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SOLVING A TWO VARIABLES FREE BOUNDARY PROBLEM ARISING IN A PERPETUAL AMERICAN EXCHANGE OPTION PRICING MODEL

Ming-Long Liu* and Hsuan-Ku Liu

Abstract. We investigate an American exchange option (AEO) pricing problem. Under the perfect market assumption, an AEO pricing problem can be modeled as a free boundary problem (FBP). The FBP is converted into an integral equation by using the Green's function. When the expiration date tends to infinity, we obtain a time-invariant constant of the exercise boundary. Moreover, we provide a pricing formula for valuating the early exercise premium of the perpetual AEO.

1. INTRODUCTION

An exchange option is an option which gives the holder a right but not obligation to exchange one asset to another. An American exchange option (AEO) is an exchange option which can be exercised at any time prior to the expiration date T. Let S_1 and S_2 be the price of asset 1 and asset 2. Under the risk neutral probability measure, the stochastic processes for the asset price is assumed to be

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i dw_i,\ i = 1, 2,$$

where r, q_i, and σ_i are the constant risk-free interest rate, the continuous dividend rate of the asset i and the volatility the i-th asset, respectively. Here, dw_1 and dw_2 are the Wiener processes of asset 1 and asset 2 respectively, and their correlation coefficient is $\mathrm{corr}(dw_1, dw_2) = \rho dt$. In this paper, we shall study the FBP arising in an AEO pricing problem.

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*Corresponding author

To derive the pricing operator of the exchange option, we first recall the European exchange option (EEO) pricing problems [8]. According to the definition of the exchange option, the final payoff of an EEO is given by

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0), \tag{1}$$

where $V(S_1, S_2, t)$ denotes the value of EEO at the time $t \leq T$. As the suggestion of Margrabe [8], the value of EEO satisfies the linear homogeneous property in S_1 and S_2, that is

$$V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t).$$

We apply Euler Theorem to the function $V(S_1, S_2, t)$ and obtain the following equation:

$$V - S_1 \frac{\partial V}{\partial S_1} - S_2 \frac{\partial V}{\partial S_2} = 0. \tag{2}$$

This means that a portfolio of holding $\frac{\partial V}{\partial S_i}$ units of asset $i,\ i = 1, 2$, becomes a replication of EEO.

By applying Itô Lemma to (2) and considering the instantaneous return with the dividend rate of both assets, we obtain the following pricing equation

$$V_t + \mathcal{L}_{BS} V = 0,\ t < T, \tag{3}$$

where the operator $\mathcal{L}_{BS}$ is defined as

$$\mathcal{L}_{BS} V = \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1 S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1 S_2} + \frac{1}{2}\sigma_2^2 S_2^2 V_{S_2 S_2} - q_1 S_1 V_{S_1} - q_2 S_2 V_{S_2}.$$

The EEO's price is then the solution of (3) with terminal condition (1).

For the AEO pricing problem, since this style of option can be exercised at any time $t < T$, the AEO pricing problem in [2] is formulated as an FBP. Let $P(S_1, S_2, t)$ be the value of an AEO, then $P(S_1, S_2, t)$ must satisfy the following inequality under the no arbitrage condition:

$$P(S_1, S_2, t) \geq \max(S_1 - S_2, 0),\ 0 \leq t \leq T.$$

Let $X_f(t)$ be the smallest value of $\frac{S_1}{S_2}$ such that

$$P(S_1, S_2, t) > \max(S_1 - S_2, 0),\ 0 \leq t \leq T.$$

Then, at any given time t, the (S_1, S_2)-plane can be separated into two distinct regions as follows:

$$\mathbf{S}(t) = \left\{ (S_1, S_2) \in \mathcal{R}^+ \times \mathcal{R}^+ | \frac{S_1}{S_2} \geq X_f(t) \right\}, \tag{4}$$

$$\mathbf{C}(t) = \left\{ (S_1, S_2) \in \mathcal{R}^+ \times \mathcal{R}^+ | \frac{S_1}{S_2} < X_f(t) \right\}, \tag{5}$$

where $\mathcal{R}^+$ denotes the set of nonnegative real numbers. The regions $\mathbf{S}$ and $\mathbf{C}$ are called the early exercise region and the holding region, and the ratio $X_f(t)$ is called the early exercise ratio.

