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THE C^*-ALGEBRAS OF SOME SOLVABLE LIE GROUPS INVOLVING CYCLIC SYMMETRIES

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Abstract. In this paper we consider the group C^*-algebras of some solvable Lie groups involving cyclic symmetries and obtain some results on their structure, stable rank, and connected stable rank for C^*-algebras.

0. Introduction

Group C^*-algebras have been of interest in some topics of C^*-algebras such as their representation theory, structure theory, and K-theory. For instance, see Dixmier [6] and Pedersen [12] for the general theory and the representation theory of C^*-algebras as well as group C^*-algebras, and see Blackadar [1] for K-theory for C^*-algebras.

On the other hand, in the unitary representation theory of Lie groups their irreducible unitary representations are crucial, and they correspond to primitive quotients of their group C^*-algebras. In this direction some remarkable results about those primitive quotient C^*-algebras have been obtained by Green [9] and Poguntke [13].

Furthermore, symmetries on C^*-algebras have also been studied. For this see Blackadar [2], Bratteli, Elliott, Evans and Kishimoto [3, 4] and [5], in which they considered symmetries on irrational or rational rotation C^*-algebras that play important roles in the C^*-algebra theory.

Moreover, the stable rank (and connected stable rank) theory for C^*-algebras was initiated by Rieffel [14], in which he proposed an interesting question of describing the stable rank of Lie group C^*-algebras in terms of Lie groups. For this question some results are obtained by Sheu [18] for certain simply connected nilpotent Lie groups, Takai and Sudo [26, 27] for all simply connected nilpotent Lie groups and solvable Lie groups of type I, and the author [19] for connected Lie groups of

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type I as well as [20-22] for certain simply connected solvable Lie groups of non type I. Those results suggest that the stable rank of (simply) connected solvable Lie group C^*-algebras can be estimated by the covering dimension of the spaces of their 1-dimensional representations. Moreover, that of the group C^*-algebras of some disconnected solvable Lie groups by the integers $\mathbb{Z}$ has been considered by the author [23-25]. It is found that their estimation becomes more complicated than the connected cases since they may have homogeneous subquotients that need to be considered in estimation.

In this paper we consider the C^*-algebras of some disconnected solvable Lie groups involving actions by the cyclic groups $\mathbb{Z}_n$ (and we call the actions cyclic symmetries) and obtain some results on their (algebraic) structure, stable rank, and connected stable rank. In Section 1 we consider the examples analogous to the real 2-dimensional $ax + b$ group C^*-algebra, i.e., the group C^*-algebra of the complex $ax + b$ group involving a cyclic symmetry and its generalization, and obtain some results on their structure, stable rank, and connected stable rank. In Section 2 we consider the examples analogous to the real 4-dimensional split oscillator group C^*-algebra, i.e., the group C^*-algebra of the complex split oscillator group involvling a cyclic symmetry and its generalization, and obtain some results on their structure, stable rank, and connected stable rank. In Section 3 we consider the examples analogous to the real 5-dimensional Mautner group C^*-algebra, i.e., the group C^*-algebra of the complex Mautner group by $\mathbb{Z}$ involvling a cyclic (and free) symmetry and its generalization, and obtain some results on the same items as before.

Roughly speaking, we find that the stable rank as well as the connected stable rank of those group C^*-algebras involving cyclic symmetries can be estimated by the covering dimension of the spaces of finite dimensional irreducible representations of homogeneous subquotients of the group C^*-algebras that depend on the cyclic symmetries themselves. Moreover, from the structure theorem obtained for the group C^*-algebra of the complex Mautner group by $\mathbb{Z}$ involvling a cyclic (and free) symmetry and its generalization we find that in general, disconnected solvable Lie group C^*-algebras involvling cyclic (and free) symmetries may have simple subquotients that are not isomorphic to noncommutative tori, and even not stably isomorphic (or Morita equivalent) to them. This phenomenon is different from that for the cases of connected solvable Lie groups and some disconnected solvable Lie groups by $\mathbb{Z}$ considered in [23-25]. In fact, see [9] and [13] for the connected case, in which it is shown that connected solvable Lie group C^*-algebras just have simple noncommutative tori, matrix algebras over $\mathbb{C}$, the C^*-algebra of compact operators, or their tensor product C^*-algebras as simple subquotients. Furthermore, the particular cases considered here could be helpful for understanding the algebraic structure, stable rank, and connected stable rank for disconnected solvable Lie group C^*-algebras in more general cases.

