rangle$, we denote by $|A|$ the cardinality of A. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \cdots, e_n$. If J is a nonempty subset of $\{0, 1, \cdots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. The notion of a generalized convex (or G-convex) space was introduced under an extra isotonic condition by Park and Kim [20, 21]. Recently, Park [19], by removing the extra condition, gives the following definition of a G-convex space.

A G-convex space (X, D, Γ) consists of a topological space X, a nonempty set D and a set-valued mapping $\Gamma : \langle D\rangle \to 2^X \setminus \{\emptyset\}$ such that for each $A = \{a_0, a_1, \cdots, a_n\} \in \langle D\rangle$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subseteq \Gamma(\{a_j : j \in J\})$ where $\Delta_J = co\{e_j : j \in J\}$. When $D = X$, write $(X, X, \Gamma) = (X, \Gamma)$. In case $D \subset X$, a subset C of (X, D, Γ) is said to be Γ-convex if for each $A \in \langle D\rangle$, $A \subset C$ implies $\Gamma(A) \subseteq C$. We define the G-convex hull of C, denoted by G-$co(C)$, as G-$co(C) = \cap\{B \subset X : C \subset B$ and B is Γ-convex$\}$.

A locally G-convex uniform space is a G-convex space (X, D, Γ) such that

(1) X is a separated uniform space with the basis β for symmetric entourages;
(2) D is a dense subset of X; and
(3) for each $V \in \beta$ and each $x \in X$, the set $V[x] = \{y \in X : (x, y) \in V\}$is Γ-convex.

Let X and Y be both topological spaces. A set-valued mapping $G : X \to 2^Y$ is said to be compact if $G(X)$ is contained in some compact subset of Y. G is said to be upper semicontinuous (resp., lower semicontinuous) on X if for each $x \in X$ and for each open set U in Y, the set $\{x \in X : G(x) \subseteq U\}$ (resp., $\{x \in X : G(x) \cap U \neq \emptyset\}$) is open in X.

Let M and N be two Hausdorff topological vector spaces and $X \subset M, Y \subset N$ be two nonempty convex subsets. Throughout this paper we always denote by $\mathcal{F}(\mathcal{X})(\mathcal{F}(\mathcal{Y}))$ the collection of all fuzzy sets on $X(Y)$. A mapping from X into $\mathcal{F}(\mathcal{Y})(\mathcal{F}(\mathcal{X}))$ is called a fuzzy mapping. If $F : X \to \mathcal{F}(\mathcal{Y})$ is a fuzzy mapping, then for each $x \in X, F(x)$(denote by F_x in the sequel) is a fuzzy set in $\mathcal{F}(\mathcal{Y})$ and $F_x(y)$ is the degree of membership of point y in F_x.

In this sequel, we denote by

$$(A)_q = \{x \in X : A(x) \geq q\}, \qquad q \in [0,1],$$

the q-cut set of $A \in \mathcal{F}(\mathcal{X})$.

We say that $\Gamma = (X_i, Y_i, A_i, B_i, P_i)_{i\in I}$ is a generalized abstract fuzzy economy, if I is a finite or an infinite set of agents, X_i and Y_i are nonempty topological space (a choice set), $A_i : X = \prod_{i\in I} X_i \to \mathcal{F}(\mathcal{X}_\rangle)$ and $B_i : \prod_{i\in I} X_i \to \mathcal{F}(\mathcal{Y}_\rangle)$ are fuzzy constraint mappings (fuzzy constraint correspondences), $P_i : X \times Y \to \mathcal{F}(\mathcal{X}_\rangle)$ is a fuzzy preference mapping (fuzzy preference correspondence), where $Y = \prod_{i\in I} Y_i$. An equilibrium for Γ is a point $(\widehat{x}, \widehat{y}) \in X \times Y$ such that for each $i \in I, \widehat{x}_i \in (A_{i\widehat{x}})_{a_i(\widehat{x})}, \widehat{y}_i \in (B_{i\widehat{x}})_{b_i(\widehat{x})}$ and $(A_{i\widehat{x}})_{a_i(\widehat{x})} \cap (P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})} = \emptyset$, where $a_i, b_i : X \to (0,1]$ and $p_i : X \times Y \to (0,1]$.

