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ON THE REGULARITY OF SOLUTION OF THE SECOND INITIAL BOUNDARY VALUE PROBLEM FOR SCHRÖDINGER SYSTEMS IN DOMAINS WITH CONICAL POINTS

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Abstract. Some results on the unique solvability and the regularity of solution of the second initial boundary value problem for strongly Schrödinger systems in finite and infinite cylinders with the bases containing conical points are given.

1. Introduction and Notations

The first initial boundary value problem for Schrödinger systems was considered in the finite cylinders $Q_T = \Omega \times (0,T),\ T < +\infty$ (see [3]) and in the infinite cylinder $Q_\infty = \Omega \times (0,+\infty)$ (see [4,5]).

The second initial boundary value problems have been dealt with for the class heat equations (see [2]) and for general second order parabolic equations in [12] in domains with edges. This problem in domains with conical points has been investigated for general hyperbolic system in [6]. In this paper, we consider such problem for general strongly Schrödinger systems in finite and infinite cylinders $Q_T = \Omega \times (0,T)$ where $\mathbf{0} < \mathbf{T} \leqslant +\infty$ and Ω is a domain containing conical points. Our main purpose is to study the regularity of solution of the mentioned problem.

As we have known, the second initial boundary value problem is absolutely different from and more difficult than the first initial boundary value problem because we do not have the Garding inequality and general Neumann boundary condition can not always be written as clearly as general Dirichlet boundary condition. Moreover the general Hardy's inequality, that always holds for $u \in \overset{o}{H}{}^m(\Omega)$ for all integer

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number m, is not true for $u \in H^m(\Omega)$ for m arbitrary. So in oder to receive our main results we have to prove some auxiliary lemmas (Lemma 1.1, Lemmas 3.1, 3.2, 3.3). In which we give some conditions to have an inequality that plays a similar role to the Garding inequality. Moreover, by dividing m to 3 cases: $m < \frac{n}{2},\ m = \frac{n}{2},\ m > \frac{n}{2}$, where n is the dimension of Ω, we manage to apply the general Hardy's inequality to serve our purpose. With the help of these lemmas, we can apply the results for elliptic boundary value problem to deal with the regularity with respect to both of time (Theorems 2.1) and spatial variables of solution (Theorems 3.1, 3.2).

Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in $\mathbb{R}^n$. We suppose that $\Gamma = \partial\Omega\backslash\{0\}$ is a smooth manifold and Ω coincides with the cone $K = \{x : \ \frac{x}{|x|} \in G\}$ in a neighborhood of the origin point 0, where G is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. We begin by introducing some notations and functional spaces which are used fluently in the rest.

Denote $Q_T = \Omega \times (0,T),\ S_T = \Gamma \times (0,T)$, for some $0 < T \leqslant +\infty;\ \ x = (x_1,...,x_n) \in \Omega,\ u(x,t) = (u_1(x,t),...,u_s(x,t))$ is a vector complex function; $\alpha = (\alpha_1,...,\alpha_n)(\alpha_i \in \mathbb{N},\ i = 1,..,n)$ is a multi-index; $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum\limits_{i=1}^{s}|D^\alpha u_i|^2,\ u_{t^j} = (\partial^j u_1/\partial t^j,...,\partial^j u_s/\partial t^j)$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$, $C_k^s = \dfrac{k!}{s!(k-s)!}$ $(0 \leqslant s \leqslant k)$.

In this paper we will use usual functional spaces: $C^\infty(\overline{\Omega}), L_2(\Omega), H^m(\Omega)$, $H^{l,k}(Q_T)$ when $T < +\infty$ and $m, l, k \in \mathbb{N}$ (see [4,5] for the precise definitions). Denote by $H^l_\beta(\Omega)$ the space of all measurable complex functions $u(x,t)$ that satisfy

$$\|u\|_{H^l_\beta(\Omega)} = \Big(\sum_{|\alpha|\leqslant l}\int_\Omega r^{2(\beta+|\alpha|-l)}|D^\alpha u|^2 dx\Big)^{\frac{1}{2}} < +\infty.$$

Moreover, when $0 < T \leqslant +\infty$ we define:
$H^{m,l}(e^{-\gamma t}, Q_T)$ $(\gamma > 0)$- the space of all measurable complex functions $u(x,t)$ that have generalized derivatives up to order m with respect to x and up to order l with respect to t with the norm

$$\|u\|_{H^{m,l}(e^{-\gamma t},Q_T)} = \Big(\int_{Q_T}\Big[\sum_{|\alpha|\leqslant m}|D^\alpha u|^2 + \sum_{j\leqslant l}|u_{t^j}|^2\Big]e^{-2\gamma t}dxdt\Big)^{\frac{1}{2}} < +\infty.$$

$H^k(e^{-\gamma t}, Q_T)$ $(\gamma > 0)$- the space of all measurable complex functions $u(x,t)$

satisfying

$$\|u\|_{H^k(e^{-\gamma t},Q_T)} = \Big(\int\limits_{Q_T} \sum_{|\alpha|+j\leqslant k} |D^\alpha u_{tj}|^2 e^{-2\gamma t} dxdt\Big)^{\frac{1}{2}} < +\infty.$$

$H^{l,k}_\beta(e^{-\gamma t}, Q_T)$- the space of all measurable complex functions $u(x,t)$ with the norm

$$\|u\|_{H^{l,k}_\beta(e^{-\gamma t},Q_T)} = \Big[\int\limits_{Q_T} \Big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)}|D^\alpha u|^2 + \sum_{j=1}^{k} |u_{tj}|^2\Big) e^{-2\gamma t} dxdt\Big]^{\frac{1}{2}} < +\infty.$$

$H^l_\beta(e^{-\gamma t}, Q_T)$-the weighted space with the norm

$$\|u\|_{H^l_\beta(e^{-\gamma t},Q_T)} = \Big(\sum_{|\alpha|+j\leqslant l} \int\limits_{Q_T} r^{2(\beta+|\alpha|+j-l)}|D^\alpha u_{tj}|^2 e^{-2\gamma t} dxdt\Big)^{\frac{1}{2}} < +\infty.$$

$L_2(e^{-\gamma t}, Q_T)$- the space of all measurable functions $u(x,t)$ that satisfy

$$\|u\|_{L_2(e^{-\gamma t},Q_T)} = \Big(\int\limits_{Q_T} |u|^2 e^{-2\gamma t} dxdt\Big)^{\frac{1}{2}} < +\infty.$$

$L_2(e^{-\gamma t}, (0,T))$- the space of all measurable functions $u(t)$ with the norm

$$\|u\|_{L_2(e^{-\gamma t},(0,T))} = \Big(\int\limits_0^T |u|^2 e^{-2\gamma t} dt\Big)^{\frac{1}{2}} < +\infty.$$

Denote by $L^\infty(0,T; L_2(\Omega))$ the space of all measurable functions $u : (0,T) \longrightarrow L_2(\Omega), t \longmapsto u(t)$ with the norm $\|u\|_\infty = \mathrm{ess}\sup\limits_{0<t<T} \|u(t)\|_{L_2(\Omega)} < +\infty$.

For convenience, in the rest of this paper we say that $u(x,t)$ belongs to some spaces if all of its components belong to that one.

We now introduce a $2m$th-order differential operator:

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p(a_{pq}(x,t)\, D^q), \tag{1.1}$$

where a_{pq} are $s \times s$ - matrices of bounded measurable complex functions defined on $\overline{Q_T}$, $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$ (a^*_{qp} denotes the transposed conjugate matric of a_{qp}). Suppose that a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,T)$ if

$|p| = |q| = m$. Moreover, we assume that the operator L satisfies a hypothesis that given as follow (see for example [6, 9]).

Hypothesis (H). *For all* $(x,t) \in Q_T$ *and* $(\eta_p)_{|p|=m} \in C^{s.m_1^*} \setminus \{0\}$, *we have*

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\eta_q\overline{\eta_p} \geqslant C_0 \sum_{|p|=m} |\eta_p|^2,$$

where C_0 *is a positive real number, independent of* $(\eta_p)_{|p|=m}$; $\quad m_1^* = \sum_{|p|=m} 1.$

Setting $\eta_p = \xi^p\eta$ with $\xi \in \mathbb{R}^n\setminus\{0\},\ \xi^p = \xi_1^{p_1}\cdots\xi_n^{p_n}$ and $\eta \in \mathbb{C}^s\setminus\{0\}$, it follows from hypothesis (H) that $\sum\limits_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geqslant C_0|\xi|^{2m}|\eta|^2,\ (x,t) \in Q_T,$ which is equivalent to the strong ellipticity of the operator L. However, one can see easily that the condition of strong ellipticity of the operator L does not imply the hypothesis (H).

In cylinder $Q_T,\ 0 < T \leqslant +\infty$, we consider *the second initial boundary value problem for the Schrödinger system*

$$i(-1)^{m-1}L(x,t,D)u - u_t = f(x,t), \qquad (x,t) \in Q_T, \tag{1.2}$$

with initial condition

$$u(x,0) = 0, \qquad x \in \Omega, \tag{1.3}$$

where $L(x,t,D)$ is the operator in (1.1); u, f are vector functions.

