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EVOLUTION OF CR YAMABE CONSTANT UNDER THE CARTAN FLOW ON A CR 3-MANIFOLD

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Abstract. In this note we derive a formula for the derivative of the CR Yamabe constant $\mathcal{Y}(J_{(t)})$, where $J_{(t)}$ is a solution of the Cartan flow on a closed CR 3-manifold. We also give some simple applications.

0. Introduction

In this paper, following the way of Chang and Lu [1], we derive a formula for the derivative of the CR Yamabe constant $\mathcal{Y}(J_{(t)})$, where $J_{(t)}$ is a solution of the Cartan flow on a closed CR 3-manifold. As an application, we show that if $(J_{(0)}, \theta)$ is of nonnegative constant Tanaka-Webster curvature and the real part (or imaginary part) of torsion along T-direction derivative vanishes for the initial data, then the Cartan flow will increase the CR Yamabe constant at later time.

To be precise, let M be a closed 3-manifold with an oriented contact structure ξ. There always exists a global contact form θ, obtained by patching together local ones with a partition of unity (see, e.g. [7, 9]). The characteristic vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. A CR structure compatible with ξ is a smooth endomorphism $J : \xi \rightarrow \xi$ such that $J^2 = -identity$. A pseudohermitian structure is a CR structure J compatible with ξ together with a global contact form θ.

Given a pseudohermitian structure (J, θ), we can choose a complex vector field Z_1, an eigenvector of J with eigenvalue i, and a complex 1-form θ^1 such that $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is dual to $\{T, Z_1, Z_{\bar{1}}\}$. It follows that $d\theta = ih_{1\bar{1}}\theta^1\wedge\theta^{\bar{1}}$ for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call such a pseudohermitian structure (J, θ) positive, and we can choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\theta = i\theta^1\wedge\theta^{\bar{1}}.$$

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We will always assume our pseudohermitian structure (J, θ) is positive and $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes C$ (and extended to tensors) given by

$$\nabla Z_1 = {\omega_1}^1 \otimes Z_1, \quad \nabla Z_{\bar{1}} = {\omega_{\bar{1}}}^{\bar{1}} \otimes Z_{\bar{1}}, \quad \nabla T = 0,$$

in which the 1-form ${\omega_1}^1$ is uniquely determined by the following equation and associated normalization condition:

$$d\theta^1 = \theta^1 \wedge {\omega_1}^1 + {A^1}_{\bar{1}} \theta \wedge \theta^{\bar{1}}, \quad {\omega_1}^1 + {\omega_{\bar{1}}}^{\bar{1}} = 0.$$

The coefficient ${A^1}_{\bar{1}}$ is called the (pseudohermitian) torsion. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}} {A^1}_{\bar{1}} = {A^1}_{\bar{1}}$. And A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Differentiating ${\omega_1}^1$ gives

$$d{\omega_1}^1 = R\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A_{11,\bar{1}} \theta^1 \wedge \theta)$$

where R is the Tanaka-Webster curvature (see [8, 9]).

We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_{,1} = Z_1 f$, $f_{,1\bar{1}} = Z_{\bar{1}} Z_1 f - {\omega_1}^1(Z_{\bar{1}}) Z_1 f$, and $f_{,0} = Tf$ for a (smooth) function f. We define the subgradient operator ∇_b and the sublaplacian operator Δ_b by

$$\nabla_b f = f_{,\bar{1}} Z_1 + f_{,1} Z_{\bar{1}}, \quad \Delta_b f = -f_{,1\bar{1}} - f_{,\bar{1}1},$$

respectively. Also we define $|\nabla_b f|^2_{J,\theta} = 2 f_{,1} f_{,\bar{1}}$ for a real-valued function f.