$\Gamma = (X_i, Y_i, P_i)$ is said to be a qualitative fuzzy game if for each $i \in I, X_i$ and Y_i are strategy sets of player i, and $P_i : X \times Y \to \mathcal{F}(\mathcal{X}_\rangle)$ is a fuzzy preference mapping (fuzzy preference correspondence) of player i. A maximal element of Γ is a point $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $(P_{i(\widehat{x},\widehat{y})})(x,y) < p_i(\widehat{x}, \widehat{y})$ for all $i \in I$ and $(x,y) \in X \times Y$, where $p_i : X \times Y \to (0,1]$.

Lemma 2.1. (see [7]). *Let $(X_i, D_i, \Gamma_i)_{i\in I}$ be a family of locally G-convex uniform spaces with each X_i having the basis β_i of symmetric entourages, $X = \prod_{i\in I} X_i, D = \prod_{i\in I} D_i$ and $\Gamma = \prod_{i\in I} \Gamma_i$. Then (X, D, Γ) is also a locally G-convex uniform space.*

Lemma 2.2. (see [10]). *Let $(X_i, D_i, \Gamma_i)_{i\in I}$ be a family of locally G-convex uniform spaces with each X_i having the basis β_i of symmetric entourages. For each $i \in I$, let $G_i : X = \prod X_i \to 2^{X_i}$ be an upper semicontinuous compact set-valued mapping with nonempty closed Γ_i-convex values. Then there exists a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $\widehat{x}_i \in G_i(\widehat{x})$ for each $i \in I$.*

3. Existence of Equilibria for Generalized Fuzzy Economies

Lemma 3.1. (see [28]). *Let X and Y be topological spaces and A be a closed (resp., open) subset of X. Suppose that $F_1 : X \to 2^Y$ and $F_2 : A \to 2^Y$ are both* *lower semicontinuous (resp., upper semicontinuous) such that* $F_2(x) \subset F_1(x)$ *for each* $x \in A$*. Then the mapping* $F : X \to 2^Y$ *defined by*

$$F(x) = \begin{cases} F_2(x), & \text{if} \quad x \in A, \\ F_1(x), & \text{if} \quad x \in X \setminus A. \end{cases}$$

is also lower semicontinuous (resp., upper semicontinuous).

Theorem 3.1. *Let* $(X_i, Y_i, A_i, B_i, P_i)_{i\in I}$ *be an abstract fuzzy economy,* $X = \prod_{i\in I} X_i, Y = \prod_{i\in I} Y_i$*, and* $a_i, b_i : X \to (0, 1], p_i : X \times Y \to (0, 1]$ *such that for each* $i \in I$*, the following conditions are satisfied:*

(1) (X_i, D_i, Γ_i) *and* (Y_i, D'_i, Γ'_i) *are locally* G*-convex uniform spaces;*

(2) *for each* $x \in X, x \to (A_{ix})_{a_i(x)} : X \to 2^{X_i}$ *and* $x \to (B_{ix})_{b_i(x)} : X \to 2^{Y_i}$ *are both upper semicontinuous compact mappings with nonempty closed* Γ_i*-convex values;*

(3) *for each* $(x, y) \in X \times Y, (x, y) \to (P_{i(x,y)})_{p_i(x,y)} : X \times Y \to 2^{X_i}$ *is a closed mapping with* Γ_i*-convex values;*

(4) *the set* $E_i = \{(x, y) \in X \times Y : (A_{ix})_{a_i(x)} \cap (P_{i(x,y)})_{p_i(x,y)} \neq \emptyset\}$ *is open in* $X \times Y$*;*

(5) *for each* $(x, y) \in X \times Y$ *with* $x_i \in (A_{ix})_{a_i(x)}$ *and* $y_i \in (B_{ix})_{b_i(x)}, x_i \notin (P_{i(x,y)})_{p_i(x,y)}$.