Thus the value of the AEO and the early exercise ratio $X_f(t)$ together are the solution of the FBP (6)-(8).

$$P_t + \mathcal{L}_{\mathcal{BS}} P = 0, \qquad (S_1, S_2) \in \mathbf{C}(t),\ 0 \le t \le T, \tag{6}$$

$$P(S_1, S_2, T) = \max(S_1 - S_2, 0), \qquad t = T, \tag{7}$$

$$P(S_1, S_2, t) = S_1 - S_2, \qquad (S_1, S_2) \in \mathbf{S}(t),\ 0 \le t \le T. \tag{8}$$

The pricing formula of European exchange option is first proposed by Margrabe [8]. He considered the case of $q_1 = q_2 = 0$ and showed that the value of AEO on the non-dividend assets price is equal to the European counterpart. The case of American option on multiple underlying assets with dividend has received gradual attention in recent years. Broadie and Detemple [2] characterized the optimal exercise regions and proposed valuation equations of different types of American options on two or more assets. However, the exact solution for these valuation equations has not been found yet. Recently, the numerical method and the simulation-based approaches are proposed to valuate the American-style options. Carr [3] generalized the Geske-Johnson approach [4] to AEO on dividend-paying assets. Longstaff and Schwartz [7] proposed the least squares Monte Carlo that can compute good price estimates very fast in practice. Rogers [9] proposed a dual representation of American option prices. and computed upper bounds on several types of American option using Monte Carlo simulation.

In this paper, we study the FBP (6)-(8) arising in an AEO pricing model. We first show that the early exercise ratio is a strictly decreasing function and $X_f(T) = \max(1, \frac{q_2}{q_1})$. The early exercise ratio of the FBP is derived to satisfy an integral equation by using the Green's function of $\mathcal{L}_{BS}$. When the time-horizon is infinite, the early exercise ratio is a time-invariant constant. Our main contribution is to obtain the value of the time-invariant constant and propose a pricing formula for valuating the exercise premium of the perpetual AEO.

The paper is organized as follows. In Section 2, we investigate two properties of the early exercise ratio for the case of $T < \infty$. In Section 3, the FBP is converted into an integral equation by using the Green's function. In Section 4, we give the formula of early exercise ratio and a pricing formula of perpetual AEO and some numerical results are given in the example. Finally, some conclusions are given in Section 5.

2. Properties of the Early Exercise Ratio

In this section, we will show that the early exercise ratio satisfies the following two properties for the case of $T < \infty$.

Proposition 2.1. *The early exercise ratio $X_f(T-\tau)$ tends to $\max(1, \frac{q_2}{q_1})$, as the remaining date tends to zero, that is*

$$\lim_{\tau \to 0} X_f(T-\tau) = \max(1, \frac{q_2}{q_1}).$$

Proof. We first show that $X_f(T-\tau) > 1$, for all τ in $(0, T)$. If $X_f(T-\tau) < 1$, by the (8), we have

$$P(S_1, S_2, \tau) = S_1 - S_2 \leq S_2(X_f(T-\tau) - 1) < 0.$$

This implies that the option price is negative at the point of early exercise ratio. There is a contradiction of that the option price is positive. When $\tau = 0$ and $S_1 > S_2$, the final payoff function is $P(S_1, S_2, T) = S_1 - S_2$. Substituting it into (6), we have

$$\frac{\partial P}{\partial \tau}(S_1, S_2, 0) = q_2 S_2 - q_1 S_1.$$

Note that $\frac{\partial P}{\partial \tau}(S_1, S_2, 0) \geq 0$ such that the AEO is kept alive until near expiry. Since the value of the alive AEO is monotone increasing as a function of remaining time τ. Thus, we have

$$\frac{S_1}{S_2} \leq \frac{q_2}{q_1}$$

for $(S_1, S_2) \in \mathbf{C}(T-\tau)$.

By the definition, the early exercise ratio $X_f(T-\tau)$ is defined by the smallest ratio of S_1 and S_2 in $\mathbf{S}(T-\tau)$. Hence, we conclude that for $q_1 < q_2$

$$\lim_{\tau \to 0^+} X_f(T-\tau) = \frac{q_2}{q_1}.$$

Proposition 2.2. *$X_f(t)$ is a strictly decreasing function.*

Proof. Note that when (P, X_f) is a solution of FBP (6)-(8), we have that $P(S_1, S_2, t)$ is a strictly decreasing function of t, that is $P_t(S_1, S_2, t) < 0$ for $(S_1, S_2) \in \mathbf{S}(t)$.

We first show that $X_f(t)$ is a nonincreasing function. Since $P(S_1, S_2, t)$ is a strictly decreasing function of t, for any time t_0 we have $P(S_1, S_2, t) < P(S_1, S_2, t_0)$ for $t < t_0$. So $\mathbf{C}(t) = \{(S_1, S_2) \in \mathcal{R}^+ \times \mathcal{R}^+ | \frac{S_1}{S_2} < X_f(t_0)\}$ contains in $\mathbf{C}(t_0)$ for all $t < t_0$, and

$$X_f(t) = \inf\{\tfrac{S_1}{S_2} | (S_1, S_2) \in \mathbf{C}(t)\} \geq \inf\{\tfrac{S_1}{S_2} | (S_1, S_2) \in \mathbf{C}(t_0)\} = X_f(t_0).$$

Next, we show that $X_f(t)$ is strictly decreasing. If not, there exist t_1 and t_2 with $0 < t_1 < t_2 < T$, we have $X_f(t_1) = X_f(t_2)$. This implies that $P(S_1, S_2, t) = S_1 - S_2$ for $\frac{S_1}{S_2} = X_f(t)$ and t in $[t_1, t_2]$. This contradicts to $P_t < 0$ for (S_1, S_2) in $\mathbf{S}(t)$ since $X_f(t)$ is the largest value of $\frac{S_1}{S_2}$ such that $P(S_1, S_2, t) > (S_1 - S_2)^+$.