Recall that the CR Yamabe constant on a closed CR 3-manifold is defined by

$$\mathcal{Y}(J) \doteqdot \inf_{u \in C^\infty(M), u>0} \frac{\int_M \left(4 |\nabla_b u|^2_{J,\theta} + Ru^2\right) \theta \wedge d\theta}{\left(\int_M u^4 \theta \wedge d\theta\right)^{1/2}}, \tag{0.1}$$

where R is the Tanaka-Webster curvature and $\theta \wedge d\theta$ is the volume form associated with (J, θ). The Euler-Lagrangian equation for a minimizer u is

$$4\Delta_b u + Ru = \mathcal{Y}(J) u^3, \tag{0.2}$$

$$\int_M u^4 \theta \wedge d\theta = 1. \tag{0.3}$$

Note that the existence of minimizer u follows from the solution of CR Yamabe problem (e.g., see the serial papers [4, 5, 6]). Given such a solution u the contact form $u^2\theta$ has constant Tanaka-Webster curvature $\mathcal{Y}(J)$. Like the Riemannian case, we define the σ invariant of M by

$$\sigma(M) = \sup_J \mathcal{Y}(J),$$

where the $\sup$ is taken over all pseudohermitian structures (J,θ) on M.

We also recall that the Cartan flow is deforming the CR structure in the direction of its Cartan tensor. Due to a result of Gray [3], without loss of generality we fix a contact structure and a contact form throughout this paper. This gives rise to the evolution equation

$$\frac{\partial}{\partial t}J_{(t)} = 2Q_{J_{(t)}}, \tag{0.4}$$

where $Q_J = 2\mathrm{Re}(iQ_{11}\theta^1 \otimes Z_{\bar{1}})$ is the Cartan tensor of the CR structure J with

$$Q_{11} = \frac{1}{6}R_{,11} + \frac{i}{2}RA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1}$$

(Lemma 2.2 in [2]), whose vanishing characterizes the spherical $J : Q_J = 0$ if and only if J is spherical.

We remark that (0.4) is a fourth order nonlinear subparabolic equation which is the negative gradient flow of the Burns-Epstein invariant and the spherical CR structures are the only equilibrium solutions . The short time existence of solutions for (0.4) is proved by adding a gauge-fixing term to the right-hand side of (0.4) (see Theorem B. in [2]).

Now we state the following result.

Proposition 0.1. *Let $J_{(t)}$, $t \in [0,\epsilon)$ for some $\epsilon > 0$, be a solution of the Cartan flow (0.4) on a closed 3-dimensional pseudohermitian manifold M. Assume that there is a C^1-family of smooth functions $u_{(t)} > 0$, $t \in [0,\epsilon)$ which satisfy*

$$4\Delta_b u_{(t)} + R_{J_{(t)}}u_{(t)} = \widetilde{\mathcal{Y}}_{(t)}u^3_{(t)}, \tag{0.5}$$

$$\int_M u^4_{(t)}\theta\wedge d\theta = 1, \tag{0.6}$$

where $\widetilde{\mathcal{Y}}$ is function of t only. Then we have

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)} = &\int_M 2\mathrm{Re}[8Q_{11}u_{,\bar{1}}u_{,\bar{1}} + \frac{2}{3}A_{\bar{1}\bar{1}}(A_{11,\bar{1}}u_{,1} + A_{11,1}u_{,\bar{1}})u]\theta\wedge d\theta \\ &+ \int_M \frac{1}{3}\left[\mathrm{Re}(iA_{11}R_{,\bar{1}\bar{1}}) + 2\left|\nabla_b A_{11}\right|^2_{J,\theta} + 7R\left|A_{11}\right|^2_{J,\theta}\right]u^2\theta\wedge d\theta \\ &- \int_M 2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + \frac{4}{3}iA_{11}A_{\bar{1}\bar{1},0})u^2\theta\wedge d\theta,\end{aligned} \tag{0.7}$$

where $u = u_{(t)}$ and Q_{11}, A_{11}, R, and ∇_b are the Cartan tensor, the torsion, the Tanaka-Webster curvature, and the subgradient of $(J_{(t)},\theta)$, respectively.

