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REGULARITY AND VANISHING MOMENTS OF MULTIWAVELETS*

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Abstract. We introduce the Wiener space and then consider wavelets which are not necessarily compactly supported but have a decay condition at infinity. Under the Wiener condition, several scaling functions and their dual functions have the same rate of decay at infinity. Furthermore, multiwavelets and their bi-orthogonal multiwavelets have the same rate of decay at infinity and the same number of vanishing moments.

1. Introduction

Wavelet theory has been explored extensively in both theory and applications in the last decade. The main advantage of wavelets is due to their time-frequency locations represented by the translates and dilates of a single function. It is well known that an orthonormal wavelet with compact support and certain regularity can not have any symmetry (see [2]). Geronimo *et al.* [3] constructed two functions whose translates and dilates form an orthonormal basis for $L^2(\mathbb{R})$. They are continuous, very good time-localized and of certain symmetry. In this section, we first define *a multiresolution analysis of multiplicity* r for any positive integer r (see [3]).

Definition 1.1. A *multiresolution analysis of multiplicity* r is a sequence of closed subspaces $\{V_k\}_{k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying the following properties:

(i) $V_k \subset V_{k+1}$ for all $k \in \mathbb{Z}$.

(ii) $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$ and $\bigcup_{k\in\mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$.

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(iii) $f \in V_k \Longleftrightarrow f(2\cdot) \in V_{k+1}$ for all $k \in \mathbb{Z}$.

(iv) Let $B_\phi = \{\phi_\alpha(\cdot + l) : \alpha = 1, \cdots, r;\ l \in \mathbb{Z}\}$. Then B_ϕ is a Riesz basis of V_0, i.e., B_ϕ is a basis of V_0 and there exist positive constants R_1 and R_2 such that

$$R_1 \sum_{\alpha=1}^{r} \sum_{l\in\mathbb{Z}} |C_\alpha(l)|^2 \le \left\| \sum_{\alpha=1}^{r} \sum_{l\in\mathbb{Z}} C_\alpha(l)\phi_\alpha(\cdot + l) \right\|_2^2 \le R_2 \sum_{\alpha=1}^{r} \sum_{l\in\mathbb{Z}} |C_\alpha(l)|^2$$

for any square summable $\{C_\alpha(l)\}_{l\in\mathbb{Z}}$.

The Wiener class, denoted by $\mathcal{M}(\mathbb{R})$, is defined as the set of all continuous functions on $\mathbb{R}$ satisfying the *Wiener condition* $\|f\| := \|f\|_w + \|\hat{f}\|_w < \infty$, where

$$\|f\|_w := \sum_{k\in\mathbb{Z}} \max_{x\in[0,1]} |f(x+k)| = \sum_{k\in\mathbb{Z}} \|f\chi_{k+[0,1]}\|_\infty < \infty.$$

Poisson summation formula is a beautiful result incorporating ideas from both Fourier series and Fourier transforms, and it has had many applications to number theory, partial differential equations and probability theory as well as wavelets. The following theorem proves that every member of the Wiener class enjoys Poisson summation formula (see [4, p. 246]).

Theorem 1.2. *Suppose that f belongs to the Wiener class $\mathcal{M}(\mathbb{R})$. Then the equalities*

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{R}} \hat{f}(k)e^{2\pi ikx}$$

and

$$\sum_{k\in\mathbb{Z}} f(k)e^{-2\theta ik\xi} = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + k)$$

hold pointwise, and all four series converge absolutely and uniformly on $[0, 1]$.

We shall assume throughout that ϕ_j and ψ_j belong to $\mathcal{M}(\mathbb{R})$ for all $j = 1, 2, \cdots, r$. Let $V_0 = \overline{span}\{T_n\phi_j : j = 1, 2, \cdots, r; n \in \mathbb{Z}\}$, $V_1 = \{D_2 f : f \in V_0\}$, where T_n is the translation operator that translates by n and D_2 is the dilation operator that dilates by 2. We let W_0 be the orthogonal complement of V_0 in V_1 and $\psi_j \in W_0$ for $j = 1, 2, \cdots, r$. Let ϕ and ψ be represented by the following vectors

$$\phi = (\phi_1, \cdots, \phi_r)^T; \qquad \psi = (\psi_1, \cdots, \psi_r)^T,$$

and let

$$\Phi(w) := \sum_{k\in\mathbb{Z}} \hat{\phi}(w+k)\hat{\phi}^*(w+k), \quad \Psi(w) := \sum_{k\in\mathbb{Z}} \hat{\psi}(w+K)\hat{\psi}^*(w+k),$$

where $\phi^* = \overline{\phi}^t,\ \psi^* = \overline{\psi}^t,\ \Phi$ and Ψ are $r \times r$ matrices.