Notation and facts. Let G be a Lie group. Denote by $C^*(G)$ the (full) group C^*-algebra of G. Let $\mathfrak{A}$ be a C^*-algebra. Denote by $\mathfrak{A} \rtimes_\alpha G$ the (full) crossed product of $\mathfrak{A}$ by an action α of G by automorphisms of $\mathfrak{A}$ ([12]). Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the C^*-algebra of all complex-valued continuous functions on X vanishing at infinity. Set $C(X) = C_0(X)$ if X is compact. Note that $C^*(G)$ is nonunital if G is not compact.

The spectrum of a C^*-algebra $\mathfrak{A}$ is the space $\mathfrak{A}^\wedge$ of equivalence classes of irreducible representations of $\mathfrak{A}$ equipped with the hull kernel topology via the map from a class $[\pi] \in \mathfrak{A}^\wedge$ to the kernel $\ker(\pi)$, that is called a primitive ideal of $\mathfrak{A}$. This correspondence is a homeomorphism if and only if $\mathfrak{A}$ is of type I (see [6] or [12]). In particular, $C_0(X)^\wedge$ consists of 1-dimensional representations of $C_0(X)$ (characters) and is homeomorphic to X, where a point x of X is identified with the (maximal) closed ideal of the elements of $C_0(X)$ vanishing at x, and the corresponding character χ_x is the evaluation at x, i.e., $\chi_x(f) = f(x)$ for $f \in C_0(X)$. Also, $(C_0(X) \otimes M_n(\mathbb{C}))^\wedge$ consists of the classes of n-dimensional irreducible representations of $C_0(X) \otimes M_n(\mathbb{C})$ (tensor product) and is homeomorphic to X via the map from $x \in X$ to $\chi_x \otimes \mathrm{id}_{M_n(\mathbb{C})}$ (tensor representation), where $\mathrm{id}_{M_n(\mathbb{C})}$ is the identity representation of the $n \times n$ matrix algebra $M_n(\mathbb{C})$ over $\mathbb{C}$. Note that $C^*(G)^\wedge$ is identified with $G^\wedge$ the unitary dual of G consisting of equivalence classes of irreducible unitary representations of G (see [6]).

Denote by $\mathrm{sr}(\mathfrak{A})$ the stable rank of a C^*-algebra $\mathfrak{A}$ and by $\mathrm{csr}(\mathfrak{A})$ the connected stable rank of $\mathfrak{A}$. Recall that $\mathrm{sr}(\mathfrak{A})$ for a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest positive integer n such that the set $L_n(\mathfrak{A})$ of all $(a_j)_{j=1}^n \in \mathfrak{A}^n$ with $\sum_{j=1}^n \mathfrak{A}a_j = \mathfrak{A}$ is dense in $\mathfrak{A}^n$, and $\mathrm{csr}(\mathfrak{A})$ is defined to be the smallest positive integer n such that for any integer $m \geq n$, $L_m(\mathfrak{A})$ is connected. The stable rank and connected stable rank for a nonunital C^*-algebra $\mathfrak{A}$ are defined to be those of its unitization $\mathfrak{A}^+$. We refer to Rieffel [14] about the ranks. For convenience to readers, we give a list of formulae about the ranks, which are often used in what follows:

First of all, note that $\mathrm{sr}(\mathfrak{A} \oplus \mathfrak{B}) = \max\{\mathrm{sr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{B})\}$ and $\mathrm{csr}(\mathfrak{A} \oplus \mathfrak{B}) = \max\{\mathrm{csr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{B})\}$ for the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$.