Then there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ *such that for each* $i \in I$*,* $\widehat{x}_i \in (A_{i\widehat{x}})_{a_i(\widehat{x})}$*,* $\widehat{y}_i \in (B_{i\widehat{x}})_{b_i(\widehat{x})}$ *and*

$$(A_{i\widehat{x}})_{a_i(\widehat{x})} \cap (P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})} = \emptyset.$$

Proof. By Lemma 2.1, $(X, D, \Gamma), (X \times Y, D \times D', \{\Gamma \times \Gamma'\})$ and for each $i \in I, (X_i \times Y_i, D_i \times D'_i, \{\Gamma_i \times \Gamma'_i\}$ are all locally G-convex uniform spaces. For each $i \in I$, define a set-valued mapping $G_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$G_i(x, y) = \begin{cases} [(A_{ix})_{a_i(x)} \cap (P_{i(x,y)})_{p_i(x,y)}] \times (B_{ix})_{b_i(x)}, & \text{if} \quad (x, y) \in E_i, \\ (A_{ix})_{a_i(x)} \times (B_{ix})_{b_i(x)}, & \text{if} \quad (x, y) \notin E_i. \end{cases}$$

By the conditions (2), (3) and Theorem 3.1.8 of Aubin and Ekeland [24], $x \to (A_{ix})_{a_i(x)} \cap (P_{i(x,y)})_{p_i(x,y)}$ is an upper semicontinuous compact mapping on $X \times Y$ with nonempty closed values. From the conditions (2)-(4), Lemma 3 of Fan [11] and Lemma 3.1, G_i is an upper semicontinuous compact mapping with nonempty closed Γ_i-convex values. By Lemma 2.2, there exists a point $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $(\widehat{x}_i, \widehat{y}_i) \in G_i(\widehat{x}, \widehat{y})$ for each $i \in I$. If for some $j \in I, (\widehat{x}, \widehat{y}) \in E_j$, then we have $\widehat{x}_j \in (A_{j\widehat{x}})_{a_j(\widehat{x})} \cap (P_{j(\widehat{x},\widehat{y})})_{p_j(\widehat{x},\widehat{y})}$ and $\widehat{y}_j \in (B_{j\widehat{x}})_{b_j(\widehat{x})}$ which contradicts the condition (5).

Hence $(\widehat{x}, \widehat{y}) \notin E_i$ for all $i \in I$. It follows from the definition of G_i that for each $i \in I, \widehat{x}_i \in (A_{i\widehat{x}})_{a_i(\widehat{x})}, \widehat{y}_i \in (B_{i\widehat{x}})_{b_i(\widehat{x})}$ and $(A_{i\widehat{x}})_{a_i(\widehat{x})} \cap (P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})} = \emptyset$.

Remark 3.1. Theorem 3.1 improves and generalizes Theorem 3.1 of Huang [13] and Theorem 3.1 of Huang [14] to a more general model of a generalized abstract fuzzy economy and locally G-convex uniform spaces. Theorem 3.1 is also an improved variant of Theorem 2 of Kim and Tan [16], Theorem 1 of Kim and Lee [15] and Theorem 1.2 of Ding, Yao and Lin [10] in locally G-convex uniform spaces.

Theorem 3.2. *Let $(X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract fuzzy economy, $X = \prod_{i\in I} X_i$, and $a_i, b_i : X \to (0,1], p_i : X \times X \to (0,1]$ such that for each $i \in I$, the following conditions are satisfied:*

(1) *(X_i, D_i, Γ_i) is a locally G-convex uniform space and C_i is a nonempty compact subset of X_i such that G-$co(C)$ is paracompact where $C = \prod_{i\in I} C_i$;*

(2) *for each $x \in X, x \to (A_{ix})_{a_i(x)} : X \to 2^{C_i}$ and $x \to (B_{ix})_{b_i(x)} : X \to 2^{C_i}$ are both upper semicontinuous compact mappings with nonempty closed Γ_i-convex values;*

(3) *for each $(x, y) \in X \times X, (x, y) \to (P_{i(x,y)})_{p_i(x,y)} : X \times X \to 2^{C_i}$ is a closed mapping with Γ_i-convex values;*