The function $u(x,t)$ is called *generalized solution* in the space $H^{m,0}(e^{-\gamma t}, Q_T)$ of the second initial boundary value problem for the Schrödinger system (1.2) and initial condition (1.3) if and only if $u(x,t) \in H^{m,0}(e^{-\gamma t}, Q_T)$, satisfying

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{Q_\tau} a_{pq}(x,t)D^q u(x,t)\overline{D^p\eta}(x,t)dxdt + \int_{Q_\tau} u(x,t)\overline{\eta_t}(x,t)dxdt$$
$$= \int_{Q_\tau} f(x,t)\overline{\eta}(x,t)dxdt \tag{1.4}$$

for each $0 < \tau < T$ and all test functions $\eta(x,t) \in H^{m,1}(Q_\tau),\ \eta(x,\tau) = 0$.

For $u(x,t) \in H^{m,0}(e^{-\gamma t}, Q_T)$ set

$$B[u,u](t) = \sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_\Omega a_{pq}(x,t)D^q u(x,t)\overline{D^p u(x,t)}dx.$$

To consider the problem we need to prove the important following lemma.

Lemma 1.1. *If the hypothesis (H) holds and* a_{pq} *is bounded in* $\overline{Q_T}$ *for all* $|p|, |q| \leqslant m$*, then there exist two constants* $\mu_0 > 0$ *and* λ_0 *such that the following inequality*

$$(-1)^m B[u,u](t) \geqslant \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)}$$

is valid for all $u \in H^{m,0}(e^{-\gamma t}, Q_T)$*,* $\gamma > 0$ *and almost all* $t \in (0,T)$*.*

Proof. It follows from the hypothesis (H) that

$$\sum_{|p|=|q|=m} \int_\Omega a_{pq}(x,t)\, D^q u \overline{D^p u} dx \geqslant C_0 \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)} \tag{1.5}$$

for all $u(x,t) \in H^{m,0}(e^{-\gamma t}, Q_T)$, almost $t \in (0,T)$, with C_0 is a constant, independent of u. Using Cauchy's inequality and (1.5) one has

$$\begin{aligned} C_0 \sum_{|p|=m} \|D^p u\|^2{}_{L_2(\Omega)} &\leqslant \sum_{|p|=|q|=m} \int_\Omega a_{pq}\, D^q u \overline{D^p u} dx \\ &= (-1)^m B[u,u](t) - (-1)^m \sum_{\substack{|p|+|q|<2m \\ |p|,|q|\leqslant m}} (-1)^{|p|} \int_\Omega a_{pq}\, D^q u \overline{D^p u} dx \\ &\leqslant (-1)^m B[u,u](t) + C(\varepsilon) \|u\|^2_{H^{m-1}(\Omega)} + \varepsilon \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)}, \end{aligned}$$

where $0 < \varepsilon < C_0, \quad C(\varepsilon) > 0$. This implies

$$\|u\|^2_{H^m(\Omega)} \leqslant C_1 (-1)^m B[u,u](t) + C_2 \|u\|^2_{H^{m-1}(\Omega)}, \text{ where } C_1, C_2 > 0. \tag{1.6}$$

In another way, because Ω coincides with the cone K in a neighborhood of the origin point 0 and $\partial\Omega \backslash \{0\}$ is a smooth manifold, it follows from [1] that for all $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that the following inequality

$$\sum_{|p|=k} \|D^p u\|^2_{L_2(\Omega)} \leqslant \varepsilon \sum_{|p|=m} \|D^p u\|^2_{L_2(\Omega)} + C(\varepsilon) \|u\|^2_{L_2(\Omega)}$$

holds for all $u \in H^m(\Omega)$, $k = 1, 2, \ldots, m-1$. This implies that for all $0 < \varepsilon < 1$, there exists $C_3 = C_3(\varepsilon)$ such that $\|u\|^2_{H^{m-1}(\Omega)} \leqslant \varepsilon \|u\|^2_{H^m(\Omega)} + C_3 \|u\|_{L_2(\Omega)}$ for all $u \in H^{m,0}(e^{-\gamma t}, Q_T)$. Hence, from (1.6) we have

$$\|u\|^2_{H^m(\Omega)} \leqslant C_1 (-1)^m B[u,u](t) + C_2 [\varepsilon \|u\|^2_{H^m(\Omega)} + C_3 \|u\|^2_{L_2(\Omega)}]$$

for all $0 < \varepsilon < \min\{1, C_0, \frac{1}{C_2}\}$. Therefore we obtain

$$(-1)^m B[u,u](t) \geqslant \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)},$$

where $\mu_0 > 0,\ \lambda_0$ are constants independent of u. This proves the lemma.

From Lemma 1.1, using the transformation $u(x,t) = e^{i\lambda_0 t} v(x,t)$ if necessary, we can assume that for all $u(x,t) \in H^{m,0}(e^{-\gamma t}, Q_T)$ the operator $L(x,t,D)$ satisfies

$$(-1)^m B[u,u](t) \geqslant \mu_0 \|u\|^2_{H^m(\Omega)},\ t \in (0,T). \tag{1.7}$$

2. Existence, Uniqueness and Smoothness with Respect to Time Variable of Solutions

In this part of the paper, we will show that the second initial boundary value problem for the Schrödinger system $(1.2) - (1.3)$ is solved uniquely in the space $H^{m,0}(e^{-\gamma t}, Q_T)$ for some $\gamma > 0$. Moreover, we will prove that the smoothness with respect to time variable of solution of this problem only depends on the smoothness of the coefficients and the right side of the systems, but does not depend on the smoothness of the boundary of domains. Indeed, denote by m^* the number of multi-indexes which have order not exceeding m. Let μ_0 be the constant in (1.7). We have the following result.

Theorem 2.1. *Suppose that*

(i) $\sup\{|\dfrac{\partial a_{pq}}{\partial t}|,\ 0 \leqslant |p|,|q| \leqslant m, (x,t) \in \overline{Q_T}\} = \mu < +\infty$

$|a_{pq},\ \dfrac{\partial^k a_{pq}}{\partial t^k}| \leqslant \mu_1,\ \mu_1 =$ *const* > 0 *for* $2 \leqslant k \leqslant h+1$,

(ii) $f_{t^k} \in L^\infty(0,T; L_2(\Omega))$, *for* $k \leqslant h+1$,

and if $h \geqslant 1$ *then* $f_{t^k}(x,0) = 0$, *for* $k \leqslant h-1$.

Then for every $\gamma > \dfrac{m^*\mu}{2\mu_0}$, *the second initial boundary value problem for* $(1.2) - (1.3)$ *has exactly one generalized solution* $u(x,t)$ *in the space* $H^{m,0}(e^{-\gamma t}, Q_T)$. *Moreover,* $u(x,t)$ *has generalized derivatives with respect to* t *up to order* h *in the space* $H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)$ *and the following estimate holds*

$$\|u_{t^h}\|^2_{H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)} \leqslant C \sum_{k=0}^{h+1} \|f_{t^k}\|^2_\infty, \tag{2.1}$$

where the constant C *does not depend on* $u,\ f$.

Proof. The uniqueness of solution of this problem can prove easily by using the Ladyzhenskaya's method [see 4, 8].

Now, we establish the existence of generalized solution of the mentioned problem by Galerkin's approximate method. Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be a basic of $H^m(\Omega)$, which is orthonormal in $L_2(\Omega)$. We find an approximate solution $u^N(x,t)$ in the form $u^N(x,t)=\sum\limits_{k=1}^{N} C_k^N(t)\varphi_k(x)$, where $\{C_k^N(t)\}_{k=1}^N$ satisfy

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_\Omega a_{pq}D^q u^N\overline{D^p\varphi_l}dx-\int_\Omega u_t^N\overline{\varphi_l}dx=\int_\Omega f\overline{\varphi_l}dx, \tag{2.2}$$

$$C_l^N(0)=0,\ l=1,\dots,N. \tag{2.3}$$

It follows from $i),\ ii)$ and $(2.2)-(2.3)$ that coefficients $C_k^N(t)$ is defined uniquely and $u^N(x,0)=0$ for all $N=1,2,\dots$

Multiplying (2.2) by $\frac{d\overline{C_l^N(t)}}{dt}$, taking sum with respect to l from 1 to N, we get

$$(-1)^m\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_\Omega a_{pq}D^q u^N\overline{D^p u_t^N}dx-i\int_\Omega u_t^N\overline{u_t^N}dx=i\int_\Omega f\overline{u_t^N}dx. \tag{2.4}$$

After adding (2.4) with its complex conjugate, integrating with respect to t from 0 to τ, $0<\tau<T$, and then integrating by parts, we arrive at

$$\begin{aligned}(-1)^m B[u^N,u^N](\tau)=&(-1)^m\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{Q_\tau}\frac{\partial a_{pq}}{\partial t}D^q u^N\overline{D^p u^N}dxdt\\&-2\mathrm{Im}\Big[\int_\Omega f(x,\tau)\overline{u^N}(x,\tau)dx-\int_{Q_\tau}f_t\overline{u^N}dxdt\Big].\end{aligned}$$

This implies by Cauchy's inequality and (1.7) that for all $0<\varepsilon<\mu_0$

$$\|u^N(x,\tau)\|^2_{H^m(\Omega)}\leqslant\frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}.\int_0^\tau\|u^N(x,t)\|^2_{H^m(\Omega)}dt+\frac{1}{\varepsilon(\mu_0-\varepsilon)}\big[\|f\|^2_\infty+\tau\|f_t\|^2_\infty\big].$$

Applying Gronwall-Bellman's inequality, one gets

$$\|u^N(x,\tau)\|^2_{H^m(\Omega)}\leqslant C_1\Big[\|f\|^2_\infty+\|f_t\|^2_\infty\Big]e^{\frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon}\tau}, \tag{2.5}$$

where $C_1=\max\{\frac{m^*\mu}{\mu_0-\varepsilon},\ \frac{1}{\varepsilon(\mu_0-\varepsilon)}\}>0$.

