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INVARIANT MEAN AND SOME CORE THEOREMS FOR DOUBLE SEQUENCES

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Abstract. In this paper we define and characterize the class (V_2^σ, V_2^σ) and establish a core theorem, where V_2^σ is the space of σ-convergent double sequences $x = (x_{jk})$. We further determine a Tauberian condition for core inclusion and core equivalence.

1. Introduction

A double sequence $x = (x_{jk})$ is said to be $convergent$ in the Pringsheim sense (or P-$convergent$) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. We shall write this as

$$\lim_{j,k\to\infty} x_{jk} = \ell,$$

where j and k tending to infinity independent of each other (cf[14]). We denote by c_2, the space of P-convergent sequences. Throughout this paper limit of a double sequence means limit in the Pringsheim sense.

A double sequence x is $bounded$ if

$$\| x \| = \sup_{j,k\geq 0} |x_{jk}| < \infty.$$

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^∞, we denote the space of double sequences which are bounded convergent and by ℓ_2^∞ the space of bounded double sequences.

Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ on ℓ_∞ is said to be an $invariant\ mean$ or a

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σ-$mean$ [16] if and only if (i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$, and (iii) $\varphi(x) = \varphi((x_{\sigma(k)}))$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the pth iterate of σ at k. Note that, a σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (see [10]). Consequently, $c \subset V_\sigma$ the set of bounded sequences all of whose σ-means are equal. We say that a sequence $x = (x_k)$ is σ-$convergent$ if and only if $x \in V_\sigma$.

The idea of σ-convergence for double sequences has recently been introduced in [2]. A double sequence $x = (x_{jk})$ of real numbers is said to be σ-$convergent$ to a number L if and only if $x \in V_2^\sigma$, where

$$V_2^\sigma = \{x \in \ell_2^\infty : \lim_{p,q\to\infty} \tau_{pqst}(x) = L \text{ uniformly in } s, t; L = \sigma\text{-}\lim x\}$$

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}$$

and $\tau_{-1,q,s,t} = \tau_{p,-1,s,t} = \tau_{-1,-1,s,t} = 0$.

For $\sigma(n) = n + 1$, the set V_2^σ is reduced to the set f_2 of almost convergent double sequences [6]. The concept of almost convergence for single sequences was introduced by Lorentz [4]. Note that $c_2^\infty \subset V_2^\sigma \subset \ell_2^\infty$.

Definition 1.1. A matrix $A = (a_{mnjk})$ is said to be σ-$regular$ if $Ax \in V_2^\sigma$ for $x = (x_{jk}) \in c_2^\infty$ with σ-$\lim Ax = \lim x$, and we denote this by $A \in (c_2^\infty, V_2^\sigma)_{reg}$.

Definition 1.2. A matrix $A = (a_{mnjk})$ is said to be σ-$multiplicative$ if $Ax \in V_2^\sigma$ for $x = (x_{jk}) \in c_2^\infty$ with σ-$\lim Ax = \alpha \lim x$, and we denote this by $A \in (c_2^\infty, V_2^\sigma)_\alpha$, where $\alpha \in \mathbb{C}$. Note that if $\alpha = 1$, then σ-multiplicative matrices are reduced to σ-regular. The class of σ-multiplicative matrices was characterized by Mursaleen and Mohiuddine [9].

Definition 1.3. A matrix $A = (a_{mnjk})$ is said to be *strongly σ-regular* if $Ax \in c_2^\infty$ for $x = (x_{jk}) \in V_2^\sigma$ with $\lim Ax = \sigma\text{-}\lim x$, and we denote this by $A \in (V_2^\sigma, c_2^\infty)_{reg}$ (see [2]).

Now, we give some new definitions:

Definition 1.4. A matrix $A = (a_{mnjk})$ is said to be V_2^σ-$regular$ if $Ax \in V_2^\sigma$ for $x = (x_{jk}) \in V_2^\sigma$ with $\sigma\text{-}\lim Ax = \sigma\text{-}\lim x$, and we denote this by $A \in (V_2^\sigma, V_2^\sigma)_{reg}$.