(4) *the set $E_i = \{(x, y) \in X \times X : (A_{ix})_{a_i(x)} \cap (P_{i(x,y)})_{p_i(x,y)} \neq \emptyset\}$ is open in $X \times X$;*

(5) *for each $(x, y) \in X \times X$ with $x_i \in (A_{ix})_{a_i(x)}$ and $y_i \in (B_{ix})_{b_i(x)}, x_i \notin (P_{i(x,y)})_{p_i(x,y)}$.*

Then there exists $(\widehat{x}, \widehat{y}) \in G$-$co(C) \times G$-$co(C)$ such that for each $i \in I, \widehat{x}_i \in (A_{i\widehat{x}})_{a_i(\widehat{x})}$, $\widehat{y}_i \in (B_{i\widehat{x}})_{b_i(\widehat{x})}$ and

$$(A_{i\widehat{x}})_{a_i(\widehat{x})} \cap (P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})} = \emptyset.$$

Proof. By Lemma 2.1, (X, D, Γ) becomes as a locally G-convex uniform space. Let $C = \prod_{i\in I} C_i$, then C is a compact subset of X. Since G-$co(C)$ is a G-convex subset of X, G-$co(C)$ is also a paracompact locally G-convex uniform space. Clearly, from (2) and (3), we obtain that $x \to (A_{ix})_{a_i(x)}$ and $x \to (B_{ix})_{b_i(x)} : G$-$co(C) \to 2^{C_i}$ are upper semicontinuous mappings with nonempty closed Γ_i-convex values, and $(x, y) \to (P_{i(x,y)})_{p_i(x,y)} : G$-$co(C) \times G$-$co(C) \to 2^{C_i}$ is a closed mapping with Γ_i-convex values. By (4), the set

$$E_i' = \{(x, y) \in G\text{-}co(C) \times G\text{-}co(C) : (A_{ix})_{a_i(x)} \cap (P_{i(x,y)})_{p_i(x,y)} \neq \emptyset\}$$

is open in G-$co(C)\times G$-$co(C)$. For each $i\in I$, define a mapping $G_i : G$-$co(C)\times G$-$co(C)\to 2^{C_i}$ by

$$G_i(x,y)=\begin{cases}[(A_{ix})_{a_i(x)}\cap(P_{i(x,y)})_{p_i(x,y)}]\times(B_{ix})_{b_i(x)}, & \text{if} \quad (x,y)\in E_i',\\ (A_{ix})_{a_i(x)}\times(B_{ix})_{b_i(x)}, & \text{if} \quad (x,y)\notin E_i'.\end{cases}$$

By using similar argument as the proof of Theorem 3.1 with $X_i=Y_i$, we can prove that there exists a point $(\widehat{x},\widehat{y})\in G$-$co(C)\times G$-$co(C)$ such that for each $i\in I,\widehat{x}_i\in(A_{i\widehat{x}})_{a_i(\widehat{x})},\widehat{y}_i\in(B_{i\widehat{x}})_{b_i(\widehat{x})}$ and $(A_{i\widehat{x}})_{a_i(\widehat{x})}\cap(P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})}=\emptyset$.

Corollary 3.1. *Let $(X_i,A_i,B_i,P_i)_{i\in I}$ be an abstract fuzzy economy, $X=\prod_{i\in I}X_i$, and $a_i,b_i:X\to(0,1],p_i:X\times Y\to(0,1]$ such that for each $i\in I$, the following conditions are satisfied:*

(1) *(X_i,D_i,Γ_i) is a compact locally G-convex uniform spaces;*
(2) *for each $x\in X,x\to(A_{ix})_{a_i(x)}:X\to 2^{X_i}$ and $x\to(B_{ix})_{b_i(x)}:X\to 2^{X_i}$ are both compact mappings with nonempty closed Γ_i-convex values;*
(3) *for each $(x,y)\in X\times X,(x,y)\to(P_{i(x,y)})_{p_i(x,y)}:X\times X\to 2^{X_i}$ is a closed mapping with Γ_i-convex values;*
(4) *the set $E_i=\{(x,y)\in X\times X:(A_{ix})_{a_i(x)}\cap(P_{i(x,y)})_{p_i(x,y)}\neq\emptyset\}$ is open in $X\times X$;*
(5) *for each $(x,y)\in X\times X$ with $x_i\in(A_{ix})_{a_i(x)}$ and $y_i\in(B_{ix})_{b_i(x)},x_i\notin(P_{i(x,y)})_{p_i(x,y)}$.*