For each $\gamma>\frac{\mu m^*}{2\mu_0}=\inf\limits_{\varepsilon\in(0,\mu_0)}\frac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}$ we can choose $\varepsilon\in(0,\mu_0)$ such that $\gamma>\frac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}$, i.e, $-2\gamma+\frac{m^*\mu+\varepsilon}{(\mu_0-\varepsilon)}<0$. Multiplying (2.5) with $e^{-2\gamma\tau}$, then integrating with respect to τ from 0 to T, we obtain

$$\|u^N\|^2_{H^{m,0}(e^{-\gamma t},Q_T)}\leqslant C\big[\|f\|^2_\infty+\|f_t\|^2_\infty\big], \tag{2.6}$$

where $C > 0$ is independent of N. Since the sequence $\{u^N\}$ is uniformly bounded in $H^{m,0}(e^{-\gamma t}, Q_T)$, we can take a subsequence, denoted also by $\{u^N\}$ for convenience, which converges weakly to a vector function $u(x,t)$ in $H^{m,0}(e^{-\gamma t}, Q_T)$. We will prove that $u(x,t)$ is a generalized solution of the problem. Since $M = \bigcup_{N=1}^{\infty}\{\sum_{l=1}^{N} d_l(t)\varphi_l(x), d_l(t) \in H^1(0,\tau),\ d_l(\tau) = 0,\ l = 1,2,...,N\}$ is dense in the space of test functions $\widehat{H}^{m,1}(Q_\tau) = \{\eta(x,t) \in H^{m,1}(Q_\tau), \eta(x,\tau) = 0\}$ for all $0 < \tau < T$ so it suffices to show that $u(x,t)$ satisfies (1.4) for all $\eta(x,t) \in M$ [1]. Taking $\eta(x,t) \in M$ arbitrarily, there exists N_0 such that η can be written in the form $\eta(x,t) = \sum_{l=1}^{N_0} d_l(t)\varphi_l(x), d_l(t) \in H^1(0,\tau),\ d_l(\tau) = 0,\ l = 1,..,N_0$. Multiplying (2.3) (with $N \geqslant N_0$) by $d_l(t)$, taking sum with respect to l from 1 to N, then integrating with respect to t from 0 to τ, we obtain

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{Q_\tau} a_{pq}D^q u^N\overline{D^p\eta}dxdt - \int_{Q_\tau} u_t^N\overline{\eta}dxdt = \int_{Q_\tau} f\overline{\eta}dxdt.$$

It is easy to check that $\int\limits_{Q_\tau} u_t^N\overline{\eta}dxdt = -\int\limits_{Q_\tau} u^N\overline{\eta_t}dxdt$, so one has

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int\limits_{Q_\tau} a_{pq}D^q u^N\overline{D^p\eta}dxdt + \int\limits_{Q_\tau} u^N\overline{\eta}_t dxdt = \int\limits_{Q_\tau} f\overline{\eta}dxdt.$$

Passing to the limit for the weakly convergent subsequence, we get

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{Q_\tau} a_{pq}D^q u\overline{D^p\eta}dxdt + \int_{Q_\tau} u\overline{\eta}_t dxdt = \int_{Q_\tau} f\overline{\eta}dxdt.$$

Hence $u(x,t)$ is a generalized solution of the second initial boundary value problem for the system $(2.3) - (2.4)$. Moreover, the weak convergence of the subsequence of $\{u^N(x,t)\}$ and (2.6) imply that this solution satisfies the following inequality

$$(2.7) \qquad \|u\|^2_{H^{m,0}(e^{-\gamma t},Q_T)} \leqslant \varliminf_{N\to\infty} \|u^N\|^2_{H^{m,0}(e^{-\gamma t},Q_T)} \leqslant C\ \left[\|f\|^2_\infty + \|f_t\|^2_\infty\right],$$

where C only depends on μ, μ_0. This completes the proof of the existence of solution.

In the following part of the proof, we shall derive that the solution u has derivatives with respect to t up to order h in $H^{m,0}(e^{-(2h+1)\gamma}, Q_T)$.

Indeed, from $i), ii)$, it follows that coefficients $C_k^N(t)$, defined uniquely by (2.3), have derivatives up to order $h+1$.

[1]The dense of the set M in the space $\widehat{H}^{m,1}(Q_\tau)$ can be proved easily by using Lemma 1.1 and arguments analogous used in the first problem (see [4,5]).

Differentiating (2.2) $(k-1)$ times with respect to t, multiplying by $\frac{d^k}{dt^k}\big(\overline{C_l^N(t)}\big)$, then taking sum with respect to l from 1 to N, we obtain

$$-i\int\limits_\Omega \left|u^N_{t^k}\right|^2 dx + (-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int\limits_\Omega a_{pq} D^q u^N_{t^{k-1}} \overline{D^p u^N_{t^k}} dx =$$

$$(-1)^{m-1} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \sum_{s=0}^{k-2} C^s_{k-1} \int\limits_\Omega \frac{\partial^{k-s-1} a_{pq}}{\partial t^{k-s-1}} D^q u^N_{t^s} \overline{D^p u^N_{t^k}} dx + i\int\limits_\Omega f_{t^{k-1}} \overline{u^N_{t^k}} dx. \tag{2.8}$$

By using $ii)$ and induction on k, we obtain

$$D^p u^N_{t^k}(x,0) = 0,\ \ k \leqslant h,\ |p| \leqslant m. \tag{2.9}$$

Now we shall prove the following inequalities

$$\left\|u^N_{t^h}(x,\tau)\right\|^2_{H^m(\Omega)} \leqslant\ Ce^{\lambda_h \tau} \sum_{k=0}^{h+1} \|f_{t^k}\|^2_\infty,\ 0<\tau<T,\ N=1,2,\ldots, \tag{2.10}$$

$$\left\|u^N_{t^h}\right\|^2_{H^{m,0}(e^{-(2h+1)\gamma t},Q_T)} \leqslant C \sum_{k=0}^{h+1} \left\|f_{t^k}\right\|^2_\infty \tag{2.11}$$

are valid with $0<\varepsilon<\mu_0$, $\lambda_s = \frac{(2s+1)m^*\mu+\varepsilon}{\mu_0-\varepsilon}$, $s \leqslant h$, where C does not depend on N, f.

It implies from $(2.5)-(2.6)$ that $(2.10)-(2.11)$ hold for $h=0$.

Now let $(2.10)-(2.11)$ be true for $h-1$, $h \geqslant 1$. We prove these also hold for h. Integrating (2.8), for $k=h+1$, with respect to t from 0 to τ, we get

$$-\, i\int\limits_{Q_\tau} \left|u^N_{t^{h+1}}\right|^2 dxdt + (-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int\limits_{Q_\tau} a_{pq} D^q u^N_{t^h} \overline{D^p u^N_{t^{h+1}}} dxdt$$

$$= (-1)^{m-1} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \sum_{s=0}^{h-1} C^s_h \int\limits_{Q_\tau} \frac{\partial^{h-s} a_{pq}}{\partial t^{h-s}} D^q u^N_{t^s} \overline{D^p u^N_{t^{h+1}}} dxdt + i\int\limits_{Q_\tau} f_{t^h} \overline{u^N_{t^{h+1}}} dxdt.$$