Definition 1.5. A matrix $A = (a_{mnjk})$ is said to be *σ-uniformly positive* if

$$\lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)}\left|\sum_{m=0}^{p}\sum_{n=0}^{q} a_{\sigma^m(s),\sigma^n(t)j,k}\right| = 1.$$

Definition 1.6. Let A and B be two V_2^σ-regular matrices and

$$(*) \qquad y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}x_{jk} \text{ and } y'_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{mnjk}x_{jk}.$$

Then A and B are said to be *σ-absolutely equivalent* on ℓ_2^∞ whenever $\sigma\text{-}\lim(y_{mn} - y'_{mn}) = 0$, i.e., either (y_{mn}) and (y'_{mn}) both tend to the same σ-limit or neither of them tends to a σ-limit, but their difference tends to σ-limit zero.

For matrix transformations of double sequences and related methods, we refer to Altay - Basar [1], Hamilton [3], Patterson [12,13], Moricz [6], Mursaleen [11], Mursaleen - Edely [7], and Mursaleen - Savas [8], Robinson [15], and Zeltser [17].

In Section 2, we define the norm on V_2^σ such that it is a Banach space, and we characterize V_2^σ-regular matrices. In Section 3, we use V_2^σ-regular matrices to establish a core theorem. Since for all $x \in \ell_2^\infty$, σ-core$\{x\} \subseteq P$-core$\{x\}$, we find a Tauberian condition for the reverse inclusion in Section 4. In Section 5, we establish a core theorem for σ-absolutely equivalent matrices.

2. Characterization

First we define the norm on V_2^σ.

Theorem 2.1. *V_2^σ is a Banach space normed by*

$$(2.1.1) \qquad \|x\| = \sup_{p,q,s,t} |\tau_{pqst}(x)|$$

Proof. It can be easily verified that (2.1.1) defines a norm on V_2^σ. We show that V_2^σ is complete.

Now, let (x^b) be a Cauchy sequence in V_2^σ. Then for each j,k, (x^b_{jk}) is a Cauchy sequence in $\mathbb{R}$. Therefore $x^b_{jk} \to x_{jk}$ (say). Put $x = (x_{jk})$, given ϵ there exists an integer $N(\epsilon) = N$ say, such that, for each $b, d > N$

$$\|x^b - x^d\| < \epsilon/2.$$

Hence

$$\sup_{p,q,s,t} |\tau_{pqst}(x^b - x^d)| < \epsilon/2,$$

then for each p, q, s, t and $b, d > N$, we have

$$|\tau_{pqst}(x^b - x^d)| < \epsilon/2.$$

Taking limit $d \to \infty$, we have for $b > N$ and for each p, q, s, t

$$|\tau_{pqst}(x^b - x)| < \epsilon/2. \tag{2.1.2}$$

Now for fixed b, the above inequality holds. Since for fixed b, $x^b \in V_2^\sigma$ we get

$$\lim_{p,q\to\infty} \tau_{pqst}(x^b) = \ell$$

uniformly in s, t. For given $\epsilon > 0$, there exist positive integers p_0, q_0 such that

$$|\tau_{pqst}(x^b) - \ell| < \epsilon/2, \tag{2.1.3}$$

for $p \geq p_0, q \geq q_0$ and for all s, t. Here p_0, q_0 are independent of s, t but depend upon ϵ. Now by using (2.1.2) and (2.1.3) we get

$$\begin{aligned} |\tau_{pqst}(x) - \ell| &= |\tau_{pqst}(x) - \tau_{pqst}(x^b) + \tau_{pqst}(x^b) - \ell| \\ &\leq |\tau_{pqst}(x) - \tau_{pqst}(x^b)| + |\tau_{pqst}(x^b) - \ell| \\ &< \epsilon, \end{aligned}$$

for $p \geq p_0, q \geq q_0$ and for all s, t.

Hence $x = (x_{jk}) \in V_2^\sigma$ and V_2^σ is complete.

This completes the proof of theorem. ∎

The class of strongly σ-regular matrices, i.e. $(V_2^\sigma, c_2^\infty)_{reg}$ has been characterized in [2].