Using Poisson's summation formula we get

$$\Phi(w) = \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} \phi(y-k)\phi^*(y)dy\right) e^{2\pi iwk} = \sum_{k\in\mathbb{Z}} (\phi * \phi^*)(k)e^{2\pi iwk},$$

and

$$\Psi(w) = \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} \psi(y-k)\psi^*(y)dy\right) e^{2\pi iwk} = \sum_{k\in\mathbb{Z}} (\psi * \psi^*)(k)e^{2\pi iwk}.$$

The matrices $\Phi(w),\ \Psi(w)$ are positive-semidefinite for all w. The following is an extension of Geronimo *et al.* [3], which they proved in the compactly supported case. Since the proof is similar, we omit it.

Theorem 1.3. *The collection* $\{T_k\phi_j : j = 1, 2, \cdots, r;\ k \in \mathbb{Z}\}$ *forms a Riesz basis for* V_0 *if and only if* $\Phi(w)$ *is positive-definite for all* w*, and* $\{T_k\psi_j : j = 1, 2, \cdots, r;\ k \in \mathbb{Z}\}$ *forms a Riesz basis for* W_0 *if and only if* $\Psi(w)$ *is positive-definite for all* w*.*

Assume that $\{T_n\phi_j : j = 1, 2, \cdots, r;\ n \in \mathbb{Z}\}$ and $\{T_n\psi_j : j = 1, 2, \cdots, r;\ n \in \mathbb{Z}\}$ are Riesz bases of V_0 and W_0, respectively. If we define

$$\tilde{\phi} = (\tilde{\phi}_1, \cdots, \tilde{\phi}_r)^T, \qquad \tilde{\psi} = (\tilde{\psi}_1, \cdots, \tilde{\psi}_r)^T$$

by

$$\hat{\tilde{\phi}} = \Phi^{-1}\hat{\phi}, \qquad \hat{\tilde{\psi}} = \Psi^{-1}\hat{\psi}, \tag{1.4}$$

where $\Phi = (\Phi_{ij})$, $\Psi = (\Psi_{ij})$ and $\Phi^{-1} = (\Phi^{-1}_{jk})$, $\Psi^{-1} = (\Psi^{-1}_{jk})$. Then by Parseval's identity, we have $\langle \phi_j, T_n\tilde{\phi}_k\rangle = \delta_{j,k}\delta_{n,0}$ (see [5]). Thus $\tilde{\phi}$ is a dual vector of ϕ. Similarly, we can prove that $\tilde{\psi}$ is a dual vector of ψ. Moreover, we can show that ϕ_j is in $\mathcal{M}(\mathbb{R})$ for all $j = 1, 2, \cdots, r$. Since $\Phi(w)$ is positive-definite and periodically continuous for all w, all the entries of $\Phi^{-1}(w)$ are in $\mathcal{M}(\mathbb{R})$. By Poisson's summation formula, we have

$$\Phi^{-1}(w) = \sum_{n\in\mathbb{Z}} e_n e^{-2\pi inw}, \tag{1.5}$$

where $(e_n)_{n\in\mathbb{Z}}$ is a sequence of $r \times r$ matrices with entries in $l^2(\mathbb{Z})$. From (1.4) and (1.5), we get

$$\tilde{\phi}(x) = \sum_{l\in\mathbb{Z}} e_l\phi(x-l). \tag{1.6}$$

This implies that $\tilde{\phi}_j$ is in $\mathcal{M}(\mathbb{R})$ for all $j = 1, 2, \cdots, r$. Similarly, we can prove that $\tilde{\psi}_j$ is in $\mathcal{M}(\mathbb{R})$ for all $j = 1, 2, \cdots, r$.