Adding this equation with its complex conjugate, then integrating by parts with respect to t, using $ii)$ and (2.9), we obtain

$$(-1)^m B[u^N_{t^h}, u^N_{t^h}](\tau) = (-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int\limits_{Q_\tau} \frac{\partial a_{pq}}{\partial t} D^q u^N_{t^h} \overline{D^p u^N_{t^h}} dxdt$$

$$+ (-1)^m 2\mathrm{Re} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} h \int\limits_{Q_\tau} \frac{\partial a_{pq}}{\partial t} D^q u^N_{t^h} \overline{D^p u^N_{t^h}} dxdt$$

$$+ (-1)^m 2\mathrm{Re} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \sum_{s=0}^{h-1} C^s_h \int\limits_{Q_\tau} \frac{\partial^{h-s+1} a_{pq}}{\partial t^{h-s+1}} D^q u^N_{t^s} \overline{D^p u^N_{t^h}} dxdt$$

$$+ (-1)^m 2\mathrm{Re} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \sum_{s=0}^{h-2} C^s_h \int\limits_{Q_\tau} \frac{\partial^{h-s} a_{pq}}{\partial t^{h-s}} D^q u^N_{t^{s+1}} \overline{D^p u^N_{t^h}} dxdt$$

$$- (-1)^m\ 2\mathrm{Re} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \sum_{s=0}^{h-1} C^s_h \int\limits_{\Omega} \frac{\partial^{h-s} a_{pq}}{\partial t^{h-s}} (x,\tau) D^q u^N_{t^s}(x,\tau) \overline{D^p u^N_{t^h}(x,\tau)} dx$$

$$- 2\mathrm{Im} \int\limits_{\Omega} f_{t^h}(x,\tau) \overline{u^N_{t^h}(x,\tau)} dx + 2\mathrm{Im} \int\limits_{Q_\tau} f_{t^{h+1}} \overline{u^N_{t^h}} dxdt.$$

For all $\varepsilon_1 > 0$, using Cauchy's inequality and (1.7), we have

$$[\mu_0 - ((2^h-1)\mu m^* + 1)\varepsilon_1] \|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)}$$
$$\leqslant [(2h+1)m^*\mu + ((2^{h+1}-2-h)\mu_1 m^* + 1)\varepsilon_1] \int\limits_0^\tau \left\|u^N_{t^h}(x,t)\right\|^2_{H^m(\Omega)} dt$$
$$+ C\Big[\sum_{k=0}^{h-1} \left\|u^N_{t^k}\right\|^2_{H^{m,0}(Q_\tau)} + \sum_{k=0}^{h-1} \left\|u^N_{t^k}(x,\tau)\right\|^2_{H^m(\Omega)} + \|f_{t^h}\|^2_\infty + \tau \left\|f_{t^{h+1}}\right\|^2_\infty\Big],$$

where C is a constant independent of $N,\ f$. Setting $\varepsilon = ((2^{h+1}-2-h)\mu_2 m^* + 1)\varepsilon_1$ where $\mu_2 = \max\{\mu,\ \mu_1\}$, one has for all $0 < \varepsilon < \mu_0$

$$\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)} \leqslant \lambda_h \int\limits_0^\tau \left\|u^N_{t^h}(x,t)\right\|^2_{H^m(\Omega)} dt + C\Big[\|f_{t^h}\|^2_\infty + \tau \left\|f_{t^{h+1}}\right\|^2_\infty$$
$$+ \sum_{k=0}^{h-1} \left\|u^N_{t^k}\right\|^2_{H^{m,0}(Q_\tau)} + \sum_{k=0}^{h-1} \left\|u^N_{t^k}(x,\tau)\right\|^2_{H^m(\Omega)}\Big].$$

Using the induction assumption, we get

$$\|u^N_{t^h}(x,\tau)\|^2_{H^m(\Omega)} \leqslant \lambda_h \int\limits_0^\tau \left\|u^N_{t^h}(x,t)\right\|^2_{H^m(\Omega)} dt + Ce^{\lambda_{h-1}\tau}(1+\tau) \sum_{k=0}^{h+1} \left\|f_{t^k}\right\|^2_\infty.$$

And then, by applying Gronwall-Bellman's inequality and noting that $\lambda_{h-1} < \lambda_h$, we obtain

$$\|u_{t^h}^N(x,\tau)\|^2_{H^m(\Omega)} \leqslant Ce^{\lambda_h \tau} \sum_{k=0}^{h+1} \|f_{t^k}\|^2_\infty. \tag{2.12}$$

We can choose $0 < \varepsilon < \mu_0$ such that $(2h+1)\gamma > \frac{\lambda_h}{2}$ for all $\gamma > \frac{\mu m^*}{2\mu_0}$ (because $\inf\limits_{0<\varepsilon<\mu_0} \frac{(2h+1)m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)} = \frac{(2h+1)m^*\mu}{2\mu_0} < (2h+1)\gamma\Big)$. After multiplying (2.12) with $e^{-2(2h+1)\,\gamma\,\tau}$, then integrating with respect to τ from 0 to T, we arrive at

$$\left\|u_{t^h}^N\right\|^2_{H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)} \leqslant C \sum_{k=0}^{h+1} \left\|f_{t^k}\right\|^2_\infty, \tag{2.13}$$

where C is a constant independent of N, f. Hence $(2.10) - (2.11)$ hold for h and $\left\{u_{t^h}^N\right\}$ is bounded in $H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)$. So we can choose a subsequence which converges weakly to a vector function $u^{(h)}$ in $H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)$.

On the other hand, it is easy to see that $\int\limits_{Q_T} u_{t^h}^N v dxdt = -(1)^h \int\limits_{Q_T} u^N v_{t^h} dxdt$, for all $v \in C_0^\infty(Q_T)$. Passing $N \to \infty$, it follows $\int\limits_{Q_T} u^{(h)} v dxdt = -(1)^h \int\limits_{Q_T} u v_{t^h} dxdt$, i.e., u has generalized derivatives up to order h with respect to t and $u_{t^h} = u^{(h)}$ in the space $H^{m,0}(e^{-(2h+1)\gamma t}, Q_T)$. Furthermore, by passing (2.13) to the limit for the weakly convergent subsequence, we receive inequality (2.1). The theorem is proved completely.

3. The Smoothness with Respect to Spatial Variable of Solutions

In this part of our paper, for simplicity we assume that the coefficients $a_{pq}(x,t)$ of the operator $L(x,t,D)$ are infinitely differentiable in $\overline{Q_T}$. Moreover, we also assume that $a_{pq}(x,t)$ and all its derivatives are bounded in $\overline{Q_T}$.

At first, we consider some base lemmas. Denote $\gamma_k = (2k+1)\gamma,\ \gamma_0 = \gamma$, where γ is the constant in Theorem 2.1, k is a nonnegative integer.

Lemma 3.1. *Let $f(x,t) \in L^\infty(0,T,L_2(\Omega))$, $u(x,t)$ be the generalized solution in $H^{m,0}(e^{-\gamma t}, Q_T)$ of the second initial boundary value problem for $(1.2) - (1.3)$ and $u_t \in L_2(Q_\tau)$ for all $\tau \in (0,T)$. Then for almost all $t \in (0,T)$ we have*

$$(-1)^m \int\limits_\Omega \sum_{|p|,|q|=1}^m (-1)^p a_{pq} D^p u \overline{D^p \chi} dx = i \int\limits_\Omega \big[u_t + f\big] \overline{\chi} dx,$$

where χ is an arbitrary function in $H^m(\Omega)$.

Proof. We consider the problem in a neighborhood of origin point when Ω coincides with the cone K, then $C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$ (see [1]). Let $\Big\{\chi_k(x) \in C^\infty(\overline{\Omega}), \quad k = 1, 2...\Big\}$ be a basic of $H^m(\Omega)$ and $\theta(s) \in C^\infty(\mathbb{R}^1)$ such that $\theta(0) = 0$ for $|s| > 1/2$, $\int_{-\infty}^{\infty} \theta(s)ds = 1$.

Fix $\tau \in (0, T)$ arbitrary. For each k, set $\eta(x, t) = h^{-1}\chi_k(x)\theta(\frac{|t-t'|}{h})$, where $0 < t' < \tau$, $0 < h < \min\{t', \tau - t'\}$. Then we have $\eta(x, \tau) = 0$ and $\eta \in H^{m,1}(Q_\tau)$. So from (1.4), one gets

$$\begin{aligned}(-1)^m \int\limits_{Q_\tau} \Big[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \chi_k}\Big] h^{-1}\theta(\frac{|t-t'|}{h}) dxdt \\ -i \int\limits_{Q_\tau} \big[u_t \overline{\chi_k} + f\overline{\chi_k}\big] h^{-1}\theta(\frac{|t-t'|}{h}) dxdt = 0.\end{aligned} \tag{3.1}$$

Denote $\xi(t) = (-1)^m \int\limits_\Omega \Big[\sum\limits_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \chi_k} - i(u_t + f)\overline{\chi_k}\Big] dx.$

Because $u \in H^{m,0}(e^{-\gamma t}, Q_T)$, $u_t \in L_2(Q_\tau)$ and $f \in L^\infty(0, T, L_2(\Omega))$ so $\xi(t) \in L_2(0, \tau)$ and (3.1) implies $\xi_h(t') = 0$ for almost all $t' \in (0, \tau)$. It follows $\xi(t) = 0$ in $(0, \tau) \setminus E_k$, where E_k is independent of χ_k and $mesE_k = 0$. Setting $E = \bigcup\limits_{k=1}^{\infty} E_k$, then $\xi(t) = 0$ in $(0, \tau) \setminus E$. Because of the dense property of the system $\{\chi_k\}_{k=1}^\infty$ in $H^m(\Omega)$, we have

$$(-1)^m \int\limits_\Omega \sum_{|p|,|q|=1}^{m} (-1)^p a_{pq} D^p u \overline{D^p \chi} dx = i \int\limits_\Omega \big[u_t + f\big]\overline{\chi} dx \tag{3.2}$$

for all $\chi \in H^m(\Omega)$ and almost everywhere $t \in (0, \tau)$. Since $\tau \in (0, T)$ arbitrary so (3.2) holds for all $\chi \in H^m(\Omega)$ and almost $t \in (0, T)$. This implies that $u(x, t)$ is a generalized solution in $H^m(\Omega)$ of the second boundary value problem for elliptic system

$$(-1)^m L(x, t, D)u = F(x, t), \quad \text{where} \quad F(x, t) = i(u_t + f). \tag{3.3}$$

The lemma is proved.