Now we characterize the matrix class (V_2^σ, V_2^σ) as well as $(V_2^\sigma, V_2^\sigma)_{reg}$. Let Z_2^σ be the subspace of V_2^σ such that $\lim_{p,q\to\infty} \tau_{pqst}(x) = 0$, uniformly in s, t, that is

$$Z_2^\sigma = \{x = (x_{jk}) \in V_2^\sigma : \lim_{p,q\to\infty} \tau_{pqst}(x) = 0, \text{ uniformly in } s, t\}. \tag{3}$$

Note that every $y \in V_2^\sigma$ can be written as

$$y = x + \ell E,$$

where $x \in Z_2^\sigma$, $\ell = \lim_{p,q} \tau_{pqst}(y)$ uniformly in s, t, and $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k.

Theorem 2.2. *A matrix $A = (a_{mnjk}) \in (V_2^\sigma, V_2^\sigma)$ if and only if* *(i)* $\|A\| = \sup_{mn} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{mnjk}| < \infty$;

(ii) $a = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} \right)_{m,n=1}^{\infty} \in V_2^{\sigma}$;

(iii) $A(\sigma - S) \in (\ell_2^{\infty}, V_2^{\sigma})$;

where S is the shift operator.

Proof. (Sufficiency). Let the conditions hold and $y = (y_{jk}) \in V_2^{\sigma}$. Then

$$y = x + \ell E \tag{2.2.1}$$

where $x = (x_{jk}) \in Z_2^{\sigma}, \ell = \lim_{p,q \to \infty} \tau_{pqst}(y)$, uniformly in s, t and $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k.

Taking A-transform in (2.2.1) we get

$$\begin{aligned} Ay &= Ax + \ell AE \\ &= Ax + \ell \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} \right)_{m,n=1}^{\infty}. \end{aligned} \tag{2.2.2}$$

If $x = (x_{jk}) \in \ell_2^{\infty}$ then by (iii) we have $A(\sigma x - x) \in V_2^{\sigma}$. Since by (i) A is bounded linear operator on ℓ_2^{∞}, we get $AZ_2^{\sigma} \subset V_2^{\sigma}$. Hence $Ax \in V_2^{\sigma}$.

Now from condition (ii) and (2.2.2), $Ay \in V_2^{\sigma}$. Therefore $A \in (V_2^{\sigma}, V_2^{\sigma})$.

Necessity. Let $A \in (V_2^{\sigma}, V_2^{\sigma})$. We know that $c_2^{\infty} \subset V_2^{\sigma} \subset \ell_2^{\infty}$ so we have $A \in (c_2^{\infty}, \ell_2^{\infty})$. Hence necessity of (i) follows. Since $E \in V_2^{\sigma}$ then $AE \in V_2^{\sigma}$. This is equivalent to

$$\left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} \right)_{m,n=1}^{\infty} \in V_2^{\sigma},$$

that is, (ii) holds. For each $x = (x_{jk}) \in \ell_2^{\infty}$, $\sigma x - x \in V_2^{\sigma}$ because

$$\varphi(\sigma x - x) = \varphi(\sigma x) - \varphi(x) = 0$$

for all σ-means φ. Hence $A(\sigma x - x) \in V_2^{\sigma}$, that is, (iii) holds.

Corollary 2.3. $A = (a_{mnjk}) \in (V_2^{\sigma}, V_2^{\sigma})_{reg}$ *if and only if conditions (i), (ii) with σ-$\lim a = 1$, and (iii) hold.*

3. Core Theorem

Let us consider the following sublinear functionals defined on ℓ_2^∞:

$$\begin{aligned} L(x) &= \limsup x, \\ Q(x) &= \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}, \\ L^*(x) &= \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{s+j,t+k}. \end{aligned}$$

For real bounded sequence $x = (x_{jk})$, we have the following cores of $x = (x_{jk})$:

P-core$\{x\}$ $= [-L(-x), L(x)]$ (see Patterson [12]),

M-core$\{x\}$ $= [-L^*(-x), L^*(x)]$ (see Mursaleen - Edely [7]),

σ-core$\{x\}$ $= [-Q(-x), Q(x)]$ (see Mursaleen-Mohiuddine [9]).