3. The Integral Equation

Before defining an explicit solution of the perpetual AEO in the next section, we first derive an integral equation by defining new variables $y_i = \frac{-1}{\sigma_i}(q_i + \frac{1}{2}\sigma_i^2)\tau + \frac{1}{\sigma_i}\ln(S_i)$, $i = 1, 2$ and $\tau = T - t$. Let $p(y_1, y_2, \tau) = P(S_1, S_2, t)$ and $x_f(\tau) = X_f(t)$, then the original pricing problem (6)-(8) can be written in the following dimensionless form:

(9) $$\frac{\partial p}{\partial \tau} = \mathcal{L}p, \quad \text{in } 0 < \tau < T,\ y_1 - \frac{\sigma_2}{\sigma_1} y_2 \le x_f(\tau),$$

(10) $$p(y_1, y_2, 0) = (e^{\sigma_1 y_1} - e^{\sigma_2 y_2})^+, \quad \text{at } \tau = 0,$$

(11) $$p(y_1, y_2, \tau) = e^{(q_1 + \frac{1}{2}\sigma_1^2)\tau + \sigma_1 y_1} - e^{(q_2 + \frac{1}{2}\sigma_2^2)\tau + \sigma_2 y_2}, \quad \text{at } y_1 - \frac{\sigma_2}{\sigma_1} y_2 = x_f(\tau),$$

where $\mathcal{L}$ is the operator given as

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + 2\rho\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2^2}\right).$$

Under this transformation, the relation between $X_f(t)$ and $x_f(\tau)$ is defined as

(12) $$x_f(\tau) = \frac{1}{\sigma_1}\left(\ln(X_f(T - \tau)) + (q_2 + \frac{1}{2}\sigma_2^2 - q_1 - \frac{1}{2}\sigma_1^2)\tau\right).$$

By imposing (11) into (9), the problem (9)-(11) can be converted into a non-homogeneous equation as follows:

(13) $$p_\tau - \mathcal{L}p = \begin{cases} 0, & \text{if } y_1 - \frac{\sigma_2}{\sigma_1} y_2 \le x_f(\tau), \\ q_1 e^{(q_1 + \frac{1}{2}\sigma_1^2)\tau + \sigma_1 y_1} - q_2 e^{(q_2 + \frac{1}{2}\sigma_2^2)\tau + \sigma_2 y_2}, & \text{if } y_1 - \frac{\sigma_2}{\sigma_1} y_2 \ge x_f(\tau). \end{cases}$$

Before solving the FBP of (6)-(8), we shall find out the representation of $x_f(t)$.

Theorem 3.1. *The early exercise ratio $x_f(t)$ satisfies the following integral equation:*

$$
\begin{aligned}
e^{(q_1+\frac{1}{2}\sigma_1^2)\tau+\sigma_1 x_f(\tau)} - e^{(q_2+\frac{1}{2}\sigma_2^2)\tau} = {} & e^{\sigma_1 x_f(\tau)+\frac{1}{2}\sigma_1^2\tau} N(\frac{\bar{a}_1}{\sigma}) - e^{\frac{1}{2}\sigma_2^2\tau} N(\frac{\bar{a}_2}{\sigma}) \\
& + e^{\sigma_1 x_f(\tau)+\frac{1}{2}\sigma_1^2\tau} \int_0^\tau q_1 e^{q_1 s} N(\frac{\bar{a}_3}{\sigma}) ds \\
& - e^{\frac{1}{2}\sigma_2^2\tau} \int_0^\tau q_2 e^{q_2 s} N(\frac{\bar{a}_4}{\sigma}) ds,
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
\bar{a}_1 &= \frac{1}{\sqrt{\tau}}(\sigma_1 x_f(\tau) + (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau), \\
\bar{a}_2 &= \frac{1}{\sqrt{\tau}}(\sigma_1 x_f(\tau) + (\rho\sigma_1\sigma_2 - \sigma_2^2)\tau), \\
\bar{a}_3 &= \frac{1}{\sqrt{\tau - s}}(\sigma_1(x_f(\tau) - x_f(s)) + (\sigma_1^2 - \rho\sigma_1\sigma_2)(\tau - s)), \\
\bar{a}_4 &= \frac{1}{\sqrt{\tau - s}}(\sigma_1(x_f(\tau) - x_f(s)) + (\rho\sigma_1\sigma_2 - \sigma_2^2)(\tau - s)),
\end{aligned}
$$

and N is the cumulative distribution function of a standard normal random variable $N(0, 1)$.