Then there exists $(\widehat{x},\widehat{y})\in X\times X$ such that for each $i\in I,\widehat{x}_i\in(A_{i\widehat{x}})_{a_i(\widehat{x})},\widehat{y}_i\in(B_{i\widehat{x}})_{b_i(\widehat{x})}$ and

$$(A_{i\widehat{x}})_{a_i(\widehat{x})}\cap(P_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})}=\emptyset.$$

Proof. By condition (2) and Corollary 3.1.9 of Aubin and Ekeland [3], for each $x\in X,x\to(A_{ix})_{a_i(x)}:X\to 2^{X_i}$ and $x\to(B_{ix})_{b_i(x)}:X\to 2^{X_i}$ are both upper semicontinuous compact mappings with nonempty closed values. Then the conclusion of Corollary 3.1 follows from Theorem 3.2 with $C_i=X_i$, for each $i\in I$.

Corollary 3.2. *Let $(X_i,P_i)_{i\in I}$ be a qualitative fuzzy game, $X=\prod_{i\in I}X_i$ and $p_i:X\times X\to(0,1]$ such that for each $i\in I$, the following conditions are satisfied:*

(1) *(X_i,D_i,Γ_i) is a compact locally G-convex uniform spaces;*
(2) *for each $(x,y)\in X\times X,(x,y)\to(P_{i(x,y)})_{p_i(x,y)}:X\times X\to 2^{X_i}$ is a closed mapping with Γ_i-convex values;*
(3) *the set $E_i=\{(x,y)\in X\times X:(P_{i(x,y)})_{p_i(x,y)}\neq\emptyset\}$ is open in $X\times X$;*
(4) *for each $(x,y)\in X\times X,x_i\notin(P_{i(x,y)})_{p_i(x,y)}$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times X$ *such that* $(P_{i(\widehat{x},\widehat{y})})(x, y) < p_i(\widehat{x}, \widehat{y})$ *for all* $i \in I, (x, y) \in X \times X$.

Proof. For each $i \in I$, let $(A_{ix})_{a_i(x)} = (B_{ix})_{b_i(x)} = X_i$ for each $x \in X$, then $x \to (A_{ix})_{a_i(x)} : X \to 2^{X_i}$ and $x \to (B_{ix})_{b_i(x)} : X \to 2^{X_i}$ are both upper semicontinuous compact mappings with nonempty closed Γ_i-convex values. The conclusion of Corollary 3.2 holds from Corollary 3.1.

Remark 3.2. Corollary 3.2 improves and generalizes theorem 3.4 of Huang [13] and Theorem 3.2 of Huang [14] to locally G-convex uniform spaces.

4. Existence of Equilibria for Generalized Abstract Economies

In this section, we shall use the results presented in Section 3 to study the existence theorems of equilibria for abstract economies.

Let I be a finite or an infinite set of agents. A generalized game $\Gamma = (X_i, Y_i, A_i, B_i, P_i)_{i\in I}$ is defined as a family of ordered systems $(X_i, Y_i, A_i, B_i, P_i)$, where X_i and Y_i are nonempty topological spaces, $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ and $B_i : X \to 2^{Y_i}$ are constraint correspondences, $P_i : X \times Y \to 2^{X_i}$ is a preference correspondence where $Y = \prod_{i\in I} Y_i$. An equilibrium for generalized game Γ is a point $(\widehat{x}, \widehat{y}) \in X \times Y$ such that for each $i \in I, \widehat{x}_i \in A_i(\widehat{x}), \widehat{y}_i \in B_i(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}, \widehat{y}) = \emptyset$.

$\Gamma = (X_i, Y_i, P_i)_{i\in I}$ is said to be qualitative game if for each $i \in I, X_i$ and Y_i are strategy sets of player i, $P_i : X \times Y \to 2^{X_i}$ is a preference correspondence of player i. A maximal element of Γ is a point $(\widehat{x}, \widehat{y}) \in X \times Y$, such that $P_i(\widehat{x}, \widehat{y}) = \emptyset$ for all $i \in I$.