For any short exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras,

$$(F1): \quad \max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\} \leq \mathrm{sr}(\mathfrak{A}) \leq \max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\},$$

$$\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\}$$

([14, Theorems 4.3, 4,4 and 4.11] and Sheu [18, Theorem 3.9]).

$$(F2): \quad \mathrm{sr}(C_0(X)) = \lfloor \dim X^+/2 \rfloor + 1, \quad \mathrm{csr}(C_0(X)) \leq \lfloor (\dim X^+ + 1)/2 \rfloor + 1,$$

where X^+ is the one point compactification of X, $\dim(\cdot)$ is the covering dimension for spaces, $\lfloor x \rfloor$ is the integer part of x, and $C_0(X)^+ \cong C(X^+)$ ([14, Proposition 1.7] and Nistor [11]).

$(F3)$: $\mathrm{sr}(M_n(\mathfrak{A})) = \lceil (\mathrm{sr}(\mathfrak{A}) - 1)/n \rceil + 1, \quad \mathrm{csr}(M_n(\mathfrak{A})) \leq \lceil (\mathrm{csr}(\mathfrak{A}) - 1)/n \rceil + 1,$

where $M_n(\mathfrak{A})$ is the $n \times n$ matrix algebra over a C^*-algebra $\mathfrak{A}$, and $\lceil x \rceil$ is equal to $\lfloor x \rfloor + 1$ for x non-integers, and $\lceil x \rceil = x$ for x integers (Rieffel [14, Theorem 6.1] and [15, Theorem 4.7]).

$(F4)$: $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \min\{\mathrm{sr}(\mathfrak{A}), 2\}, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq \min\{\mathrm{csr}(\mathfrak{A}), 2\}$

for any C^*-algebra $\mathfrak{A}$, where $\mathbb{K}$ is the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space ([14, Theorems 3.6 and 6.4], [18, Theorem 3.10], and [11]).

$(F5)$: For a unital C^*-algebra $\mathfrak{A}$, if $\mathrm{sr}(\mathfrak{A}) \leq n$, then the canonical map from $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ to $GL_{n+1}(\mathfrak{A})/GL_{n+1}(\mathfrak{A})_0$ (for $GL_n(\mathfrak{A})$ the group of invertible elements of $M_n(\mathfrak{A})$ and its connected component $GL_n(\mathfrak{A})_0$ with the identity matrix) is an isomorphism, so that the K_1-group $K_1(\mathfrak{A}) \cong GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ (Rieffel [15, Theorem 2.10]), where by definition $K_1(\mathfrak{A}) = GL_\infty(\mathfrak{A})/GL_\infty(\mathfrak{A})_0$ for $GL_\infty(\mathfrak{A})$ the union of $GL_n(\mathfrak{A})$ $(n \geq 1)$ with the canonical inclusion (see [1]). Note that $K_1(\mathfrak{A}) \cong K_1(\mathfrak{A}^+)$ for any C^*-algebra $\mathfrak{A}$. For a nonunital C^*-algebra $\mathfrak{A}$, we write $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ for $GL_n(\mathfrak{A}^+)/GL_n(\mathfrak{A}^+)_0$. Also, $GL_n(\mathfrak{A})$ and $GL_n(\mathfrak{A})_0$ can be replaced with $U_n(\mathfrak{A})$ the group of unitaries of $M_n(\mathfrak{A})$ and its connected component $U_n(\mathfrak{A})_0$ with the identity matrix, respectively.

If a C^*-algebra $\mathfrak{A}$ has connected stable rank 1, then $K_1(\mathfrak{A}) \cong 0$ (by (F3)). Hence, if $K_1(\mathfrak{A}) \not\cong 0$, then $\mathrm{csr}(\mathfrak{A}) \geq 2$.