By definition of a multiresolution analysis of multiplicity r, we have the equations

$$\textbf{(1.7)} \qquad \phi\Big(\frac{x}{2}\Big) = \sum_{n\in\mathbb{Z}} P_n \phi(x+n), \qquad \psi\Big(\frac{x}{2}\Big) = \sum_{n\in\mathbb{Z}} Q_n \phi(x+n),$$

where $(P_n)_{n\in\mathbb{Z}}$ and $(Q_n)_{n\in\mathbb{Z}}$ are sequences of $r \times r$ matrices with entries in $l^2(\mathbb{Z})$. Moreover, we can prove that the entries of $(P_n)_{n\in\mathbb{Z}}$ and $(Q_n)_{n\in BbbZ}$ belong to $l^1(\mathbb{Z})$.

Theorem 1.8. *Let ϕ_j, ψ_j belong to $\mathcal{M}(\mathbb{R})$ and satisfy Equations* (1.6)*. If $P_n = (P_{ij}(n))$ and $Q_n = (Q_{ij}(n))$, then $\{P_{ij}(n)\}_{n\in\mathbb{Z}}$ and $\{Q_{ij}(n)\}_{n\in\mathbb{Z}}$ must be in $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$.*

Proof. Since

$$\phi_i\Big(\frac{x}{2}\Big) = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{r} P_{il}(n)\phi_l(x+n),$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}} \phi_i\Big(\frac{x}{2}\Big)\overline{\tilde{\phi}_j(x+k)}dx &= \sum_{n\in\mathbb{Z}} \sum_{l=1}^{r} P_{il}(n) \int_{\mathbb{R}} \phi_l(x+n)\overline{\tilde{\phi}_j(x+k)}dx \\ &= \sum_{n\in\mathbb{Z}} \sum_{l=1}^{r} P_{il}(n)\delta_{l,j}\delta_{n,k} = P_{ij}(k). \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |P_{ij}(k)| &= \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}} \phi_i\Big(\frac{x}{2}\Big)\overline{\tilde{\phi}_j(x+k)}dx \right| \\ &\leq \int_{\mathbb{R}} \left|\phi_i\Big(\frac{x}{2}\Big)\right| \sum_{k\in\mathbb{Z}} \left|\overline{\tilde{\phi}_j(x+k)}\right| dx \\ &\leq 2\|\phi_i\|_1 \|\tilde{\phi}_j\|_w < \infty. \end{aligned}$$

Consequently, the sequence $\{P_{ij}(n)\}_{n\in\mathbb{Z}}$ belongs to $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$. Similarly, we can prove that $\{Q_{ij}(n)\}_{n\in\mathbb{Z}}$ belongs to $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$. $\square$

Taking the Fourier transform of both sides of Equations (1.7) and letting

$$P(u) = \sum_{v\in\mathbb{Z}} P_v e^{2\pi i v u}, \qquad Q(u) = \sum_{v\in\mathbb{Z}} Q_v e^{2\pi i v u},$$

we obtain the relations

$$\hat{\phi}(u) = P\Big(\frac{u}{2}\Big)\hat{\phi}\Big(\frac{u}{2}\Big), \qquad \hat{\psi}(u) = Q\Big(\frac{u}{2}\Big)\hat{\phi}\Big(\frac{u}{2}\Big), \tag{1.9}$$

where P, Q are $r \times r$ matrices with entries in $C[0,1]$ since $\{P_{ij}(n)\}_{n\in\mathbb{Z}}$ and $\{Q_{ij}(n)\}_{n\in\mathbb{Z}}$ belong to $l^1(\mathbb{Z})$.

2. Necessary Conditions for the Decay Rate and Regularity

In this section, we continue to use the notations in Section 1. First we prove that $\tilde{\phi}_j$ has the same decay rate as ϕ_j for all $j = 1, 2, \cdots, r$. For this purpose, a lemma is needed. Let $\sigma(x) = (1+|x|)^\rho$ where $\rho > 0$ and $\|\sigma f\|_w = \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(x+k) f(x+k)|$.

Lemma 2.1. *If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ satisfy $\sum_{k\in\mathbb{Z}} \sigma(k)(|a_k| + |b_k|) < \infty$, then $\sum_{n\in\mathbb{Z}} \sigma(n)(|c_n|) < \infty$, where $c_n = \sum\limits_{k+l=n} a_k b_l$ for all $n \in \mathbb{Z}$.*

Proof.
$$\begin{aligned}\sum_{n\in\mathbb{Z}} \sigma(n)(|c_n|) &= \sum_{n\in\mathbb{Z}} \sigma(n) \Big|\sum_{k+l=n} a_k b_l\Big| \\ &\le \sum_{n\in\mathbb{Z}} \sum_{k+1=n} \sigma(k)\sigma(n-k)|a_k b_l| \\ &\le \Big[\sum_{k\in\mathbb{Z}} \sigma(k)|a_k|\Big]\Big[\sum_{l\in\mathbb{Z}} \sigma(l)|b_l|\Big] < \infty. \qquad \square\end{aligned}$$