Lemma 3.2. *Let $u(x, t)$ be a solution in $H^{m,0}(e^{-\gamma t}, K_T)$ of the second initial boundary value problem for $(1.2) - (1.3)$ such that $u(x, t) = 0$ when $|x| > R$ =constant. Moreover, we assume that $f, f_t, f_{tt} \in L^\infty(0, T, L_2(K)), f(x, 0) = 0$. Then for almost all $t \in [0, T)$, we have*

(i) *if* $m < \dfrac{n}{2}$ *then* $u \in H^{2m}_m(K)$,

(ii) *if* $m = \dfrac{n}{2}$ *then* $u \in H^{2m}_{m+\varepsilon}(K)$, *where* $\varepsilon > 0$ *arbitrary.*

Proof. Because $f, f_t, f_{tt} \in L^\infty(0,T,L_2(K)), f(x,0) = 0$, from Theorem 2.1 we have $u_t \in H^{m,0}(e^{-\gamma_1 t}, K_T)$. That follows for each $\tau \in (0,T)$, $u_t \in L_2(K_\tau)$. Following Lemma 3.1, $u(x,t)$ is a solution of the second boundary value problem for elliptic system (3.3), where $F = i(u_t + f) \in L_2(K)$ for almost $t \in (0,\tau)$.

Denote $\Omega^k = \{x \in \Omega : 2^{-k} \leqslant |x| \leqslant 2^{-k+1}\}$, $k = 1, 2, ...$. Let k_0 be large enounugh such that $2^{-k_0+2} < R$. From the theory of the regular of solution of the boundary value problem for elliptic systems in smooth domains and near the piece smooth boundary of domain (see [11] for reference), we have $u \in H^{2m}(\Omega_{k_0})$ for almost $t \in (0,\tau)$ and the following inequality holds

$$\int\limits_{\Omega_{k_0}} |D^\alpha u(x,t)|^2 dx \leqslant C\Big[\int\limits_{\Omega_{k_0-1}\cup\Omega_{k_0}\cup\Omega_{k_0+1}} |F(x,t)|^2 dx + \int\limits_{\Omega_{k_0-1}\cup\Omega_{k_0}\cup\Omega_{k_0+1}} |u(x,t)|^2 dx\Big], |\alpha| \leqslant 2m,$$

where C is a positive constant. By choosing $k_1 > k_0$ and setting $x = (2^{k_0}/2^{k_1})x'$, one has

$$\int\limits_{\Omega_{k_0}} |D^\alpha u(x',t)|^2 dx' \leqslant C \int\limits_{\Omega_{k_0-1}\cup\Omega_{k_0}\cup\Omega_{k_0+1}} \Big[|F(x',t)|^2 \left(\frac{2^{k_0}}{2^{k_1}}\right)^{4m} dx' + |u(x',t)|^2\Big] dx'.$$

Return to the variable x, we get

$$\left(\frac{2^{k_0}}{2^{k_1}}\right)^{2|\alpha|} \int\limits_{\Omega_{k_1}} |D^\alpha u(x,t)|^2 dx \leqslant C \int\limits_{\Omega_{k_1-1}\cup\Omega_{k_1}\cup\Omega_{k_1+1}} \Big[|F(x,t)|^2 \left(\frac{2^{k_0}}{2^{k_1}}\right)^{4m} + |u(x,t)|^2\Big] dx. \tag{3.4}$$

Case 1. $m < \frac{n}{2}$. Then

$$\int\limits_K r^{-2m}|u|^2 dx \leqslant C \int\limits_K r^{-n}|u|^2 dx < +\infty. \tag{3.5}$$

It follows from (3.4) that

$$\int\limits_{\Omega_{k_1}} r^{2(|\alpha|-m)}|D^\alpha u|^2 dx \leqslant C \int\limits_{\Omega_{k_1-1}\cup\Omega_{k_1}\cup\Omega_{k_1+1}} \Big[|F(x,t)|^2 r^{2m} + r^{-2m}|u|^2\Big] dx,$$

where C is a constant independent of u, f, k_1. Taking sum with respect to $k_1 > k_0$, one has

$$\sum_{k_1>k_0}\int_{\Omega_{k_1}} r^{2(|\alpha|-m)}|D^\alpha u|^2dx \leqslant C\sum_{k_1\geqslant k_0}\int_{\Omega_{k_1}}\Big[|F(x,t)|^2r^{2m}+r^{-2m}|u|^2\Big]dx.$$

This implies

$$\int_{\bigcup\limits_{k>k_0}\Omega_k} r^{2(|\alpha|-m)}|D^\alpha u|^2dx \leqslant C\int_{\bigcup\limits_{k\geqslant k_0}\Omega_k}[|F(x,t)|^2r^{2m}+r^{-2m}|u|^2]dx.$$

Because in out of a neighborhood of conical point K is a smooth domain, so we have

$$\int_K r^{2(|\alpha|-m)}|D^\alpha u|^2dx \leqslant C\int_K[|F(x,t)|^2+r^{-2m}|u|^2]dx \tag{3.6}$$

for all $|\alpha| \leqslant 2m$, almost all $t \in (0,\tau)$. From $(3.5), (3.6)$ and $F \in L_2(K)$ we receive $u \in H^{2m}_m(K)$ for almost all $t \in (0,T)$.

Case 2. $m = \dfrac{n}{2}$. Since $u \in H^{m,0}(e^{-\gamma t}, K_T)$ so for almost all $t \in (0,T)$ one has $\int\limits_K r^0|D^\beta u|^2dx < \infty, |\beta| = m$. This implies $\int\limits_K r^\delta|D^\beta u|^2dx \leqslant R^\delta\int\limits_K|D^\beta u|^2dx < +\infty$, where $\delta > 0$ arbitrary.

For all $\delta \in (0,2m)$ we have $\delta > 0 = m - \dfrac{n}{2}$, so it follows from Lemma 7.1.1, page 268 in [9] that

$$\int_K r^{2(\delta/2-m)}|u|^2dx \leqslant C\sum_{|\beta|=m}\int_K r^\delta|D^\beta u|^2dx \leqslant C\sum_{|\beta|=m}\int_K|D^\beta u|^2dx < +\infty. \tag{3.7}$$

From the inequality (3.4), for all $|\alpha| \leqslant 2m$ one gets

$$\begin{aligned}&\int_{\Omega_{k_1}} r^{2(|\alpha|-m+\frac{\delta}{2})}|D^\alpha u|^2dx\\ &\leqslant C\int_{\Omega_{k_1-1}\cup\Omega_{k_1}\cup\Omega_{k_1+1}}\Big[|F(x,t)|^2r^{2m+\delta}+r^{-2m+\delta}|u|^2\Big]dx,\end{aligned} \tag{3.8}$$

where C is a constant independent of u, f, k_1.

By using analogous arguments used in the proof of case 1), from (3.7), (3.8) we have

$$\int\limits_K r^{2(m+\frac{\delta}{2}-|\alpha|-2m)}|D^\alpha u|^2 dx \leqslant C\int\limits_K \Big[|F(x,t)|^2+\sum_{|\beta|=m}|D^\beta u|^2\Big]dx < +\infty \tag{3.9}$$

for all $|\alpha| \leqslant 2m$, almost all $t \in (0,T)$. That is $u \in H^{2m}_{m+\frac{\delta}{2}}(K)$. The lemma is proved.