1. The Complex (Generalized) $ax + b$ Groups Involving Cyclic Symmetries

Let $\mathbb{C} \rtimes_\alpha \mathbb{Z}_n$ be the semi-direct product of $\mathbb{C}$ by an action α of $\mathbb{Z}_n$ defined by $\alpha_1(z) = e^{2\pi i/n} z$ for $z \in \mathbb{C}$ and $1 \in \mathbb{Z}_n$. We call it an n-cyclic symmetry on $\mathbb{C}$. Let $C^*(\mathbb{C} \rtimes_\alpha \mathbb{Z}_n)$ be the group C^*-algebra of $\mathbb{C} \rtimes_\alpha \mathbb{Z}_n$. Then it is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{C}) \rtimes_\alpha \mathbb{Z}_n$, where the action α of $\mathbb{Z}_n$ on $C^*(\mathbb{C})$ is induced from that on $\mathbb{C}$. We also call it an n-cyclic symmetry on $C^*(\mathbb{C})$.

Note also that the semi-direct product $\mathbb{C} \rtimes_\alpha \mathbb{Z}_n$ can be identified with the group of all the following matrices:

$$A_{1,n} = \left\{ \begin{pmatrix} e^{2\pi i t/n} & z \\ 0 & 1 \end{pmatrix} \quad \text{for } t \in \mathbb{Z}_n, z \in \mathbb{C} \right\}$$

Thus we call $A_{1,n} = \mathbb{C} \rtimes_\alpha \mathbb{Z}_n$ the complex $ax + b$ group involving the n-cyclic symmetry α.

Now consider the structure for $C^*(\mathbb{C}) \rtimes_\alpha \mathbb{Z}_n$. By the Fourier transform, $C^*(\mathbb{C}) \rtimes_\alpha \mathbb{Z}_n$ is isomorphic to $C_0(\mathbb{C}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n$, where $C_0(\mathbb{C})$ is the C^*-algebra of all complex-valued continuous functions on $\mathbb{C}$ vanishing at infinity and the dual action $\alpha^\wedge$ is induced from the duality between $\mathbb{C}$ and its dual group $\mathbb{C}$. Since the origin of $\mathbb{C}$ is fixed under $\alpha^\wedge$, we have the following short exact sequence:

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C_0(\mathbb{C}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C^*(\mathbb{Z}_n) \to 0,$$

and $C^*(\mathbb{Z}_n) \cong \mathbb{C}^n$. Furthermore, since the action of $\mathbb{Z}_n$ is trivial on the radius direction in $\mathbb{C} \setminus \{0\}$,

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z}_n)$$

where $\mathbb{R}_+$ is the space of all positive real numbers, and

$$C(\mathbb{T}) \rtimes \mathbb{Z}_n \cong C(\mathbb{T}/\mathbb{Z}_n) \otimes (\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong C(\mathbb{T}) \otimes M_n(\mathbb{C})$$

since the action on $\mathbb{T}$ is n-cyclic, where $\mathbb{T}/\mathbb{Z}_n$ is the orbit space under the action of $\mathbb{Z}_n$ on $\mathbb{T}$.

Summing up the above argument we obtain

Theorem 1.1. *Let $A_{1,n} = \mathbb{C} \rtimes_\alpha \mathbb{Z}_n$ be the complex $ax+b$ group involving the n-cyclic symmetry α. For the group C^*-algebra $C^*(A_{1,n})$ of $A_{1,n}$, we have the following exact sequence*:

$$0 \to C_0(\mathbb{R}_+) \otimes C(\mathbb{T}) \otimes M_n(\mathbb{C}) \to C^*(A_{1,n}) \to \mathbb{C}^n \to 0.$$

Using the structure theorem above we obtain

Theorem 1.2. *The group C^*-algebra $C^*(A_{1,n})$ of $A_{1,n}$ has stable rank 2, and*

$$\mathrm{sr}(C^*(A_{1,n})) = 2 = \dim C^*(A_{1,n})_n^\wedge,$$

where $C^*(A_{1,n})_n^\wedge$ means the space of all n-dimensional irreducible representations of $C^*(A_{1,n})$ up to unitary equivalence, and is homeomorphic to $\mathbb{R}_+ \times \mathbb{T}$.