Theorem 2.2. *Suppose that ϕ_i belongs to $\mathcal{M}(\mathbb{R})$ with $\|\sigma\phi_i\|_w < \infty$ for all $i = 1, 2, \cdots, r$. Let $a_{ij}(k) = \phi_i * \phi_j^*(k)$. Then $\sum\limits_{k\in\mathbb{Z}} \sigma(k)|a_{ij}(k)| < \infty$ fol all $i, j = 1, 2, \cdots, r$.*

Proof.

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sigma(k)|a_{ij}(k)| &= \sum_{k\in\mathbb{Z}} \sigma(k) \Big|\int_{\mathbb{R}} \phi_i(x)\phi_j(x+k)\Big| dx \\ &\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |(\sigma\phi_i)(x)|\, |(\sigma\phi_j)(x+k)| dx \\ &\le \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \int_0^1 |(\sigma\phi_i)(x+l)|\, |(\sigma\phi_j)(x+k+l)| dx \\ &\le \|\sigma\phi_i\|_w \|\sigma\phi_j\|_w < \infty. \qquad \square\end{aligned}$$

Using Theorem 2.2 we can prove that ψ_i and $\tilde{\phi}_i$ have the same decay rate.

Theorem 2.3. *If ϕ_i belongs to $\mathcal{M}(\mathbb{R})$ with $\|\sigma\phi_i\|_w < \infty$, then the function $\tilde{\phi}_i$ defined by Equation* (1.4) *satisfies $\|\sigma\tilde{\phi}_i\|_w < \infty$ for all $i = 1, 2, \cdots, r$.*

Proof. By Poisson's summation formula, we know that

$$\Phi(w) = \sum_{k\in\mathbb{Z}} \phi * \phi^*(k)e^{2\theta iwk}.$$

Let $\Phi_{ij}(w) = \sum_{k\in\mathbb{Z}} a_{ij}(k)e^{2\pi iwk}$, where $a_{ij}(k) = \phi_i * \phi_j^*(k)$. Using Theorem 2.2, we obtain a sequence $\{a_{ij}(k)\}_{k\in\mathbb{Z}}$ satisfying

$$\sum_{k\in\mathbb{Z}} \sigma(k)|a_{ij}(k)| < \infty$$

for all $i, j = 1, 2, \cdots, r$. Since $\Phi(w)$ is positive-definite and periodically continuous for all w, we have $\det(\Phi(w)) \geq m$ for all w, where $m > 0$. If we let A_{ij} be the matrix obtained from Φ by deleting its i-th row and j-th column, then

$$\Phi_{ij}^{-1}(w) = \frac{(-1)^{i+j}det(A_{ji}(w))}{det(\Phi(w))} \leq \frac{1}{m}(-1)^{i+j}det(A_{ji}(w)).$$

Thus Φ_{ij}^{-1} is bounded. Moreover, we have that Φ_{ij}^{-1} is the finite linear combination of the entries of $\Phi(w)$. Let $\Phi_{ij}^{-1}(w) = \sum_{k\in\mathbb{Z}} c_{ij}(k)e^{-2\pi iwk}$. Then by Lemma 2.1,

$$\sum_{k\in\mathbb{Z}} \sigma(k)|c_{ij}(k)| < \infty$$

for all $i, j = 1, 2, \cdots, r$. Using Equations (1.5) and (1.6), we have

$$\tilde{\phi}(x) = \sum_{l\in\mathbb{Z}} e(l)\phi(x - l),$$

where $e(l) = (c_{ij}(l))_{i,j=1}^r$. Hence for all $i = 1, 2, \cdots, r$, we obtain

$$\begin{aligned}
\|\sigma\tilde{\phi}_i\|_w &= \sum_{m\in\mathbb{Z}} \max_{0\leq x\leq 1} |\sigma(x+m) \sum_{l\in\mathbb{Z}} \sum_{k=1}^{r} c_{ik}(l)\phi_k(x - l + m)| \\
&\leq \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \sum_{k=1}^{r} \max_{0\leq x\leq 1} |(\sigma\phi_k)(x + m - l)|\, |\sigma(l)c_{ik}(l)| \\
&\leq \sum_{k=1}^{r} \left[\|\sigma\phi_k\|_w \left(\sum_{l\in\mathbb{Z}} \sigma(l)|c_{ik}(l)| \right) \right] < \infty.
\end{aligned}$$