Proof. Let θ be a fixed contact form and let $h \doteqdot \frac{du}{dt}$. Note that

$$\widetilde{\mathcal{Y}}_{(t)}=\int_M \left(4\left|\nabla_b u\right|_{J,\theta}^2 + Ru^2\right)\theta\wedge d\theta.$$

By applying the computation on pages 231-232 in [2] (with $E_1{}^{\bar{1}}$ replaced by iQ_{11}), we have

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)} &= \int_M [16\mathrm{Re}(Q_{11}u_{,\bar{1}}u_{,\bar{1}}) + 8u\Delta_b h]\theta\wedge d\theta \\ &\quad + \int_M [-2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u^2 + 2Rhu]\theta\wedge d\theta,\end{aligned} \tag{0.8}$$

where we have used

$$\frac{\partial}{\partial t}\left|\nabla_b u\right|_{J,\theta}^2 = 4\mathrm{Re}(Q_{11}u_{,\bar{1}}u_{,\bar{1}} + h_{,1}u_{,\bar{1}})$$

and

$$\frac{\partial}{\partial t}R = -2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11}).$$

Taking derivative d/dt of (0.5), we obtain

$$\begin{aligned}&4\Delta_b h - 16\mathrm{Re}(Q_{11}u_{,\bar{1}\bar{1}} + Q_{11,\bar{1}}u_{,\bar{1}}) - 2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u + Rh \\ &= \left[\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)}\right]u^3 + 3\widetilde{\mathcal{Y}}_{(t)}hu^2.\end{aligned}$$

Multiplying this by $2u$, we get

$$\begin{aligned}8u\Delta_b h + 2Rhu &= 32\mathrm{Re}(Q_{11}u_{,\bar{1}\bar{1}}u + Q_{11,\bar{1}}u_{,\bar{1}}u) \\ &\quad +4\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u^2 \\ &\quad + \left[\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)}\right]2u^4 + 6\widetilde{\mathcal{Y}}_{(t)}hu^3.\end{aligned}$$

Substituting this into (0.8) to eliminate h, we obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)} &= \int_M 16\mathrm{Re}(Q_{11}u_{,\bar{1}}u_{,\bar{1}} + 2Q_{11}u_{,\bar{1}\bar{1}}u + 2Q_{11,\bar{1}}u_{,\bar{1}}u)\theta\wedge d\theta \\ &\quad + \int_M 2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u^2\theta\wedge d\theta + 2\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)} \\ &\quad +6\widetilde{\mathcal{Y}}_{(t)}\int_M hu^3\theta\wedge d\theta.\end{aligned}$$

Integrating by parts, we have

$$\frac{d}{dt}\widetilde{\mathcal{Y}}_{(t)} = \int_M [16\mathrm{Re}(Q_{11}u_{,\bar{1}}u_{,\bar{1}}) - 2\mathrm{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u^2]\theta \wedge d\theta, \tag{0.9}$$

where we used

$$\int_M (Q_{11}u_{,\bar{1}}u_{,\bar{1}} + Q_{11}u_{,\bar{1}\bar{1}}u + Q_{11,\bar{1}}u_{,\bar{1}}u)\theta \wedge d\theta = 0$$

and

$$\int_M hu^3\theta \wedge d\theta = 0$$

which is obtained by taking derivative d/dt of (0.6).

Next by using the commutation relation for pseudohermitian covariant derivative

$$A_{11,1\bar{1}} - A_{11,\bar{1}1} = iA_{11,0} + 2A_{11}R, \tag{0.10}$$

the Cartan tensor Q_{11} can also be represented by

$$Q_{11} = \frac{1}{6}[R_{,11} + 7iRA_{11} - 8A_{11,0} - 2i(A_{11,\bar{1}1} + A_{11,1\bar{1}})].$$

Hence we have

$$\begin{aligned} &-2\mathrm{Re}(iA_{\bar{1}\bar{1}}Q_{11}) \\ = &\frac{1}{3}\mathrm{Re}\left[-iA_{\bar{1}\bar{1}}R_{,11} + 7R|A_{11}|^2_{J,\theta} + 8iA_{\bar{1}\bar{1}}A_{11,0} - 2A_{\bar{1}\bar{1}}(A_{11,\bar{1}1} + A_{11,1\bar{1}})\right]. \end{aligned}$$

The proposition follows from integrating by parts to the last term of above equation into (0.9). ∎

We say a function f is *basic* if $Tf = 0$. We have the following Corollary.