Lemma 4.1. (see [6]). *Let* X *and* Y *be topological spaces and* $F : X \to 2^Y$ *be a multifunction. Define a fuzzy mapping* $A : X \to \mathcal{F}(\mathcal{Y})$ *by*

$$x \to A_x(\cdot) = \mathcal{X}_{F(x)}(\cdot),$$

where $\mathcal{X}_E(\cdot)$ *is the characteristic function on set* E*, then we have*

$$(A_x)_{a(x)} = F(x), \quad \forall x \in X,$$

where $a : X \to (0, 1]$ *is a function such that* $a(x) \equiv 1$ *for all* $x \in X$ *and*

$$(A_x)_{a(x)} = \{y \in Y : A_x(y) \geq a(x)\}.$$

Theorem 4.1. *Let* $(X_i, Y_i, A_i, B_i, P_i)_{i\in I}$ *be an abstract economy,* $X = \prod_{i\in I} X_i$ *and* $Y = \prod_{i\in I} Y_i$ *such that for each* $i \in I$*, the following conditions are satisfied:*

(1) *(X_i, D_i, Γ_i) and (Y_i, D_i', Γ_i') are locally G-convex uniform spaces;*

(2) *$A_i : X \to 2^{X_i}$ and $B_i : X_i \to 2^{Y_i}$ are both upper semicontinuous compact mappings with nonempty closed Γ_i-convex values;*

(3) *$P_i : X \times Y \to 2^{X_i}$ is a closed mapping with Γ_i-convex values;*

(4) *the set $E_i = \{(x, y) \in X \times Y : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is open in $X \times Y$;*

(5) *for each $(x, y) \in X \times Y$ with $x_i \in A_i(x)$ and $y_i \in B_i(x), x_i \notin P_i(x, y)$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ such that for each $i \in I, \widehat{x}_i \in A_i(\widehat{x}), \widehat{y}_i \in B_i(\widehat{x})$ and

$$A_i(\widehat{x}) \cap P_i(\widehat{x}, \widehat{y}) = \emptyset.$$

Proof. Define three fuzzy mappings : $\widetilde{A}_i : X \to \mathcal{F}(\mathcal{X}_\rangle), \widetilde{\mathcal{B}}_\rangle : \mathcal{X} \to \mathcal{F}(\mathcal{Y}_\rangle)$ and $\widetilde{P}_i : X \times Y \to \mathcal{F}(\mathcal{X}_\rangle)$ by:

$$\begin{aligned} &\widetilde{A}_{ix}(\cdot) = \mathcal{X}_{A_i(x)}(\cdot), && x \in X, i \in I, \\ &\widetilde{B}_{ix}(\cdot) = \mathcal{X}_{B_i(x)}(\cdot), && x \in X, i \in I, \\ &\widetilde{P}_{i(x,y)}(\cdot, \cdot) = \mathcal{X}_{P_i(x,y)}(\cdot, \cdot), && (x, y) \in X \times Y, i \in I, \end{aligned}$$

where $\mathcal{X}_E$ is a characteristic function on E. By Lemma 4.1, we have

$$\begin{aligned} &(\widetilde{A}_{ix})_{a_i(x)} = A_i(x), && \text{for all } i \in I, x \in X, \\ &(\widetilde{B}_{ix})_{b_i(x)} = B_i(x), && \text{for all } i \in I, x \in X, \\ &(\widetilde{P}_{i(x,y)})_{p_i(x,y)} = P_i(x, y), && \text{for all } i \in I, (x, y) \in X \times Y, \end{aligned}$$

where $a_i(x) = b_i(x) = p_i(x, y) = 1$ for all $i \in I, (x, y) \in X \times Y$.