Moreover, we obtain $\mathrm{csr}(C^*(A_{1,n})) \le 2$ and $K_1(C^*(A_{1,n})) \cong 0$ and

$$GL_k(C^*(A_{1,n}))/GL_k(C^*(A_{1,n}))_0 \cong 0 \quad (k \ge 2).$$

Proof. Using the structure of Theorem 1.1 and the formula (F1) we have

$$\max\{\mathrm{sr}(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{sr}(\mathbb{C}^n)\}$$
$$\le \mathrm{sr}(C^*(A_{1,n})) \le \max\{\mathrm{sr}(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{sr}(\mathbb{C}^n), \mathrm{csr}(\mathbb{C}^n)\}.$$

Furthermore, we have $\mathrm{sr}(\mathbb{C}^n) = 1$, $\mathrm{csr}(\mathbb{C}^n) = 1$ and

$$\mathrm{sr}(C_0(\mathbb{R} \times \mathbb{T}) \otimes M_n(\mathbb{C})) = \lceil \lfloor 2/2 \rfloor / n \rceil + 1 = 2$$

by the formulae (F2) and (F3). On the other hand,

$$\mathrm{csr}(C^*(A_{1,n})) \leq \max\{\mathrm{csr}(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{csr}(\mathbb{C}^n)\}$$

by (F1), and $\mathrm{csr}(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})) \leq \lceil \lfloor (2+1)/2 \rfloor / n \rceil + 1 = 2$ by (F2) and (F3). Hence $\mathrm{csr}(C^*(A_{1,n})) \leq 2$.

On the other hand, using the fundamental result of the equivariant K-theory for crossed products by $\mathbb{Z}_n$ and the Bott periodicity (see [1, Theorems 11.7.1 and 11.9.4]) we compute the K_1-group $K_1(C^*(A_{1,n}))$ of $C^*(A_{1,n})$ as follows:

$$\begin{aligned} K_1(C^*(A_{1,n})) &\cong K_1(C_0(\mathbb{C}) \rtimes \mathbb{Z}_n) \cong K_1^{\mathbb{Z}_n}(C_0(\mathbb{C})) \\ &\cong K_1^{\mathbb{Z}_n}(\mathbb{C}) \cong K_1(C^*(\mathbb{Z}_n)) \cong \oplus^n K_1(\mathbb{C}) \cong 0, \end{aligned}$$

where $K_1^{\mathbb{Z}_n}(\cdot)$ means the equivariant K-theory by $\mathbb{Z}_n$. By (F5), we obtain $GL_k(C^*(A_{1,n}))/GL_k(C^*(A_{1,n}))_0 \cong 0$ for $k \geq 2$ as $\mathrm{sr}(C^*(A_{1,n})) = 2$. ■

Remark. It is very likely that $\mathrm{csr}(C^*(A_{1,n})) = 1$.

Its generalization. Let $\mathbb{C}^m \rtimes_\alpha \mathbb{Z}_n$ be the semi-direct product of $\mathbb{C}^m$ by an action α of $\mathbb{Z}_n$ defined by $\alpha_1((z_j)) = (e^{2\pi i/n} z_j)$ for $(z_j) \in \mathbb{C}^m$ and $1 \in \mathbb{Z}_n$. We call it an n-cyclic symmetry on $\mathbb{C}^m$. Let $C^*(\mathbb{C}^m \rtimes_\alpha \mathbb{Z}_n)$ be the group C^*-algebra of $\mathbb{C}^m \rtimes_\alpha \mathbb{Z}_n$. Then it is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{C}^m) \rtimes_\alpha \mathbb{Z}_n$, where the action α of $\mathbb{Z}_n$ on $C^*(\mathbb{C}^m)$ is induced from that on $\mathbb{C}^m$. We also call it an n-cyclic symmetry on $C^*(\mathbb{C}^m)$.