Corollary 0.2. *Let (M, J_0, θ) be a closed 3-dimensional pseudohermitian manifold of nonnegative constant Tanaka-Webster curvature and the real part (or imaginary part) of torsion is basic. Let $J_{(t)}$ be the solution of the Cartan flow (0.4) with $J_{(0)} = J_0$ and assume that there is a C^1-family of smooth functions $u_{(t)} > 0$, $t \in [0, \epsilon)$ for some $\epsilon > 0$ satisfy the assumption in Proposition 0.1. Then we have $\frac{d}{dt}|_{t=0}\widetilde{\mathcal{Y}}_{(t)} \geq 0$ and the equality holds if and only if J_0 is spherical.*

Proof. Integrating by parts, we obtain

$$\int_M \mathrm{Re}(Q_{11,\bar{1}\bar{1}} + \frac{4}{3}iA_{11}A_{\bar{1}\bar{1},0})\theta \wedge d\theta = 0,$$

which follows from the fact that $\text{Re}(A_{11})$ (or $\text{Im}(A_{11})$) is basic. Now since $\nabla_b u_{(0)} = 0$ and $R \geq 0$ is a constant, we have

$$\frac{d}{dt}|_{t=0}\widetilde{\mathcal{Y}}_{(t)} = \frac{1}{3}[u_{(0)}]^2 \int_M [2\,|\nabla_b A_{11}|^2_{J,\theta} + 7R\,|A_{11}|^2_{J,\theta}]\theta \wedge d\theta \geq 0.$$

Suppose the above equality holds. If R is positive, then $A_{11} = 0$. If R is zero, then by the fact $\nabla_b A_{11} = 0$ and the commutation relation (0.10) will imply $A_{11,0} = 0$. All these implies that $Q_{11} = 0$ and therefore J is spherical. Conversely, if J is spherical, that is $Q_{11} = 0$. Then from (0.9), we get

$$\frac{d}{dt}|_{t=0}\widetilde{\mathcal{Y}}_{(t)} = \int_M [16\text{Re}(Q_{11}u_{,\bar{1}}u_{,\bar{1}}) - 2\text{Re}(Q_{11,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}Q_{11})u^2]\theta \wedge d\theta = 0.$$

This implies the Corollary. ∎

Note that $\widetilde{\mathcal{Y}}_{(t)}$ in Corollary 0.2 may not be equal to the CR Yamabe constant $\mathcal{Y}(J_{(t)})$ even if $J_{(0)}$ satisfies $\widetilde{\mathcal{Y}}_{(0)} = \mathcal{Y}(J_{(0)})$. If we assume that the real part (or imaginary part) of torsion of $(J_{(0)}, \theta)$ is basic and $(J_{(t)}, \theta)$ has unit volume and constant Tanaka-Webster curvature $\mathcal{Y}(J_{(t)})$, we have the following result, which says that infinitesimally the Cartan flow will try to increase the CR Yamabe constant.

Corollary 0.3.

(i) *Let (M, J_0, θ) be a closed 3-dimensional pseudohermitian manifold of non-negative constant Tanaka-Webster curvature and the real part (or imaginary part) of torsion is basic. Let $J_{(t)}$ be the solution of the Cartan flow (0.4) with $J_{(0)} = J_0$ and assume that there is a C^1-family of smooth functions $u_{(t)} > 0$, $t \in [0, \epsilon)$ for some $\epsilon > 0$ with constant $u_{(0)}$ such that $(J_{(t)}, \theta)$ has unit volume and constant Tanaka-Webster curvature $\mathcal{Y}(J_{(t)})$. Then we have $\frac{d}{dt}|_{t=0}\mathcal{Y}(J_{(t)}) \geq 0$ and the equality holds if and only if J_0 is spherical.*

(ii) *Furthermore, if J_0 further satisfies $\mathcal{Y}(J_0) = \sigma(M)$, then J_0 must be spherical.*

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