Lemma 3.3. *Assume that $u(x,t)$ is a generalized solution in $H^{m,0}(e^{-\gamma t}, K_T)$ of the second initial boundary value problem for $(1.2)-(1.3)$, such that $u(x,0)=0$ for $|x|>R=$ constant. Moreover, assume that $f_{t^k} \in L^\infty(0,T,L_2(K))$, $k \leqslant h+2$; $f_{t^k}(x,0)=0$, $k \leqslant h$. Then for almost $t \in [0,T)$ we have:*

(i) *if $m > \dfrac{n}{2}$ and n is odd, then $u = \sum\limits_{|\alpha|\leqslant[m-\frac{n}{2}]} c_\alpha(t)x^\alpha + u_0$, where $u_0 \in H^{2m}_m(K)$, $d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t},(0,T)), j \leqslant h,\ |\alpha| \leqslant [m-\frac{n}{2}]$,*

(ii) *if $m > \dfrac{n}{2}$ and n is even, then $u = \sum\limits_{|\alpha|\leqslant[m-\frac{n}{2}-1]} c_\alpha(t)x^\alpha + u_0$, where $u_0 \in H^{2m}_{m+\varepsilon}(K)$, $\varepsilon > 0$ arbitrary, $d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t},(0,T)), j \leqslant h;\ |\alpha| \leqslant [m-\frac{n}{2}-1]$.*

Proof. (i) *Let $m > \dfrac{n}{2}$ and n be odd.*

Set $l = [m-\dfrac{n}{2}]$. Because $u \in H^{m,0}(e^{-\gamma t}, K_T)$, so for almost $t \in [0,T)$ we have

$$\int\limits_K r^0 |D^{\beta_1}D^{\beta_2}u|^2 dx < +\infty, |\beta_1| = m-l-1, |\beta_2| = l+1.$$

It is clear that $(m-l-1)-\frac{n}{2} < 0$, so by applying Lemma 7.1.1 of [9] we obtain

$$\int\limits_K r^{-2(m-l-1)}|D^{\beta_2}u|^2 dx \leqslant C\sum_{|\beta_1|=m-l-1}\int\limits_K |D^{\beta_1}D^{\beta_2}u|^2 dx \leqslant C\sum_{|\alpha|=m}\int\limits_K |D^\alpha u|^2 dx < +\infty.$$

Since $-2(m-l-1) < 2-n$, then by setting $v = \dfrac{\partial^l u}{\partial x_1^{l_1}...\partial x_n^{l_n}}$, $l_1+...+l_n = l$ and applying Lemma 4.10, page 244 of [7], we receive $v = C^l_{l_1...l_n}(t) + u^{(l)}$, where $u^{(l)}$ satisfies

$$\int_K r^{-2(m-l-1)-2}|u^{(l)}|^2 dx \leqslant C\sum_{|\alpha|=m}\int\limits_K |D^\alpha u|^2 dx < +\infty. \tag{3.10}$$

It implies from (3.10) that $u^{(l)} \in H^0_{l-m}(K)$ and then $C^l_{l_1...l_n}(t) \in L_2(e^{-\gamma t}, (0,T))$. By setting $v^{(l)} = u - \sum\limits_{l_1+...+l_n=l} \dfrac{C^l_{l_1...l_n}(t)x_1^{l_1}...x_n^{l_n}}{l_1!...l_n!}$, then $\partial^l v^{(l)}/\partial x_1^{l_1}...\partial x_n^{l_n} = u^{(l)}$. From (3.10) one has

$$\int\limits_K r^{-2(m-l-1)-2}|\frac{\partial^l v^{(l)}}{\partial x_1^{l_1}...\partial x_n^{l_n}}|^2 dx \leqslant C \sum_{|\alpha|=m} \int\limits_K |D^\alpha u|^2 dx < +\infty.$$

It can be rewritten in the form $\int\limits_K r^{-2(m-l-1)-2}\left|\nabla\left(\frac{\partial^{l-1}v^{(l)}}{\partial x_1^{(l-1)_1}...\partial x_n^{(l-1)_n}}\right)\right|^2 dx < +\infty$, where $(l-1)_1 + ... + (l-1)_n = l-1$. Applying Lemma 4.10 of [7] again, one gets $\frac{\partial^{(l-1)}v^{(l)}}{\partial x_1^{(l-1)_1}...\partial x_n^{(l-1)_n}} = C^{(l-1)}_{(l-1)_1...(l-1)_n}(t) + u^{(l-1)}$, where $u^{(l-1)}$ satisfies

$$\int\limits_K r^{-2(m-l-1)-4}|u^{(l-1)}|^2 dx \leqslant C\sum_{|\alpha|=m} \int\limits_K |D^\alpha u|^2 dx < +\infty. \tag{3.11}$$

It follows that $u^{(l-1)} \in H^0_{l-m-1}(K)$ and $C^{(l-1)}_{(l-1)_1...(l-1)_n}(t) \in L_2(e^{-\gamma t}, (0,T))$. Setting $v^{(l-1)} = v^{(l)} - \sum\limits_{(l-1)_1+...+(l-1)_n=l-1} \dfrac{C^{l-1}_{(l-1)_1...(l-1)_n}(t)x_1^{(l-1)_1}...x_n^{(l-1)_n}}{(l-1)_1!...(l-1)_n!}$, we have $\partial^{l-1}v^{(l-1)}/\partial x_1^{(l-1)_1}...\partial x_n^{(l-1)_n} = u^{(l-1)}$. From (3.11) one has

$$\int\limits_K r^{-2(m-l-1)-4}\left|\partial^{l-1}v^{(l-1)}/\partial x_1^{(l-1)_1}...\partial x_n^{(l-1)_n}\right|^2 dx \leqslant C\sum_{|\alpha|=m} \int\limits_K |D^\alpha u|^2 dx < +\infty.$$

Repeating that arguments we receive a functions sequence $\{v^{(j)}\}$ with

$$v^{(j)} = v^{(j+1)} - \sum_{j_1+...+j_n=j} \frac{C^j_{j_1...j_n}(t)x_1^{j_1}...x_n^{j_n}}{j_1!...j_n!},\ j = 1,...,l;\ \ v^{(l+1)} = u.$$

It follows that

$$u = \sum_{|\alpha| \leqslant [m-\frac{n}{2}]} c_\alpha(t)x^\alpha + u^{(0)}, \tag{3.12}$$

where $u^{(0)}$ satisfies

$$\int\limits_K r^{-2m}|u^{(0)}|^2 dx \leqslant C \sum_{|\alpha|=m} \int\limits_K |D^\alpha u|^2 dx < +\infty, \tag{3.13}$$

and $c_\alpha(t) \in L_2(e^{-\gamma t}, (0,T))$ for all $|\alpha| \leqslant l$.

It is easy to see from (3.12) that $u^{(0)}$ is a generalized solution of the second boundary value problem for the following system

$$(-1)^m L(x,t,D)u^{(0)} = F_1,$$

where $F_1 = F + (-1)^{m-1} \sum\limits_{|\alpha| \leqslant [m-\frac{n}{2}]} c_\alpha(t) L(x,t,D)x^\alpha \in L_2(K)$ for almost all $t \in (0,T)$ and $u^{(0)}(x,0) = 0$. From the inequality (3.4) and (3.13), repeating arguments that used to prove the inequality (3.6), we obtain

$$\begin{aligned}\int\limits_K r^{2(|\alpha|-m)}|D^\alpha u^{(0)}|^2 dx &\leqslant C \int\limits_K [|F_1(x,t)|^2 r^{2m} + r^{-2m}|u^{(0)}|^2]dx \\ &\leqslant C \int\limits_K [|F_1(x,t)|^2 + \sum_{|\alpha|=m} |D^\alpha u(x,t)|^2]dx < +\infty.\end{aligned}$$

This implies $u^{(0)} \in H^{2m}_m(K)$ for almost all $t \in [0,T)$.

Now we prove $d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t}, (0,T)), j \leqslant h$. We consider some following cases.

Case 1. $|\alpha| = l = [m - \dfrac{n}{2}]$. From (3.12) we have

$$D^\alpha u = d_\alpha c_\alpha(t) + D^\alpha u^{(0)},\ |\alpha| = l,\ d_\alpha = constant \neq 0. \tag{3.14}$$

In another way, from Theorem 2.1, Lemma 3.1 and the assumptions of this lemma, there exist $u_t \in H^{m,0}(e^{-\gamma_1 t}, Q_T)$, $u_{tt} \in H^{m,0}(e^{-\gamma_2 t}, Q_T)$ and u_t is a solution of elliptic problem for the system

$$(-1)^m L u_t = F_t + (-1)^{m-1} L_t u,$$

where $L_t = \sum\limits_{|p|,|q| \leqslant m} D^p(\dfrac{\partial a_{pq}}{\partial t} D^q)$. It is easy to see that $F_t + (-1)^{m-1} L_t u \in L_2(K)$ for almost $t \in (0,T)$. Repeating arguments used above for function u we get

$$D^\alpha u_t = d_\alpha \widetilde{c}_\alpha(t) + D^\alpha \widetilde{u}^{(0)},\ \text{where } \widetilde{u}^{(0)} \in H^{2m}_m(K),\ \widetilde{c}_\alpha(t) \in L_2(e^{-\gamma_1 t}, (0,T)).$$

Since $u(x,0) = 0$, then

$$D^\alpha u = d_\alpha \int_0^t \widetilde{c}_\alpha(t)dt + \int_0^t D^\alpha \widetilde{u}^{(0)} dt. \tag{3.15}$$

Comparing (3.14), (3.15) one gets

$$I(t) = d_\alpha \left[\int_0^t \widetilde{c}_\alpha(t)dt - c_\alpha(t) \right] = D^\alpha u^{(0)} - \int_0^t D^\alpha \widetilde{u}^{(0)} dt.$$

Because $u^{(0)}$, $\widetilde{u}^{(0)}$ are in $H^{2m}_m(K)$, so $\int_K r^{2(m+|\alpha|-2m)}|I(t)|^2dx < +\infty$. It implies that $|I(t)|^2 \int_K r^{-n}dx < \int_K r^{2(m+|\alpha|-2m)}|I(t)|^2dx < +\infty$ because of $|\alpha|-m < -\dfrac{n}{2}$. Thus, $I(t) = 0$ or $\int_0^t \widetilde{c}_\alpha(t)dt = c_\alpha(t)$ for almost $t \in (0,T)$. Therefore $dc_\alpha(t)/dt = \widetilde{c}_\alpha(t) \in L_2(e^{-\gamma_1 t},(0,T))$.