Note also that the semi-direct product $\mathbb{C}^m \rtimes_\alpha \mathbb{Z}_n$ can be identified with the group $A_{m,n}$ of all the following $(m+1) \times (m+1)$ matrices:

$$A_{m,n} = \left\{ \begin{pmatrix} e^{2\pi i t/n} & & & z_1 \\ & \ddots & & \vdots \\ & & e^{2\pi i t/n} & z_m \\ 0 & & & 1 \end{pmatrix} \quad \text{for } t \in \mathbb{Z}_n, z_j \in \mathbb{C} (1 \leq j \leq m) \right\}$$

Thus we call it the complex m-dimensional generalized $ax+b$ group involving the n-cyclic symmetry α.

Now consider the structure for $C^*(\mathbb{C}^m) \rtimes_\alpha \mathbb{Z}_n$. Using the same method as before, we have that:

$$0 \to C_0(\mathbb{C}^m \setminus \{0_m\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C^*(\mathbb{Z}_n) \to 0,$$

where $0_m \equiv \{0\} \times \cdots \times \{0\}$ is the origin of $\mathbb{C}^m$, and $C^*(\mathbb{Z}_n) \cong \mathbb{C}^n$. Moreover, since the following n subspaces X_{1j} $(1 \leq j \leq m)$:

$$X_{11} = (\mathbb{C} \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}, \cdots, X_{1m} = \{0\} \times \cdots \times \{0\} \times (\mathbb{C} \setminus \{0\})$$

are invariant under $\alpha^\wedge$ and disjoint and closed in $\mathbb{C}^m \setminus \{0_m\}$, we have

$$0 \to C_0(X_2) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C_0(\mathbb{C}^m \setminus \{0_m\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to \oplus_{j=1}^m (C_0(\mathbb{C}\setminus\{0\}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n) \to 0,$$

where X_2 is the complement of $\sqcup_{j=1}^m X_{1j}$ in $\mathbb{C}^m \setminus \{0_m\}$. Inductively, we can construct the following:

$$0 \to C_0(X_{k+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to C_0(X_k) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \to \oplus_{j=1}^{{}_m\mathrm{C}_k} (C_0(X_{kj}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n) \to 0$$

for $2 \leq k \leq m-1$, where ${}_m\mathrm{C}_k$ means the combination of k from m, and the subspaces X_{kj} are defined by

$$X_{k1} = (\mathbb{C} \setminus \{0\})^k \times (\{0\})^{m-k}, \cdots, X_{k_m\mathrm{C}_k} = (\{0\})^{m-k} \times (\mathbb{C} \setminus \{0\})^k.$$

Indeed, note that the subspaces X_{kj} are invariant under $\alpha^\wedge$ and disjoint and closed in X_k so that X_{k+1} is defined to be the complement of $\sqcup_{j=1}^{{}_m\mathrm{C}_k} X_{kj}$ in X_k. In particular, we have $X_m = X_{m1} = (\mathbb{C} \setminus \{0\})^m$.

Furthermore, since the action of $\mathbb{Z}_n$ is trivial on the radius direction in $\mathbb{C} \setminus \{0\}$ of each X_{kj},

$$C_0(X_{kj}) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \cong C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\alpha^\wedge} \mathbb{Z}_n \cong C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z}_n),$$

where $\mathbb{R}_+$ is the space of all positive real numbers, and

$$C(\mathbb{T}^k) \rtimes \mathbb{Z}_n \cong C(\mathbb{T}^k/\mathbb{Z}_n) \otimes (\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong C(\mathbb{T}^k) \otimes M_n(\mathbb{C})$$

since the action on $\mathbb{T}^k$ is n-cyclic, where $\mathbb{T}^k/\mathbb{Z}_n$ is the orbit space under the action of $\mathbb{Z}_n$ on $\mathbb{T}^k$.