In order to prove the theorem, we need the following two lemma:

Lemma 3.1. [6]. *The Green's function $\phi(x, y, \tau)$ for (13) is given by*

$$
\begin{aligned}
&\phi(y_1, y_2, \tau; \xi_1, \xi_2, s) \\
&= \frac{1}{2\pi(\tau - s)} \frac{1}{\sqrt{1-\rho^2}} exp(-\frac{(y_1-\xi_1)^2 - 2\rho(y_1-\xi_1)(y_2-\xi_2) + (y_2-\xi_2)^2}{2(1-\rho^2)(\tau-s)}).
\end{aligned}
$$

Lemma 3.2. *Let*

$$
\varphi(u_1, u_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} exp(-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)})
$$

be the probability density function of the standard bivariate normal distribution with covariant correlation ρ. Then, we have

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{a+\frac{\sigma_1}{\sigma_2}u_2} \varphi(u_1, u_2) du_1 du_2 = \int_{-\infty}^{\frac{\sigma_1}{\sigma}a} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = N\left(\frac{\sigma_1}{\sigma}a\right),
$$

for any real number a.

Proof.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{a+\frac{\sigma_2 u_2}{\sigma_1}} \frac{1}{2\pi\sqrt{1-\rho^2}}\varphi(u_1,u_2)du_1du_2\\
&=\int_{-\infty}^{a}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{\sigma_1^2 v_1^2}{2\sigma^2}}e^{-\frac{\sigma^2}{2\sigma_1^2(1-\rho^2)}\left(v_2+\frac{\sigma_1\sigma_2-\sigma_1\rho}{\sigma^2}v_1\right)^2}dv_2dv_1\\
&=\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sigma_1 v_1)^2}{2\sigma^2}}\frac{\sigma_1}{\sigma}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dwdv_1\\
&=\int_{-\infty}^{\frac{a\sigma_1}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv=N(\frac{a\sigma_1}{\sigma})
\end{aligned}$$

where $v_1=u_1-\frac{\sigma_2}{\sigma_1}u_2$, $v_2=u_2$, and $w=\frac{\sigma(v_2+\frac{\sigma_1\sigma_2-\sigma_1\rho}{\sigma}v_1)}{\sigma_1\sqrt{1-\rho^2}}$. ■

Proof of Theorem 3.1. We apply Green's function to $p(x,y,\tau)$ as well as the fact that ϕ is in a domain bounded by the optimal exercise boundary and the line $\tau=0$ obtaining that

$$\begin{aligned}
&p(y_1,y_2,\tau)\\
&=\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty}(e^{\sigma_1\xi_1}-e^{\sigma_2\xi_2})\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1d\xi_2\\
&\quad+\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty}(q_1e^{\sigma_1\xi_1}-q_2e^{\sigma_2\xi_2})\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1d\xi_2ds
\end{aligned}\tag{15}$$

We separate (15) into following four integrals:

$$\begin{aligned}
&p(y_1,y_2,\tau)\\
&=\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty}e^{\sigma_1\xi_1}\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1d\xi_2\\
&\quad-\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty}e^{\sigma_2\xi_2}\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1d\xi_2\\
&\quad+\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty}q_1e^{\sigma_1\xi_1}\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1d\xi_2ds\\
&\quad-\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty}q_2e^{\sigma_2\xi_2}\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1d\xi_2ds\\
&=I^{(1)}(y_1,y_2,\tau)-I^{(2)}(y_1,y_2,\tau)+I^{(3)}(y_1,y_2,\tau)-I^{(4)}(y_1,y_2,\tau).
\end{aligned}\tag{16}$$

In order to rewrite the integrals $I^{(1)}$-$I^{(4)}$, we let $\sqrt{\tau}u_1-\sigma_1\tau=y_1-\xi_1$ and $\sqrt{\tau}u_2-\rho\sigma_1\tau=y_2-\xi_2$ in $I^{(1)}$, $\sqrt{\tau}u_1-\rho\sigma_2\tau=y_1-\xi_1$ and$\sqrt{\tau}u_2-\sigma_2\tau=y_2-\xi_2$ in

$I^{(2)}$, $\sqrt{\tau-s}u_1-\sigma_1(\tau-s)=y_1-\xi_1$ and$\sqrt{\tau-s}u_2-\rho\sigma_1(\tau-s)=y_2-\xi_2$ in $I^{(3)}$ and $\sqrt{\tau-s}u_1-\rho\sigma_2(\tau-s)=y_1-\xi_1$ and$\sqrt{\tau-s}u_2-\sigma_2(\tau-s)=y_2-\xi_2$ in $I^{(4)}$, and then the integrals $I^{(1)}$-$I^{(4)}$ can be written as the following equations:

(17)
$$\begin{aligned}
I^{(1)} &= e^{(\sigma_1 y_1+\frac{1}{2}\sigma_1^2\tau)}\int_{-\infty}^{\infty}\int_{-\infty}^{a_1(y_1,y_2,\tau)+bu_2}\varphi(u_1,u_2)du_1du_2,\\
I^{(2)} &= e^{(\sigma_2 y_2+\frac{1}{2}\sigma_2^2\tau)}\int_{-\infty}^{\infty}\int_{-\infty}^{a_2(y_1,y_2,\tau)+bu_2}\varphi(u_1,u_2)du_1du_2,\\
I^{(3)} &= e^{(\sigma_1 y_1+\frac{1}{2}\sigma_1^2\tau)}\int_0^{\tau}q_1e^{q_1s}\int_{-\infty}^{\infty}\int_{-\infty}^{a_3(y_1,y_2,\tau,x_f(\tau))+bu_2}\varphi(u_1,u_2)du_1du_2ds,\\
I^{(4)} &= e^{(\sigma_2 y_2+\frac{1}{2}\sigma_2^2\tau)}\int_0^{\tau}q_2e^{q_2s}\int_{-\infty}^{\infty}\int_{-\infty}^{a_4(y_1,y_2,\tau,x_f(\tau))+bu_2}\varphi(u_1,u_2)du_1du_2ds,
\end{aligned}$$

where

$$\begin{aligned}
a_1(y_1,y_2,\tau) &= \frac{1}{\sqrt{\tau}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\sigma_1-\rho\sigma_2)\tau),\\
a_2(y_1,y_2,\tau) &= \frac{1}{\sqrt{\tau}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\rho\sigma_2-\frac{\sigma_2^2}{\sigma_1})\tau),\\
a_3(y_1,y_2,\tau,s,x_f(s)) &= \frac{1}{\sqrt{\tau-s}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\sigma_1-\rho\sigma_2)(\tau-s)-x_f(s)),\\
a_4(y_1,y_2,\tau,s,x_f(s)) &= \frac{1}{\sqrt{\tau-s}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\rho\sigma_2-\frac{\sigma_2^2}{\sigma_1})(\tau-s)-x_f(s)),\\
b &= \frac{\sigma_2}{\sigma_1}.
\end{aligned}$$

Here, $\varphi(u_1,u_2)$ is a probability density function of the bivariate standard normal distribution with covariant correlation ρ. By using Lemma 3.2, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{a_i+bu_2}\varphi(u_1,u_2)du_1du_2=\mathrm{N}(\frac{\sigma_1 a_i}{\sigma}),\ i=1,2,3,4.$$

Hence from (17), we have

(18)
$$\begin{aligned}
I^{(1)} &= e^{(\sigma_1 y_1+\frac{1}{2}\sigma_1^2\tau)}N\left(\frac{\sigma_1 a_1(y_1,y_2,\tau)}{\sigma}\right),\\
I^{(2)} &= e^{(\sigma_2 y_2+\frac{1}{2}\sigma_2^2\tau)}N\left(\frac{\sigma_1 a_2(y_1,y_2,\tau)}{\sigma}\right),\\
I^{(3)} &= e^{(\sigma_1 y_1+\frac{1}{2}\sigma_1^2\tau)}\int_0^{\tau}q_1e^{q_1s}N\left(\frac{\sigma_1 a_3(y_1,y_2,\tau,s,x_f(s))}{\sigma}\right)ds,\\
I^{(4)} &= e^{(\sigma_2 y_2+\frac{1}{2}\sigma_2^2\tau)}\int_0^{\tau}q_2e^{q_2s}N\left(\frac{\sigma_1 a_4(y_1,y_2,\tau,s,x_f(s))}{\sigma}\right)ds.
\end{aligned}$$

If the value of $\frac{S_1}{S_2}$ reaches the early exercise ratio at the first time, that is $\frac{S_1}{S_2} = X_f(T-\tau)$ or $\sigma_1 y_1 - \sigma_2 y_2 = \sigma_1 x_f(\tau)$, it is optimal to exercise the AEO. By (11), (16) and (18), we get (14). ∎

After replacing $x_f(\tau)$ by $X_f(\tau)$, Theorem 3.1. can be rewritten in the following form:

Theorem 3.2. *The early exercise ratio $X_f(t)$ satisfies the following integral equation:*

$$
\begin{aligned}
X_f(T-\tau) - 1 = {} & X_f(T-\tau)e^{-q_1\tau}N(\hat{a}_1) - e^{-q_2\tau}N(\hat{a}_2) \\
& + X_f(T-\tau)e^{-q_1\tau}\int_0^\tau q_1 e^{q_1 s}N(\hat{a}_3)ds \\
& - e^{-q_2\tau}\int_0^\tau q_2 e^{q_2 s}N(\hat{a}_4)ds,
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\hat{a}_1 &= \frac{1}{\sigma\sqrt{\tau}}(\ln X_f(T-\tau) + \frac{1}{2}(\sigma^2 - 2q_1 + 2q_2)\tau), \\
\hat{a}_2 &= \frac{1}{\sigma\sqrt{\tau}}(\ln X_f(T-\tau) - \frac{1}{2}(\sigma^2 + 2q_1 - 2q_2)\tau), \\
\hat{a}_3 &= \frac{1}{\sigma\sqrt{\tau-s}}(\ln \frac{X_f(T-\tau)}{X_f(T-s)} + \frac{1}{2}(\sigma^2 - 2q_1 + 2q_2)(\tau-s)), \\
\hat{a}_4 &= \frac{1}{\sigma\sqrt{\tau-s}}(\ln \frac{X_f(T-\tau)}{X_f(T-s)} - \frac{1}{2}(\sigma^2 + 2q_1 - 2q_2)(\tau-s)), \\
\sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.
\end{aligned}
$$

4. An Explicit Solution of Perpetual AEO

A perpetual option is an option which does not have an expiry date but rather an infinite time horizon. Namely, the expiry date T for a perpetual option is equal to infinity at the initial. The early exercise ratio for the perpetual AEO in (19) is a time-invariant constant [5], denoted as $X_f(\infty)$. Since τ is any point in $(0,T)$ and $0 < s < \tau$, $X_f(T-\tau)$ and $X_f(T-s)$ in (19) are both replaced by $X_f(\infty)$. Therefore, the early exercise ratio $X_f(\infty)$ satisfies the following equation

$$
\begin{aligned}
X_f(\infty) - 1 = {} & X_f(\infty)e^{-q_1\tau}N(\hat{a}_1) - e^{-q_2\tau}N(\hat{a}_2) \\
& + X_f(\infty)e^{-q_1\tau}\int_0^\tau q_1 e^{q_1 s}N(\hat{a}_3)ds - e^{-q_2\tau}\int_0^\tau q_2 e^{q_2 s}N(\hat{a}_4)ds,
\end{aligned}
\tag{20}
$$

where

$$\hat{a}_1 = \frac{\ln X_f(\infty) + \delta_1\tau}{\sigma\sqrt{\tau}},\ \hat{a}_2 = \frac{\ln X_f(\infty) - \delta_2\tau}{\sigma\sqrt{\tau}},$$
$$\hat{a}_3 = \frac{\delta_1}{\sigma}\sqrt{\tau - s},\ \hat{a}_4 = \frac{-\delta_2}{\sigma}\sqrt{\tau - s},$$
$$\delta_1 = \frac{1}{2}(\sigma^2 - 2q_1 + 2q_2),\ \delta_2 = \frac{1}{2}(\sigma^2 + 2q_1 - 2q_2).$$

Theorem 4.1. *The value of the early exercise ratio* $X_f(\infty)$ *is*

$$X_f(\infty) = \frac{\left(1 + \sqrt{\frac{\delta_2^2}{\delta_2^2 + 2q_2\sigma^2}}\right)}{\left(1 - \sqrt{\frac{\delta_1^2}{\delta_1^2 + 2q_1\sigma^2}}\right)}. \tag{21}$$

Proof. Let $u = \tau - s$, then $\hat{a}_3 = \frac{\delta_1}{\sigma}\sqrt{u}$. Using integration by parts to the third term on the RHS of (20), we obtain

$$\begin{aligned} &\int_0^\tau q_1 e^{q_1(\tau-u)} N\left(\frac{\delta_1}{\sigma}\sqrt{u}\right) du \\ &= -N(\frac{\delta_1}{\sigma}\sqrt{\tau}) + \frac{1}{2}e^{q_1\tau} + \frac{1}{2}e^{q_1\tau}\frac{\delta_1}{\sigma}\int_0^\tau \frac{1}{\sqrt{2\pi u}} e^{-(q_1 + (\frac{\delta_1}{\sqrt{2}\sigma})^2)u} du. \end{aligned} \tag{22}$$

Applying the same argument to the forth term in (20), we also obtain

$$\begin{aligned} &\int_0^\tau q_2 e^{q_2(\tau-u)} N\left(-\frac{\delta_2}{\sigma}\sqrt{u}\right) du \\ &= -N(-\frac{\delta_2}{\sigma}\sqrt{\tau}) + \frac{1}{2}e^{q_2\tau} - \frac{1}{2}e^{q_2\tau}\frac{\delta_2}{\sigma}\int_0^\tau \frac{1}{\sqrt{2\pi u}} e^{-(q_2 + (\frac{\delta_2}{\sqrt{2}\sigma})^2)u} du. \end{aligned} \tag{23}$$