□

Corollary 2.4. *If ϕ_j satisfies a 2-scale dilation equation*

$$\phi\Big(\frac{x}{2}\Big) = \sum_{n\in\mathbb{Z}} P_n\phi(x + n),$$

where $P_n = (P_{ij}(n))$ *for all* $i, j = 1, 2, \cdots, r$, *and* $\|\sigma\phi_i\|_w < \infty$ *for all* $i = 1, 2, \cdots, r$, *then* $\sum_{n\in\mathbb{Z}} \sigma(n)|P_{ij}(n)| < \infty$ *for all* $i, j = 1, 2, \cdots, r$.

Proof. Since

$$P_{ij}(n) = \int_{\mathbb{R}} \phi_i\Big(\frac{x}{2}\Big)\overline{\tilde{\phi}_j(x+n)}dx,$$

we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \sigma(n)|P_{ij}(n)| &\leq \sum_{n\in\mathbb{Z}} \sigma(n) \int_{\mathbb{R}} \Big|\phi_i\Big(\frac{x}{2}\Big)\tilde{\phi}_j(x+n)\Big| dx \\
&\leq \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \Big|\sigma(x)\phi_i\Big(\frac{x}{2}\Big)\Big| \; |\sigma(x+n)\tilde{\phi}_j(x+n)|dx \\
&\leq \|\sigma\tilde{\phi}_j\|_w \int_{\mathbb{R}} \frac{\sigma(x)}{\sigma(\frac{x}{2})}\sigma\Big(\frac{x}{2}\Big) \Big|\phi_i\Big(\frac{x}{2}\Big)\Big| dx \\
&\leq \|\sigma\tilde{\phi}_j\|_w \|h\|_\infty \left[2\int_{\mathbb{R}} \sigma(x)|\phi_i(x)|dx\right] \\
&\leq \|\sigma\tilde{\phi}_j\|_w \|h\|_\infty \left[2\int_0^1 \sum_{l\in\mathbb{Z}} \sigma(x+l)|\phi_i(x+l)|dx\right] \\
&\leq 2\|\sigma\tilde{\phi}_j\|_w \|h\|_\infty \|\sigma\phi_i\|_w < \infty,
\end{aligned}$$

since $h(x) = \frac{\sigma(x)}{\sigma(\frac{x}{2})}$ is bounded. □

In the following theorem, we will show that the function ψ_j has the same decay rate as the function ϕ_j for all $j = 1, 2, \cdots, r$.

Theorem 2.5. *If* ϕ_j *and* ψ_j *belong to* $\mathcal{M}(\mathbb{R})$ *with* $\|\sigma\phi_j\|_w < \infty$ *for all* $j = 1, 2, \cdots, r$, *and satisfy Equations* (1.7), *then* $\|\sigma\psi_j\|_w < \infty$ *for all* $j = 1, 2, \cdots, r$.

Proof. Let $Q_n = (Q_{ij}(n))$ and $P_n = (P_{ij}(n))$ for all $i, j = 1, 2, \cdots, r$. By Theorem 1.8, we know that $\{Q_{ij}(n)\}_{n\in\mathbb{Z}}$ and $\{P_{ij}(n)\}_{n\in\mathbb{Z}}$ belong to $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$. Hence, we have

$$\phi_j(x) = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{r} P_{jl}(n)\phi_l(2x+n)$$

and

$$\psi_j(x) = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{r} Q_{jl}(n)\phi_l(2x+n).$$