Summing up the above argument we obtain

Theorem 1.3. *Let $A_{m,n} = \mathbb{C}^m \rtimes_\alpha \mathbb{Z}_n$ be the complex m-dimensional generalized $ax+b$ group involving the n-cyclic symmetry α. For the group C^*-algebra $C^*(A_{m,n})$ of $A_{m,n}$, we have the following exact sequences*:

$$0 \to \mathfrak{I}_{k+1} \to \mathfrak{I}_k \to \mathfrak{I}_k/\mathfrak{I}_{k+1} \to 0 \quad \textit{for } 0 \leq k \leq m$$

of its closed ideals such that $\mathfrak{I}_0 = C^*(A_{m,n})$, $\mathfrak{I}_{m+1} = \{0\}$, $\mathfrak{I}_0/\mathfrak{I}_1 \cong \mathbb{C}^n$ and

$$\mathfrak{I}_k/\mathfrak{I}_{k+1} \cong \oplus_{j=1}^{{}_m\mathrm{C}_k} [C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}^k) \otimes M_n(\mathbb{C})] \quad \textit{for } 1 \leq k \leq m.$$

Using the structure theorem above we obtain

Theorem 1.4. *For the group C^*-algebra $C^*(A_{m,n})$ of $A_{m,n}$,*

$$\mathrm{sr}(C^*(A_{m,n})) = \lceil m/n \rceil + 1 = \mathrm{sr}(C_0(\mathbb{R}_+^m \times \mathbb{T}^m) \otimes M_n(\mathbb{C})),$$

where $m = \dim(\mathbb{R}_+^m \times \mathbb{T}^m)/2$, and $2m$ is the $($maximum$)$ dimension of the spaces $\mathbb{R}_+^k \times \mathbb{T}^k$ $(1 \leq k \leq m)$ of n-dimensional irreducible representations of $C^*(A_{m,n})$ up to unitary equivalence. If $n \geq m$, then $\mathrm{sr}(C^*(A_{m,n})) = 2$.

Moreover, we have

$$\mathrm{csr}(C^*(A_{m,n})) \leq \lceil m/n \rceil + 1,$$

and $K_1(C^*(A_{m,n})) \cong 0$, and if $n \geq m$, then

$$GL_k(C^*(A_{m,n}))/GL_k(C^*(A_{m,n}))_0 \cong 0 \quad (k \geq 2).$$

Proof. Using the structure of Theorem 1.3 and the formula (F1) we have

$$\max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(\mathbb{C}^n)\} \leq \mathrm{sr}(C^*(A_{m,n})) \leq \max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(\mathbb{C}^n), \mathrm{csr}(\mathbb{C}^n)\},$$

$$\max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\} \leq \mathrm{sr}(\mathfrak{I}_k)$$

$$\leq \max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\}.$$

Furthermore, we have $\mathrm{sr}(\mathbb{C}^n) = 1$, $\mathrm{csr}(\mathbb{C}^n) = 1$, and

$$\mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) = \mathrm{sr}(C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}^k) \otimes M_n(\mathbb{C}))$$

$$= \lceil \lfloor 2k/2 \rfloor /n \rceil + 1 = \lceil k/n \rceil + 1,$$

$$\mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) = \mathrm{csr}(C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}^k) \otimes M_n(\mathbb{C}))$$

$$\leq \lceil \lfloor (2k+1)/2 \rfloor /n \rceil + 1 = \lceil k/n \rceil + 1$$

using the formulae (F2) and (F3). Therefore, we obtain

$$\mathrm{sr}(C^*(A_{m,n})) = \mathrm{sr}(\mathfrak{I}_m) = \lceil m/n \rceil + 1.$$

Moreover, by (F1) we obtain

$$\mathrm{csr}(\mathfrak{I}_k) \leq \max\{\mathrm{csr}(\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\}$$

for $0 \leq k \leq m$. Hence, it follows that

$$\mathrm{csr}(C^*(A_{m,n})) = \mathrm{csr}(\mathfrak{I}_0) \leq \max_{0 \leq k \leq m} \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}),$$

and $\mathrm{csr}(C^*(A_{m,n})) \leq \lceil m/n \rceil + 1$.