Substituting (22)-(23) into (20), and letting the remaining time τ tend to infinity, since the terms $e^{-q_1\tau}N\left(\frac{\delta_1\sqrt{\tau}}{\sigma}\right)$ and $e^{-q_2\tau}N\left(\frac{-\delta_2\sqrt{\tau}}{\sigma}\right)$ both tend to zero, then we get the following equation

$$\begin{aligned} &\left(\frac{1}{2} - \frac{\delta_1}{2\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi u}} e^{-(\frac{\delta_1^2}{2\sigma^2} + q_1)u} du\right) X_f(\infty) \\ &= \left(\frac{1}{2} + \frac{\delta_2}{2\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi u}} e^{-(\frac{\delta_2^2}{2\sigma^2} + q_2)u} du\right). \end{aligned} \tag{24}$$

By the well-known result

$$\frac{1}{\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{u}} e^{-cu} du \to \sqrt{\frac{1}{2c}},\ \text{as } \tau \to \infty$$

and (24), we get (21).

Note that $X_f(\infty)$ in the above theorem is a constant when the parameters σ_1, σ_2, q_1, q_2 and ρ are given. Substituting $X_f(\infty)$ into (16), the early exercise premium for the perpetual AEO can be obtained as follows:

Theorem 4.2. *The early exercise premium of the perpetual AEO is*

$$\left(\frac{S_1}{2}\left(\frac{\delta_1}{D_1}-1\right)e^{-\left(\frac{D_1+\delta_1}{\sigma^2}\right)}+\frac{S_2}{2}\left(\frac{\delta_2}{D_2}+1\right)e^{-\left(\frac{D_2-\delta_2}{\sigma^2}\right)}\right)\left(\frac{S_1}{S_2X_f(\infty)}\right),$$

where $D_i=\sqrt{\delta_i^2+2q_i\sigma^2}$, $i=1,2$.

In order to prove this theorem, we need to use the formula of the moment generating function of Inverse Gaussian distribution. A brief of this formula is given as follows, see Berg [3]:

Lemma 4.1. *Let X be an Inverse Gaussian random variable with mean μ and variance μ^2/ν and its probability density funciton is written as*

$$h(x|\mu,\nu)=\left(\frac{\mu\nu}{2\pi x^3}\right)^{1/2}e^{-\frac{\nu(x-\mu)^2}{2\mu x}}$$

then the moment generating function of Inverse Gaussian random variable is

$$\mathrm{E}\left[e^{tX}\right]=\int_0^\infty e^{tx}h(x|\mu,\nu)dx=e^{\nu-\eta(t)},$$

where $\eta(t)=\sqrt{\nu^2-2t\mu\nu}$.

Proof of Theorem 4.2. We first express (16) in terms of S_1, S_2 and $X_f(T-\tau)$ as follows:

$$\begin{aligned}&P(S_1,S_2,\tau)\\ &= S_1e^{-q_1\tau}N\left(\frac{\ln(\frac{S_1}{S_2})+\delta_1\tau}{\sigma\sqrt{\tau}}\right)-S_2e^{-q_2\tau}N\left(\frac{\ln(\frac{S_1}{S_2})-\delta_2\tau}{\sigma\sqrt{\tau}}\right)\\ &\quad+q_1S_1e^{-q_1\tau}\int_0^\tau e^{q_1s}N\left(\frac{a_3}{\sigma}\right)ds-q_2S_2e^{-q_2\tau}\int_0^\tau e^{q_2s}N\left(\frac{a_4}{\sigma}\right)ds,\end{aligned}\tag{25}$$

where

$$\begin{aligned}a_3&=\frac{1}{\sqrt{\tau-s}}\left(\ln\left(\frac{S_1}{S_2X_f(\tau-s)}\right)+\delta_1(\tau-s)\right),\\ a_4&=\frac{1}{\sqrt{\tau-s}}\left(\ln\left(\frac{S_1}{S_2X_f(\tau-s)}\right)-\delta_2(\tau-s)\right).\end{aligned}$$

Replacing $X_f(\tau - s)$ by $X_f(\infty)$ and letting $u = \tau - s$, the early exercise premium of the perpetual AEO is reduced to

$$P(S_1, S_2) = S_1 \int_0^\infty q_1 e^{-q_1 u} N(\frac{a_3}{\sigma}) du - S_2 \int_0^\infty q_2 e^{-q_2 u} N(\frac{a_4}{\sigma}) du, \tag{26}$$

where

$$a_3(S_1, S_2) = \frac{1}{\sqrt{u}} (\ln(\frac{S_1}{S_2 X_f(\infty)}) + \delta_1 u),$$

$$a_4(S_1, S_2) = \frac{1}{\sqrt{u}} (\ln(\frac{S_1}{S_2 X_f(\infty)}) - \delta_2 u),$$

when the remaining time τ tends to infinity. Here, the first two terms of RHS in (25) are both converge to zero since $\mathrm{N}(x)$ is bounded, and $e^{-q_1 \tau}$, $e^{-q_2 \tau}$ both converge to zero.