Since $\|\sigma\phi_j\|_w < \infty$, we have

$$\begin{aligned}
\|\sigma\phi_j\|_w &= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(x+k)\phi_j(x+k)| \\
&= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} \left|\sigma(x+k)\sum_{n\in\mathbb{Z}}\sum_{l=1}^{r} P_{jl}(n)\phi_l(2x+2k+n)\right| \\
&\le \sum_{l=1}^{r}\sum_{n\in\mathbb{Z}} |P_{jl}(n)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(x+k)\phi_l(2x+2k+n)| \\
&\le \sum_{l=1}^{r}\sum_{n\in\mathbb{Z}} \sigma(n)|P_{jl}(n)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(2x+2k+n)\phi_l(2x+2k+n)| \\
&\le \sum_{l=1}^{r}\sum_{n\in\mathbb{Z}} \sigma(n)|P_{jl}(n)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |(\sigma\phi_l)(2x+k)| \\
&\le 2\sum_{l=1}^{r}\sum_{n\in\mathbb{Z}} \sigma(n)|P_{jl}(n)|\ \|\sigma\phi_l\|_w < \infty,
\end{aligned}$$

and so

$$\begin{aligned}
\|\sigma\psi_j\|_w &= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(x+k)\psi_j(x+k)| \\
&= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} \left|\sigma(x+k)\sum_{n\in\mathbb{Z}}\sum_{l=1}^{r} Q_{jl}(n)\phi_l(2x+2k+n)\right| \\
&\le \sum_{l=1}^{r}\sum_{n\in\mathbb{Z}} |Q_{jl}(n)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma(x+k)\phi_l(2x+2k+n)| < \infty,
\end{aligned}$$

since $\{Q_{ij}(n)\}_{n\in\mathbb{Z}}$ belongs to $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$. Thus, we get $\|\sigma\psi_j\|_w < \infty$ for all $j = 1, 2, \cdots, r$. □

The following corollary is an application of Theorem 2.3.

Corollary 2.6. *Under the hypotheses of Theorem* 2.3*, the function* $\tilde{\psi}_j$ *defined by Equation* (1.4) *satisfies the condition* $\|\sigma\tilde{\psi}_j\|_w < \infty$ *for all* $j = 1, 2, \cdots, r$.

We extend the result of Corollary 2.4 to the following.

Corollary 2.7. *Under the hypotheses of Theorem* 2.5*, we have*

$$\sum_{n\in\mathbb{Z}} \sigma(n)|Q_{ij}(n)| < \infty$$

for all $i, j = 1, 2, \cdots, r$.

Next, we want to show that $\tilde{\phi}_j$ and ψ_j have the same regularity as ϕ_j for all $j = 1, 2, \cdots, r$. We let $C^\rho(\mathbb{R})$ be the set of functions f which satisfy the condition $\|\sigma\hat{f}\|_1 < \infty$ (see [2, p. 216]).

Theorem 2.8. *Suppose that ϕ_j and ψ_j belong to $\mathcal{M}(\mathbb{R})$ with $\|\sigma\hat{\phi}_j\|_w < \infty$ for all $j = 1, 2, \cdots, r$, and satisfy Equation* (1.7)*. Then $\tilde{\phi}_j$ and ψ_j belong to the class $C^\rho(\mathbb{R}) \bigcap \mathcal{M}(\mathbb{R})$.*

Proof. Let $\Phi^{-1}(w) = (\Phi_{ij}^{-1}(w))$. Then $\{\Phi_{ij}^{-1}(w)\}$ is bounded for all $i, j = 1, 2, \cdots, r$. By (1.4), we have

$$\hat{\tilde{\phi}}_j = \sum_{i=1}^{r} \Phi_{ji}^{-1} \hat{\phi}_i \,,$$

and thus

$$\|\sigma\hat{\tilde{\phi}}_j\|_w = \left\| \sum_{i=1}^{r} \Phi_{ji}^{-1} \sigma\hat{\phi}_i \right\|_w \le M_j \sum_{i=1}^{r} \|\sigma\hat{\phi}\|_w < \infty,$$

where $M_j = \max\limits_{1\le i\le r} |\Phi_{ji}^{-1}|$. Hence $\tilde{\phi}_j \in C^\rho(\mathbb{R})$. Next, by (1.7), we have that

$$\hat{\psi}_j(u) = \sum_{v\in\mathbb{Z}} \sum_{i=1}^{r} Q_{ji}(v)\hat{\phi}_i\Big(\frac{u}{2}\Big)e^{\pi i v u},$$

where $\{Q_{ij}(v)\}_{v\in\mathbb{Z}}$ belongs to $l^1(\mathbb{Z})$ for all $i, j = 1, 2, \cdots, r$. Thus