The following theorem is a double sequence version of Thoerem 3 of Mishra - Satpathy - Rath [5].

Theorem 3.1. *For every $x \in V_2^\sigma$,*

$$Q(Ax) \le Q(x) \ (\textit{or } \sigma\text{-}core\{Ax\} \subset \sigma\text{-}core\{x\}) \tag{3.1.1}$$

if and only if

(*i*) *A is V_2^σ-regular;*

(*ii*) $\limsup_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\beta(p,q,j,k,s,t)| = 1$*;*

where

$$\beta(p,q,j,k,s,t) = \frac{1}{(p+1)(q+1)} \sum_{m=0}^{p}\sum_{n=0}^{q} a_{\sigma^m(s),\sigma^n(t).j.k}.$$

Proof. (Necessity). Let (3.1.1) hold for all $x = (x_{jk}) \in V_2^\sigma$. Then

$$-Q(-x) \le -Q(-Ax) \le Q(Ax) \le Q(x)$$

i.e.

$$\sigma\text{-}\liminf x \le -Q(-Ax) \le Q(Ax) \le \sigma\text{-}\limsup x.$$

If $x \in V_2^\sigma$ then we have

$$-Q(-Ax) = Q(Ax) = \sigma\text{-}\lim x$$

i.e.

$$\sigma\text{-}\lim(Ax) = \sigma\text{-}\lim x.$$

Hence A is V_2^σ-regular, i.e. (i) holds.

Now by [11, Lemma 2.1], there is $x = (x_{jk}) \in \ell_2^\infty$ such that $\|x\| \leq 1$ and

$$\begin{aligned} &\limsup_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(p,q,j,k,s,t)x_{jk} \\ &= \limsup_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\beta(p,q,j,k,s,t)|. \end{aligned} \tag{3.1.2}$$

Hence if we define $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 1\ ;\ \text{if } j = k \\ 0\ ;\ \text{otherwise;} \end{cases}$$

then

$$\begin{aligned} 1 = q(Ax) &= \liminf_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\beta(p,q,j,k,s,t)| \\ &\leq Q(Ax) \leq Q(x) \leq \|x\| \leq 1 \end{aligned}$$

and hence (ii) is satisfied, where

$$q(x) = \liminf_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} \Big/ (p+1)(q+1).$$

Sufficiency. We know that $c_2^\infty \subset V_2^\sigma$. Thus by Theorem 2 in [9]

$$Q(Ax) \leq L(x).$$

Hence for $z \in Z_2^\sigma$, we get

$$Q(Ax + Az) \leq L(x + z).$$

Taking infimum over $z \in Z_2^\sigma$, we get

$$\inf_{z\in Z_2^\sigma} Q(Ax + Az) \leq \inf_{z\in Z_2^\sigma} \limsup_{p,q\to\infty}(x_{pq} + z_{pq}) = W(x),\ \text{say.}$$

Thus

$$\sup_{s,t} \limsup_{p,q\to\infty} \tau_{pqst}(Ax) + \inf_{z\in Z_2^\sigma} \inf_{s,t} \liminf_{p,q\to\infty} \tau_{pqst}(Az) \leq W(x). \tag{3.1.3}$$

Since $Az \in V_2^\sigma$, we can write

$$Az = \bar{z} + \ell E,$$

where $\bar{z} \in Z_2^\sigma$, $\ell = \sigma\text{-}\lim Az$ $(= \sigma\text{-}\lim z$, since A is V_2^σ-regular).

Now operating τ_{pqst} on both sides, we have

$$\tau_{pqst}(Az) = \tau_{pqst}(\bar{z}) + \tau_{pqst}(\ell E).$$

By σ-regularity we have

$$(3.1.4). \quad \liminf_{p,q\to\infty} \tau_{pqst}(Az) = \lim_{p,q\to\infty} \tau_{pqst}(\bar{z}) + \ell \lim_{p,q\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(p,q,j,k,s,t)$$

By definition of Z_2^σ

$$\lim_{p,q\to\infty} \tau_{pqst}(\bar{z}) = 0$$

uniformly in s, t. Also

$$\lim_{p,q\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(p,q,j,k,s,t) = 1.$$

From (3.1.4) we have

$$(3.1.5) \qquad \liminf_{p,q\to\infty} \tau_{pqst}(Az) = \ell$$

uniformly in s, t. Using (3.1.5) and (3.1.3) we get

$$Q(Ax) + \ell \leq W(x)$$

that is

$$Q(Ax) \leq W(x).$$

As $W(x) = Q(x)$, we get

$$Q(Ax) \leq Q(x).$$

This completes the proof of the Theorem.