This shows that from an abstract economy $\Gamma = (X_i, Y_i, A_i, B_i, P_i)_{i \in I}$, we obtain an abstract fuzzy economy $\widetilde{\Gamma} = (X_i, Y_i, \widetilde{A}_i, \widetilde{B}_i, \widetilde{P}_i)_{i \in I}$. By hypotheses of Γ, the abstract fuzzy economy $\widetilde{\Gamma}$ satisfies all hypotheses of Theorem 3.1. Therefore there exists a point $(\widehat{x}, \widehat{y}) \in X \times Y$ such that for each $i \in I, \widehat{x}_i \in (\widetilde{A}_{i\widehat{x}})_{a_i(\widehat{x})}, \widehat{y}_i \in (\widetilde{B}_{i\widehat{x}})_{b_i(\widehat{x})}$ and $(\widetilde{A}_{i\widehat{x}})_{a_i(\widehat{x})} \cap (\widetilde{P}_{i(\widehat{x},\widehat{y})})_{p_i(\widehat{x},\widehat{y})} = \emptyset$. By Lemma 4.1, we have $\widehat{x}_i \in A_i(\widehat{x}), \widehat{y}_i \in B_i(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}, \widehat{y}) = \emptyset$.

Similarly, we can prove the following results.

Theorem 4.2. *Let $(X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy and $X = \prod_{i \in I} X_i$ such that for each $i \in I$, the following conditions are satisfied:*

(1) *(X_i, D_i, Γ_i) is a locally G-convex uniform space and C_i is a nonempty compact subset of X_i such that G-$co(C)$ is paracompact where $C = \prod C_i$;*

(2) *$A_i : X \to 2^{C_i}$ and $B_i : X_i \to 2^{C_i}$ are both upper semicontinuous compact mappings with nonempty closed Γ_i-convex values;*

(3) *$P_i : X \times X \to 2^{C_i}$ is a closed mapping with Γ_i-convex values;*

(4) *the set $E_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is open in $X \times Y$;*

(5) *for each $(x, y) \in X \times X$ with $x_i \in A_i(x)$ and $y_i \in B_i(x), x_i \notin P_i(x, y)$.*

Then there exists $(\widehat{x}, \widehat{y}) \in G\text{-}co(C) \times G\text{-}co(C)$ such that for each $i \in I, \widehat{x}_i \in A_i(\widehat{x}), \widehat{y}_i \in B_i(\widehat{x})$ and

$$A_i(\widehat{x}) \cap P_i(\widehat{x}, \widehat{y}) = \emptyset.$$

Corollary 4.1. *Let $(X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy and $X = \prod_{i \in I} X_i$ such that for each $i \in I$, the following conditions are satisfied:*

(1) *(X_i, D_i, Γ_i) is a compact locally G-convex uniform space;*

(2) *$A_i : X \to 2^{X_i}$ and $B_i : X_i \to 2^{X_i}$ are both compact mappings with nonempty closed Γ_i-convex values;*

(3) *$P_i : X \times X \to 2^{X_i}$ is a closed mapping with Γ_i-convex values;*

(4) *the set $E_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is open in $X \times Y$;*

(5) *for each $(x, y) \in X \times X$ with $x_i \in A_i(x)$ and $y_i \in B_i(x), x_i \notin P_i(x, y)$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in I, \widehat{x}_i \in A_i(\widehat{x}), \widehat{y}_i \in B_i(\widehat{x})$ and

$$A_i(\widehat{x}) \cap P_i(\widehat{x}, \widehat{y}) = \emptyset.$$

Corollary 4.2. *Let $(X_i, P_i)_{i \in I}$ be an abstract economy and $X = \prod_{i \in I} X_i$ such that for each $i \in I$, the following conditions are satisfied:*

(1) *(X_i, D_i, Γ_i) is a compact locally G-convex uniform spaces;*

(2) *for each $p_i(x, y) : X \times X \to 2^{X_i}$ is a closed mapping with Γ_i-convex values;*

(3) *the set $E_i = \{(x, y) \in X \times X : P_i(x, y) \neq \emptyset\}$ is open in $X \times X$;*

(4) *for each $(x, y) \in X \times X, x_i \notin P_i(x, y)$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times X$ such that $P_i(\widehat{x}, \widehat{y}) \neq \emptyset$ for all $i \in I$.

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