Assume that there exist $d^jc_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t},(0,T)), j \leqslant h-1$. We will prove that $d^{j+1}c_\alpha(t)/dt^{j+1} \in L_2(e^{-\gamma_{j+1}t},(0,T))$. Indeed, because of results of Theorem 2.1 we have $u_{t^{j+1}} \in H^{m,0}(e^{-\gamma_{j+1}t},K_T), j \leqslant h-1$. By using similar arguments used in the proof of (3.12), one gets $D^\alpha u_{t^{j+1}} = d_\alpha \widehat{c_\alpha}(t) + D^\alpha \widehat{u}^{(0)}$, where $\widehat{u}^{(0)} \in H^{2m}_m(K),\ \widehat{c_\alpha}(t) \in L_2(e^{-\gamma_{j+1}t},(0,T))$. From the induction hypotheses, we have $D^\alpha u_{t^j} = d_\alpha \dfrac{d^jc_\alpha(t)}{dt^j} + D^\alpha u^{(0)}_{t^j}$. This implies $\dfrac{d^{j+1}c_\alpha(t)}{dt^{j+1}} = \widehat{c_\alpha}(t) \in L_2(e^{-\gamma_{j+1}t},(0,T))$.

Case 2. $|\alpha| = l-1 = [m-\dfrac{n}{2}]-1$. From (3.12) we have

$$D^\alpha u = d_\alpha c_\alpha(t)x + d_{\alpha+1}c_{\alpha+1}(t) + D^\alpha u^{(0)}, \tag{3.16}$$

where $d_\alpha, d_{\alpha+1} = constant \neq 0$. Using similar method that applied to prove (3.12), one gets

$$D^\alpha u_t = d_\alpha \widetilde{c}_\alpha(t)x + d_{\alpha+1}\widetilde{c}_{\alpha+1}(t) + D^\alpha \widetilde{u}^{(0)},$$

where $\widetilde{u}^{(0)} \in H^{2m}_m(K),\ \widetilde{c}_\alpha(t) \in L_2(e^{-\gamma_1 t},(0,T))$. So we have

$$D^\alpha u = d_\alpha \int_0^t \widetilde{c}_\alpha(t)xdt + d_{\alpha+1}\int_0^t \widetilde{c}_{\alpha+1}(t)dt + \int_0^t D^\alpha \widetilde{u}^{(0)}dt. \tag{3.17}$$

From case 1 one has $d_{\alpha+1}\int_0^t \widetilde{c}_{\alpha+1}(t)dt = d_{\alpha+1}c_{\alpha+1}(t)$. Combining (3.16) and (3.17) we obtain

$$J_1(x,t) = d_\alpha \left[\int_0^t \widetilde{c}_\alpha(t)dt - c_\alpha(t)\right]x = D^\alpha u^{(0)} - \int_0^t D^\alpha \widetilde{u}^{(0)}dt.$$

Because $u^{(0)}$, $\widetilde{u}^{(0)}$ are in $H^{2m}_m(K)$, so $\int_K r^{2(m+|\alpha|-2m)}|J_1(x,t)|^2dx < +\infty$. Setting $J(t) = d_\alpha\left[\int_0^t \widetilde{c}_\alpha(t)dt - c_\alpha(t)\right]$, we have $\int_K r^{2(|\alpha|-m)}|J(t)|^2|x|^2dx < +\infty$. This implies $\int_K r^{2(|\alpha|-m+1)}|J(t)|^2dx < +\infty$. Since $|\alpha|-m+1 < -\frac{n}{2}$ so $|J(t)|^2\int_K r^{-n}dx < +\infty$. Hence $J(t) = 0$ or $dc_\alpha(t)/dt = \widetilde{c}_\alpha(t) \in L_2(e^{-\gamma_1 t},(0,T))$.

The same results about $d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t}, (0,T))$ can be received by using induction method on $j,\ j \leqslant h$.

When $|\alpha| < [m - \dfrac{n}{2}] - 1$, by induction on $|\alpha|$ we obtain $d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t}, (0,T))$ for all $j \leqslant h, |\alpha| \leqslant [m - \dfrac{n}{2}]$.

(ii) *Let* $m > \dfrac{n}{2}$ *and* n *be even.*

Then $m = \dfrac{n}{2} + k + 1$, k is a nonnegative integer. Because $u \in H^{m,0}(e^{-\gamma t}, K_T)$, so for almost all $t \in [0,T)$ we have

$$\int\limits_K r^0 |D^{\beta_1} D^{\beta_2} u|^2 dx < +\infty, |\beta_1| = m - k - 1, |\beta_2| = k + 1.$$

It follows

$$\int_K r^\delta |D^{\beta_1} D^{\beta_2} u|^2 dx \leqslant \sum_{|\alpha|=m} \int\limits_K |D^\alpha u|^2 dx < +\infty, \quad \text{where } \delta > 0 \text{ arbitrary.}$$

Using similar method used in the part *(i)*, we achieve $u = \sum\limits_{|\alpha| \leqslant [m - \frac{n}{2} - 1]} c_\alpha(t) x^\alpha + u^{(0)}$, where $u^{(0)} \in H^{2m}_{m+\delta/2}(K), d^j c_\alpha(t)/dt^j \in L_2(e^{-\gamma_j t}, (0,T)),\ j \leqslant h, |\alpha| \leqslant [m - \dfrac{n}{2} - 1]$. The lemma is proved completely.

Let ω be a local coordinate system on S^{n-1}. The principal part of the operator $L(x,t,D)$ at origin point 0 can be rewritten in the form:

$$L_0(0,t,D) = r^{-2m} Q(\omega, t, rD_r, D_\omega), (D_r = i\partial/\partial r), \tag{3.18}$$

where Q is a linear operator with smooth coefficients.

Denote by $\lambda(t)$ an eigenvalue of Neumann problem for the system

$$Q(\omega, t, \lambda(t), D_\omega) v(\omega) = 0,\ \omega \in G. \tag{3.19}$$

It is well known in [9] that for every $t \in (0,T)$, the spectrum of this problem is on enumerable set of eigenvalue.

Theorem 3.1. *Let* $u(x,t)$ *be a generalized solution in* $H^{m,0}(e^{-\gamma t}, K_T)$ *of the second initial boundary value problem for* $(1.2) - (1.3)$ *such that* $u \equiv 0$ *when* $|x| > R =$ *constant. In addition, suppose that the strip*

$$m - \frac{n}{2} \leqslant \mathit{Im}\lambda \leqslant 2m - \frac{n}{2}$$

does not contain any point of the spectrum of Neumann problem for system (3.19) *for all* $t \in [0,T)$. *Then*

(i) *if* $f_{t^k} \in L^{\infty}(0, T, L_2(K)), k \leqslant 2m+1$, *and* $f_{t^k}(x, 0) = 0, k \leqslant 2m-1$, *then* $u \in H_0^{2m}(e^{-\gamma_{2m}t}, K_T)$ *for* $m \leqslant \dfrac{n}{2}$.

(ii) *if* $f_{t^k} \in L^{\infty}(0, T, L_2(K)), k \leqslant 2m+2$, *and* $f_{t^k}(x, 0) = 0, k \leqslant 2m$, *then* $u \in H^{2m}(e^{-\gamma_{2m}t}, K_T)$ *for* $m > \dfrac{n}{2}$.

Proof. We rewrite (1.2) in the form

$$(-1)^m L_0(0, t, D)u = \widehat{F}(x, t), \text{ where} \tag{3.20}$$

$$\widehat{F}(x, t) = i(u_t + f) + (-1)^m [L_0(0, t, D) - L(x, t, D)]u. \tag{3.21}$$

(i) $m \leqslant \dfrac{n}{2}$. We need to prove

$$\sum_{|\alpha|+k \leqslant 2m} \int_{K_T} r^{2(|\alpha|+k-2m)} |D^{\alpha} u_{t^k}|^2 e^{-2\gamma_{2m}t} dxdt < +\infty. \tag{3.22}$$

Case 1. $k = 2m$. From Theorem 2.1 we have $\int\limits_{K_T} |u_{t^{2m}}|^2 e^{-2\gamma_{2m}t} dxdt < +\infty$, so (3.22) is valid for $k = 2m$.

Case 2. $k \leqslant 2m-1$.