Applying integration by parts to the first integral of (26), we obtain that

$$\begin{aligned} & q_1 \int_0^\infty e^{-q_1 u} N\left(\frac{\delta_1 u + \delta_1}{\sigma \sqrt{u}}\right) du \\ = & \frac{-A}{2\sigma} \left[\int_0^\infty e^{-q_1 u} \left(\frac{1}{2\pi u^3}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du \right] \\ & + \frac{\delta_1}{2\sigma} \left[\int_0^\infty e^{-q_1 u} \left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du \right], \end{aligned} \tag{27}$$

where $A = \ln\left(\frac{S_1}{S_2 X_f(\infty)}\right)$.

Let $\mu_1 = \frac{-A}{\delta_1}$ and $\nu_1 = \frac{-A\delta_1}{\sigma^2}$, then $\mu_1 \nu_1 = \frac{A^2}{\sigma^2}$. Using Lemma 4.1 to the first term of RHS in (27), we obtain that

$$\int_0^\infty e^{-q_1 u} \left(\frac{\mu_1 \nu_1}{2\pi u^3}\right)^{1/2} e^{-\nu_1 \frac{(u-\mu_1)^2}{2\mu_1 u}} du = e^{\left(\nu_1 - \sqrt{\nu_1^2 + 2q_1 \nu_1 \mu_1}\right)} = e^{\frac{\delta_1 + D_1}{\sigma^2} A}, \tag{28}$$

where $D_1 = \sqrt{\delta_1^2 + 2q_1 \sigma^2}$.

By expanding $(\delta_1 u + A)^2$ and letting $u = v^2$, we have

$$\frac{\delta_1}{2\sigma} \int_0^\infty e^{-q_1 u} \left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du = \frac{\delta_1 e^{-\frac{\delta_1 A}{\sigma^2}}}{\sigma} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\left(\alpha_1 v^2 + \beta v^{-2}\right)} dv$$

in (27), where $\alpha_1 = \frac{\delta_1^2}{2\sigma^2} + q_1$ and $\beta = \frac{A^2}{2\sigma^2}$. Note that the following identity can be obtained from the integral table:

$$\int_0^\infty e^{-(\alpha_1 v^2 + \beta v^{-2})} dv = \frac{1}{2} \sqrt{\frac{\pi}{\alpha_1}} e^{-2\sqrt{\alpha_1 \beta}}$$

for any positive real number α_1, β. Hence, we obtain

$$\frac{\delta_1}{2\sigma}\int_0^\infty e^{-q_1 u}\left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}}du = \frac{\delta_1}{2D_1}e^{-\frac{(D_1+\delta_1)A}{\sigma^2}}. \tag{29}$$

in (27). By (27), (28) and (29), we get

$$\int_0^\infty q_1 e^{-q_1 u} N(\frac{a_3}{\sigma})du = \frac{1}{2}\left(\frac{\delta_1}{D_1}-1\right)e^{-\left(\frac{D_1+\delta_1}{\sigma^2}\right)}\left(\frac{S_1}{S_2 X_f(\infty)}\right).$$

in (26).

The same process can also be applied to the second integral of (26) and we have derived an explict pricing formula for the early exercise premium of AEO. ■

To demostrate the pricing formula in Theorem 4.2, let us consider the following example.

Example 4.1. The current price and the constant volatility of asset 1 are given as 40 and 40%. For asset 2, the current price and the constant volatility are given as 35 and 40%. The correlation coefficient between two assets is given as $\rho = 10\%$. Consider the following three cases: (q_1, q_2)=(0.1, 0.01), (0.05, 0.01) and (0.02, 0.01). From case 1 to case 3, the early exercise ratio are 2.1644, 3.5219, and 10.0365, respectively, and the early exercise premium are 12.5432, 9.2213, and 3.4067, respectively.

5. Conclusions

In this paper, the AEO pricing problem has been modeled as an FBP. We found that the early exercise ratio is a strictly decreasing function and $X_f(T) = \max(1, \frac{q_2}{q_1})$. We have proposed an integral equation for the early exercise ratio of AEO. For the perpetual AEO, we obtained the value of early exercise ratio and an explicit pricing formula. However, the exact solution of this integral equation has not been found for the finite time horizon. One of the future studies is to provide an asymptotic formula for the early exercise ratio when the time horizon is finite.

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Ming-Long Liu
Department of Mathematical Sciences,
National Chengchi University,
Taipei, Taiwan
E-mail: mlliu@nccu.edu.tw

Hsuan-Ku Liu
Department of Mathematics and Imformation Education,
National Taipei University of Education,
Taipei, Taiwan
E-mail: HKLiu.nccu@gmail.com