$$\begin{aligned}
\|\sigma\hat{\psi}_j\|_w &= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} |\sigma\hat{\psi}_j(x+k)| \\
&= \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} \left| \sigma(x+k) \sum_{v\in\mathbb{Z}} \sum_{i=1}^{r} e^{\pi i v(x+k)} Q_{ji}(v)\hat{\phi}_i\left(\frac{x+k}{2}\right) \right| \\
&\le \sum_{v\in\mathbb{Z}} \sum_{i=1}^{r} |Q_{ji}(v)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} \left| \sigma(x+k)\hat{\phi}_i\left(\frac{x+k}{2}\right) \right| \\
&\le \sum_{v\in\mathbb{Z}} \sum_{i=1}^{r} |Q_{ji}(v)| \sum_{k\in\mathbb{Z}} \max_{0\le x\le 1} \left| \sigma(x+k)\hat{\phi}_i(x+k) \right| \\
&\le \sum_{i=1}^{r} \sum_{v\in\mathbb{Z}} |Q_{ji}(v)| \; \|\sigma\hat{\phi}_i\|_w < \infty,
\end{aligned}$$

since $\{Q_{ij}(v)\}_{v\in\mathbb{Z}}$ is absolutely convergent. Thus $\psi_j \in C^\rho(\mathbb{R})$. $\square$

The following is a corollary to Theorem 2.8.

Corollary 2.9. *Under the hypotheses of Theorem* 2.8*, the function $\tilde{\psi}_j$ defined by Equation* (1.4) *has the same regularity as ϕ_j, that is, $\tilde{\psi}_j \in C^\rho(\mathbb{R})$ for all $j = 1, 2, \cdots, r$.*

Finally, we want to show that the function $\tilde{\psi}_j$ defined by Equation (1.4) has the same number of vanishing moments as ψ_j, for all $j = 1, 2, \cdots, r$.

Definition 2.10. Let $\sigma(x) = (1 + |x|)^k$, $k > m + 1$, and let D^m be the space $\{f : \sigma|f| \text{ is bounded on } \mathbb{R}\}$. A function f in D^m has N vanishing moments if f satisfies

$$M_p(f) = \int_{\mathbb{R}} x^p f(x)dx = 0 \quad for\ all \quad 0 \le p \le N, \quad but \quad \int_{\mathbb{R}} x^{N+1} f(x)dx \ne 0,$$

where p is a positive integer and $N \le m$.

Theorem 2.11. *Let $\psi_j(x) \in D^N$, and let ψ_j and $\tilde{\psi}_j$ have N_1 and N_2 vanishing moments, respectively, for all $j = 1, 2, \cdots, r$, where $N_, N_2 \le N$. Then $N_1 = N_2$.*

Proof. Since $\tilde{\psi}_j$ has N_2 vanishing moments, we have

$$\textbf{(2.12)} \quad \int_{\mathbb{R}} x^p \tilde{\psi}_j(x)dx = 0 \quad for\ all \quad 0 \le p \le N_2 \quad but \quad \int_{\mathbb{R}} x^{N_2+1} \tilde{\psi}_j(x)dx \ne 0.$$

By Equations (1.4), we obtain

$$\begin{aligned} \int_{\mathbb{R}} x^p \tilde{\psi}_j(x)dx &= \int_{\mathbb{R}} x^p \left(\sum_{i=1}^{r} \sum_{l \in \mathbb{Z}} d_{ji}(l)\psi_i(x-l) \right) dx \\ &= \sum_{i=1}^{r} \sum_{l \in \mathbb{Z}} d_{ji} \int_{\mathbb{R}} x^p \psi_i(x-l)dx = 0 \end{aligned}$$

for all $0 \le p \le N_1$, since $\{d_{ji}(l)\}_{l \in \mathbb{Z}}$ is absolutely convergent and ψ_j has the N_1 vanishing moments. By (2.12), we have $N_1 \le N_2$. On the other hand, using Equations (1.4) again we can get

$$\psi_j(x) = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} b_{ji}(k)\tilde{\psi}_i(x+k), \qquad where \quad b_{ij}(k) = \psi_i * \psi_j^*(k).$$

Similarly, we can prove that $N_2 \le N_1$. Hence $N_1 = N_2$. □

Finally, we want to construct a multiresolution analysis of multiplicity 2 (see [6]).