4. Tauberian Condition

Since σ-core$\{x\} \subseteq$ P-core$\{x\}$, we find here the condition (Tauberian) for the reverse inclusion.

Theorem 4.1. *For $x = (x_{jk}) \in \ell_2^\infty$, if*

$$(4.1.1) \qquad \lim_{s,t}(x_{st} - x_{\sigma(s),\sigma(t)}) = 0$$

holds, then P-core$\{x\} \subseteq \sigma$*-core*$\{x\}$.

Proof. By the definition of P-core and σ-core, we have to show that $L(x) \leq Q(x)$. Let $Q(x) = \ell$. Then, for given $\epsilon > 0$, for all j, k, s, t and for large p, q it follows from the definition of Q that

$$\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} < \ell + \epsilon/2. \tag{4.1.2}$$

Now we have
(4.1.3)

$$x_{st} = x_{st} - \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} + \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}$$

$$\leq \left| x_{st} - \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} \right| + \ell + \epsilon/2.$$

Since (4.1.1) holds, for given $\epsilon > 0$ we have $|x_{st} - x_{\sigma^j(s),\sigma^k(t)}| < \epsilon/2$ for all $j, k \geq 0$. Thus

$$\left| x_{st} - \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} \right|$$

$$= \frac{1}{(p+1)(q+1)} \left| (p+1)(q+1)x_{st} - \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} \right|$$

$$\leq \frac{1}{(p+1)(q+1)} (p+1)(q+1)|x_{st} - x_{\sigma^j(s),\sigma^k(t)}|, \ j, k = \cdots$$

$$< \epsilon/2.$$

Taking $\limsup_{s,t}$ in (4.1.3), we get $L(x) \leq \ell + \epsilon$, since ϵ is arbitrary. Hence $L(x) \leq Q(x)$. This complete the proof. ∎

In case $\sigma(n) = n + 1$ in Theorem 4.1, we have

Corollary 4.2. *For* $x = (x_{jk}) \in \ell_2^\infty$, *if*

$$\lim_{s,t} (x_{st} - x_{s+1,t+1}) = 0 \tag{4.2.1}$$

holds, then P-core$\{x\} \subseteq$ *M-core*$\{x\}$.

Corollary 4.3. *If the condition (4.1.1) holds and* x *is* σ*-convergent, then* x *is convergent.*

Corollary 4.4. *If the condition (4.2.1) holds and x is almost convergent, then x is convergent.*

5. Core Theorems for Absolutely Equivalent Matrices

First we prove the following useful lemma.

Lemma 5.1. *For $x, y \in \ell_2^\infty$, if $\sigma\text{-}\lim|x-y| = 0$, then σ-core$\{x\}$ = σ-core$\{y\}$.*

Proof. If $\sigma\text{-}\lim|x-y| = 0$ then $\sigma\text{-}\lim(x-y) = \sigma\text{-}\lim(-x+y) = 0$. By definition of σ-core, we have

$$Q(x-y) = -Q(-x+y) = 0.$$

Since Q is sublinear,

$$0 = -Q(-x+y) \le -Q(-x) - Q(y).$$

Therefore,

$$Q(y) \le -Q(-x).$$

Also

$$-Q(-x) \le Q(x),$$

this implies that $Q(y) \le Q(x)$. By an argument similar as above, we can show that

$$Q(x) \le Q(y).$$

This completes the proof. ∎

Theorem 5.2. *Let $A = (a_{mnjk})$ be a V_2^σ-regular matrix. Then, $Q(Ax) \le Q(x)$ for all $x = (x_{jk}) \in \ell_2^\infty$ if and only if there is a V_2^σ-regular matrix $B = (b_{mnjk})$ such that B is a σ-uniformly positive and σ-absolutely equivalent with A on ℓ_2^∞.*