• If $m < \frac{n}{2}$ then by following the Lemma 3.2 we have $u \in H_{\underline{m}}^{2m}(K)$ for almost $t \in (0, T)$. In another way, because a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in (0, T)$ if $|p| = |q| = m$ then $|a_{pq}(x, t) - a_{pq}(0, t)| \leqslant C|x|$, for all $t \in [0, T)$ and C is a constant. Therefore, from (3.21) and the hypotheses of this theorem, one gets $\widehat{F} \in H_{m-1}^0(K)$. Since in the strip $m - \dfrac{n}{2} \leqslant \mathrm{Im}\lambda \leqslant m + 1 - \dfrac{n}{2}$ there is no spectral point of Neumann problem for (3.19) for all $t \in [0, T)$, then following Theorem 3.2 page 37 in paper [10], one gets $u \in H_{m-1}^{2m}(K)$ and satisfies

$$\|u\|^2_{H^{2m}_{m-1}(K)} \leqslant C\Big[\|F\|^2_{H^0_{m-1}(K)} + \|u\|^2_{H^{2m}_{\underline{m}}(K)}\Big],$$

for almost $t \in (0, T)$, where C is a positive constant.

From $u \in H_{m-1}^{2m}(K)$, by using similar arguments as giving above, we have $F \in H_{m-2}^0(K)$. In another way, there is no spectral point of Neumann problem for (3.19) for all $t \in [0, T)$ in the strip $m+1-\dfrac{n}{2} \leqslant \mathrm{Im}\lambda \leqslant m+2-\dfrac{n}{2}$. Hence following the results of elliptic problem (see Theorem 3.2 in [10]) one gets $u \in H_{m-2}^{2m}(K)$ satisfying

$$\|u\|^2_{H^{2m}_{m-1}(K)} \leqslant C\Big[\|F\|^2_{H^0_{m-2}(K)} + \|u\|^2_{H^{2m}_{\underline{m}}(K)}\Big].$$

Repeating the arguments above we receive $u \in H_0^{2m}(K)$ and the inequality

$$\|u\|^2_{H_0^{2m}(K)} \leqslant C\Big[\|F\|^2_{L_2(K)} + \|u\|^2_{H_m^{2m}(K)}\Big] \tag{3.23}$$

holds for almost $t \in (0,T)$. It follows that

$$\begin{aligned}\|u\|^2_{H_0^{2m,0}(e^{-\gamma_{2m}t},K_T)} &\leqslant C\left[\int_0^T \|F\|^2_{L_2(K)}e^{-2\gamma_{2m}t}dt + \int_0^T \|u\|^2_{H_m^{2m}(K)}e^{-2\gamma_{2m}t}dt\right]\\ &\leqslant C\sum_{k=0}^{2}\|f_{t^k}\|^2_\infty.\end{aligned}$$

Then (3.22) is valid for $k=0$.

We assume that $u_{t^k} \in H_0^{2m,0}(e^{-\gamma_{2m}t},K_T)$ for all $k \leqslant s-1,\ s \leqslant 2m-1$.

Denote $L_{t^s} = \sum\limits_{|p|,|q|\leqslant m} D^p(a_{pq})_{t^s}D^q, \quad v = u_{t^s}$. Differentiating (1.2) k times with respect to t, we have

$$(-1)^m Lv = i(u_{t^{s+1}} + f_{t^s}) + (-1)^{m-1}\sum_{k=1}^{s} C_s^k L_{t^k}u_{t^{s-k}} = F_s.$$

Repeating the arguments used for function u and the induction assumption we receive $F_s \in L_2(K)$ for almost $t \in (0,T)$ and $v \in H_0^{2m,0}(e^{-\gamma_{2m}t},K_T)$. This implies $u_{t^s} \in H_0^{2m,0}(e^{\gamma_{2m}t},K_T)$ for all $s \leqslant 2m-1$. Therefore, if $m < \frac{n}{2}$ then from case 1 and case 2 we have $u \in H_0^{2m}(e^{-\gamma_{2m}t},K_T)$.

• If $m = \frac{n}{2}$ then it follows from the Lemma 3.2 that $u \in H^{2m}_{m+\varepsilon}(K)$ for all $\varepsilon \in (0,1)$, almost all $t \in (0,T)$. Because the straight line $\mathrm{Im}\lambda = m - \frac{n}{2}$ does not contain any point from the spectrum of Neumann problem for (3.19) for all $t \in (0,T)$ then for each $t \in (0,T)$ there exists $\varepsilon(t) > 0$ such that the strip $m - \varepsilon(t) - \dfrac{n}{2} \leqslant \mathrm{Im}\lambda \leqslant m - \dfrac{n}{2}$ does not contain any spectral point of Neumann problem for (3.19). In another way, we also have $\widehat{F} \in H_m^0(K)$. This implies from Theorem 3.2 of [10] that $u \in H_m^{2m}(K)$ satisfying the inequality $\|u\|^2_{H_m^{2m}(K)} \leqslant C\Big[\|F\|^2_{H_m^0(K)} + \|u\|^2_{H^{2m}_{m+\varepsilon}}\Big]$. Repeating the proof for the case $m < \frac{n}{2}$ we achieve $u_{t^s} \in H_0^{2m,0}(e^{-\gamma_{2m}t},K_T)$ for all $s \leqslant 2m-1$. So for $m = \frac{n}{2}$, $u \in H_0^{2m}(e^{-\gamma_{2m}t},K_T)$ too.

(ii) $m > \dfrac{n}{2}$.

Case 1. If n is odd then there exists a nonnegative integer l such that $\dfrac{n}{2} + l < m < \dfrac{n}{2} + l + 1$. From Lemma 3.3 we get

$$u = \sum_{|\alpha|\leqslant[m-\frac{n}{2}]} c_\alpha(t)x^\alpha + u_0,\ u_0 \in H_m^{2m}(K). \tag{3.25}$$

We rewrite (1.2) in the form $(-1)^{m-1}Lu_0=F+(-1)^m\sum\limits_{|\alpha|\leqslant l}c_\alpha(t)L(x,t,D)x^\alpha$. By using analogous arguments in the proof of the part (i), we get $u_0\in H_0^{2m}(e^{-\gamma_{2m}t},K_T)$. Because $d^jc_\alpha(t)/dt^j\in L_2(e^{-\gamma_j t},(0,T)), j\leqslant 2m$, we have $u\in H^{2m}(e^{-\gamma_{2m}t},K_T)$.

Case 2. If n is even then $m=\dfrac{n}{2}+l+1$, l is a nonegative integer. From Lemma 3.3 one has $u=\sum_{|\alpha|\leqslant[m-\frac{n}{2}-1]}c_\alpha(t)x^\alpha+u_0,\quad u_0\in H^{2m}_{m+\varepsilon}(K),\varepsilon>0$. Because the straight line Im$\lambda=m-\dfrac{n}{2}$ does not contain any point from the spectrum of Neumann problem for the system (3.19) for all $t\in[0,T)$, so for each $t\in[0,T)$ there exists an $\varepsilon(t)$ such that in the strip $m-\varepsilon(t)-\frac{n}{2}\leqslant$ Im$\lambda\leqslant m-\dfrac{n}{2}$ there is no eigenvalue of Neumann problem for system (3.19). In other hand, u_0 is a solution of the second boundary value problem for system $(-1)^mL(0,t,D)u_0=F_0$, where $F_0=i[u_t+f]+(-1)^{m-1}[L(0,t,D)-L(x,t,D)]u_0\in H^0_m(K)$. Hence from the results of elliptic problem (see [10]) and above arguments, one gets $u_0\in H^{2m}_m(K)$ for all most $t\in[0,T)$. And then, by using similar arguments as used in the case 1, we receive that $u_0\in H_0^{2m}(e^{-\gamma_{2m}t},K_T)$ and $d^jc_\alpha(t)/dt^j\in L_2(e^{-\gamma_{2m}t},(0,T))$. This implies $u\in H^{2m}(e^{-\gamma_{2m}t},K_T)$.

The theorem is proved completely.

From Lemma 3.1 and similar method used in Theorem 3.1 we achieve the regular of the solution of the second initial boundary value problem for $(1.2)-(1.3)$. It is given as follow.

Theorem 3.2. *Let $u(x,t)$ be a generalized solution of the second initial boundary value problem for the system $(1.2)-(1.3)$ in the space $H^{m,0}(e^{-\gamma t},Q_T)$ and assume that in the strip*

$$m-\frac{n}{2}\leqslant \mathit{Im}\lambda\leqslant 2m-\frac{n}{2}$$

there is no point from the spectrum of Neumann problem for system (3.19) for all $t\in[0,T)$. Then

(i) *if $f_{t^k}\in L^\infty(0,T,L_2(\Omega)),k\leqslant 2m+1$, and $f_{t^k}(x,0)=0,k\leqslant 2m-1$, then $u\in H_0^{2m}(e^{-\gamma_{2m}t},Q_T)$ for $m\leqslant\dfrac{n}{2}$.*

(ii) *if $f_{t^k}\in L^\infty(0,T,L_2(\Omega)),k\leqslant 2m+2$, and $f_{t^k}(x,0)=0,k\leqslant 2m$, then $u\in H^{2m}(e^{-\gamma_{2m}t},Q_T)$ for $m>\dfrac{n}{2}$.*

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