Example 2.13. Let $\mathcal{T}_{n,r}(\mathbb{Z})$ denote the space of cardinal B-spline functions of degree n on $\mathbb{R}$ with integer knots of multiplicity r. Write $V_0 = \mathcal{T}_{n,r}(\mathbb{Z}) \cap L^2(\mathbb{R})$ for some integers n, r with $1 \le r \le n+1$. Let $N_k^{n,r}$ denote the B-spline in $\mathcal{T}_{n,r}(\mathbb{Z})$ with support on $[t_k,\, t_{k+n+1}]$ and knots $x_k, \cdots, x_{k+n+1}$, where for $j \in \mathbb{Z}$ we define $x_k = j,\ jr \le k \le (j+1)r - 1$. In particular, we assume that $N_0^{3,2}$ and $N_1^{3,2}$ belong to $\mathcal{T}_{3,2}(\mathbb{Z})$. It was shown (see [6]) that $N_0^{3,2}$ and $N_1^{3,2}$ are in $C^1(\mathbb{R})$ with support $[0,2]$ and the Fourier transformed vector is

$$\hat{N}_3^2(w) = \frac{2}{(2\pi i w)^4} \begin{pmatrix} \frac{-15}{2} + 6\pi i w + (6 + 8\pi i w)e^{-2\pi i w} + \frac{3}{2}e^{-4\pi i w} \\ \frac{3}{2} + (6 - 12\pi i w)e^{-2\pi i w} - (\frac{15}{2} + 6\pi i w)e^{-4\pi i w} \end{pmatrix}.$$

Thus, $\hat{N}_3^2(w) = O(w^{-2})$ as $w \to \infty$. $N_0^{3,2}$ and $N_1^{3,2}$ belong to $\mathcal{M}(\mathbb{R})$. This implies that N_3^2 satisfies a 2-scale dilation equation. Also, $\hat{N}(w) = P(\frac{w}{2})\hat{N}(\frac{w}{2})$, where

$$P_3^2(z) = \frac{1}{144} \begin{pmatrix} 9z^{-1} + 128 + 9z & 53z^{-1} + 80 + z \\ z^{-1} + 80 + 53z & 9z^{-1} + 128 + 9z \end{pmatrix}$$

and $z = e^{-2\pi i w}$. Since $\Phi(w) = \sum_{k\in\mathbb{Z}} \hat{N}(w+k)\hat{N}^*(w+k)$, we have

$$\Phi_3^2(z) = \frac{1}{560} \begin{pmatrix} 9z^{-1} + 128 + 9z & 53z^{-1} + 80 + z \\ z^{-1} + 80 + 53z & 9z^{-1} + 128 + 9z \end{pmatrix}.$$

We know that $\Phi(w)$ is Hermitian and positive-semidefinite so that $\Phi(u)$ is positive-definite since $\det \Phi(w) = \frac{1}{560}(\frac{53}{2}\cos(4\pi w) + 2160\cos 2\pi w + 9210) > 0$ for all w. Hence, $\Phi(w)$ is positive-definite for all w. By Theorem 1.3, the collection $\{N_0^{3,2}(\cdot + k),\, N_1^{3,2}(\cdot + k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0. The dual vector $\tilde{N}_3^2$ of N_3^2 can be defined by

$$\tilde{N}_3^2 := (\tilde{N}_0^{3,2}, \tilde{N}_1^{3,2})^T = \Phi^{-1} N_3^2$$

with $(N_j^{3,2},\, T_n \tilde{N}_k^{3,2}) = \delta_{j,k}\delta_{n,0}$. In order to construct a compactly supported spline wavelet of multiplicity 2 in W_0 (see [5]), we first construct functions $g_i,\ i = 0,1$, having support $[0,3]$ in the space $U := \{f \in \mathcal{T}_{7,2}(\frac{1}{2}\mathbb{Z}) \colon f^{(i)}(k) = 0,\ k \in \mathbb{Z},\ j = 0,1\}$ by defining

$$g_i(x) = \sum_{j=0}^{4} c_j N_{i+j}^{7,2}(2x),$$

where c_j can be calculated from $g_i^{(j)}(k) = 0, i, j = 0, 1, k \in \mathbb{Z}$. Applying the differentiation recurrence relation for B-spline, we then define $\psi_i \in W_0,\ i = 0,1$, by letting

$$\psi_i(x) = g_i^{(4)}(x) = \sum_{j=0}^{8} d_j N_{i+j}^{3,2}(2x),$$

for some constants d_j. The functions $\psi_i,\ i = 0,1$, are also in $\mathcal{M}(\mathbb{R}) \cap C(\mathbb{R})$ and $\hat{\psi}_i(w) = O(w^{-2})$ as $w \to \infty$. $\square$

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