Proof. Let there be a V_2^σ-regular matrix B such that B is σ-uniformly positive and σ-absolutely equivalent with A on ℓ_2^∞. Then, by $(*)$ in Definition 1.6 and σ-absolutely equivalence of A and B, we have

$$\begin{aligned}
&\sigma\text{-}\lim|y_{mn} - y'_{mn}| \\
&= \lim_{p,q\to\infty}\sup_{s,t}\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+1)(q+1)}\sum_{m=0}^{p}\sum_{n=0}^{q}[a_{\sigma^m(s),\sigma^n(t)j,k} - b_{\sigma^m(s),\sigma^n(t)j,k}]x_{jk}\right| \\
&\le \|x\|\lim_{p,q\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+1)(q+1)}\left|\sum_{m=0}^{p}\sum_{n=0}^{q}[a_{\sigma^m(s),\sigma^n(t)j,k} - b_{\sigma^m(s),\sigma^n(t)j,k}]\right| \\
&= 0
\end{aligned}$$

uniformly in s,t. Now, by Lemma 5.1, σ-core$\{Ax\} = \sigma$-core$\{Bx\}$ for all $x \in \ell_2^\infty$.

By Theorem 3.1, we have $Q(Ax) \leq Q(x)$, since x is arbitrary.

Conversely, let $Q(Ax) \leq Q(x)$ for all $x \in \ell_2^\infty$. Then by Theorem 3.1, A is σ-uniformly positive.

Now we define a matrix $B = (b_{mnjk})$ as

$$b_{mnjk} = \frac{1}{2}(a_{mnjk} + a_{m,n,j+1,k+1})$$

for all $m,n,j,k \in \mathbb{N}$. Then it is easy to see that B is V_2^σ-regular since A is V_2^σ-regular, and

$$\sigma\text{-}\lim(Ax) = \sigma\text{-}\lim(Bx). \tag{5.2.1}$$

Further

$$\begin{aligned}
&\lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)} \left| \sum_{m=0}^{p}\sum_{n=0}^{q} b_{\sigma^m(s),\sigma^n(t)j,k} \right| \\
&\leq \frac{1}{2}\Bigg[\lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)} \left| \sum_{m=0}^{p}\sum_{n=0}^{q} a_{\sigma^m(s),\sigma^n(t)j,k} \right| \\
&\quad + \lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)} \left| \sum_{m=0}^{p}\sum_{n=0}^{q} a_{\sigma^m(s),\sigma^n(t)j+1,k+1} \right| \Bigg].
\end{aligned} \tag{5.2.2}$$

Since B is V_2^σ-regular, we have by (5.2.2) that

$$\lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)} \left| \sum_{m=0}^{p}\sum_{n=0}^{q} b_{\sigma^m(s),\sigma^n(t)j,k} \right| \leq 1.$$

Thus B is σ-uniformly positive. Further, it follows from (5.2.1) that A and B are σ-absolutely equivalent.

This completes the proof. ■

When $\sigma(n) = n+1$ in Lemma 5.1 and Theorem 5.2, we have the following corollaries:

Corollary 5.3. *Let $x, y \in \ell_2^\infty$. If f_2-$\lim |x-y| = 0$ then*

$$\text{M-core}\{x\} = \text{M-core}\{y\}.$$

Corollary 5.4. *Let A be a $(f_2, f_2)_{reg}$-matrix. Then $L^*(Ax) \leq L^*(x)$ for all $x \in \ell_2^\infty$ if and only if there is a $(f_2, f_2)_{reg}$-matrix B such that*

$$\lim_{p,q\to\infty} \sup_{s,t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(p+1)(q+1)} \left| \sum_{m=0}^{p}\sum_{n=0}^{q} b_{s+m,t+n,j,k} \right| = 1$$

and B is f_2-absolutely equivalent with A on ℓ_2^∞.

For $(f_2, f_2)_{reg}$-matrices, see Mursaleen [11].

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