On the other hand, using the fundamental result of the equivariant K-theory for crossed products by $\mathbb{Z}_n$ and the Bott periodicity (see [1, Theorems 11.7.1 and 11.9.4]) we compute the K_1-group $K_1(C^*(A_{m,n}))$ of $C^*(A_{m,n})$ as follows:

$$\begin{aligned} K_1(C^*(A_{m,n})) &\cong K_1(C_0(\mathbb{C}^m) \rtimes \mathbb{Z}_n) \cong K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}^m)) \\ &\cong K_1^{\mathbb{Z}_n}(\mathbb{C}) \cong K_1(C^*(\mathbb{Z}_n)) \cong \oplus^n K_1(\mathbb{C}) \cong 0. \end{aligned}$$

If $m \le n$, then $\mathrm{sr}(C^*(A_{m,n})) = 2$. Thus, using (F5) we imply the last isomorphism in the statement. ■

2. The Complex (Generalized) Split Oscillator Groups Involving Cyclic Symmetries

We define the complex 3-dimensional split oscillator group $S_{3,n}$ involving an n-cyclic symmetry to be the group consisting of all the following matrices:

$$S_{3,n} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & e^{2\pi i t/n} & z_2 \\ 0 & 0 & 1 \end{pmatrix} = (z_3, z_2, z_1, t) \quad \text{for } t \in \mathbb{Z}_n, z_1, z_2, z_3 \in \mathbb{C} \right\}$$

Note that it is isomorphic to the semi-direct product $H_3^{\mathbb{C}} \rtimes_\alpha \mathbb{Z}_n$, where $H_3^{\mathbb{C}}$ is the complex 3-dimensional Heisenberg group defined by the matrices $(z_3, z_2, z_1, 0) = (z_3, z_2, z_1)$ for $z_1, z_2, z_3 \in \mathbb{C}$, and the action α of $\mathbb{Z}_n$ is defined by

$$\alpha_t(z_3, z_2, z_1) = t(z_3, z_2, z_1)(-t) = (z_3, e^{2\pi i t/n} z_2, e^{-2\pi i t/n} z_1)$$

for $(0,0,0,t) = t \in \mathbb{Z}_n$. Note also that $H_3^{\mathbb{C}}$ is isomorphic to the semi-direct product $\mathbb{C}^2 \rtimes_\beta \mathbb{C}$, where the action β of $\mathbb{C}$ is defined by $\beta_{z_1}(z_3, z_2) = (z_3 + z_1 z_2, z_2)$ for $z_1, z_2, z_3 \in \mathbb{C}$.

Let $C^*(S_{3,n})$ be the group C^*-algebra of $S_{3,n}$. Then it is isomorphic to the crossed product C^*-algebra $C^*(H_3^{\mathbb{C}}) \rtimes_\alpha \mathbb{Z}_n$, where $C^*(H_3^{\mathbb{C}})$ is the group C^*-algebra of $H_3^{\mathbb{C}}$. Since $H_3^{\mathbb{C}} \cong \mathbb{C}^2 \rtimes_\beta \mathbb{C}$, we have $C^*(H_3^{\mathbb{C}}) \cong C^*(\mathbb{C}^2) \rtimes_\beta \mathbb{C}$. By the Fourier transform, $C^*(\mathbb{C}^2) \rtimes_\beta \mathbb{C}$ is isomorphic to $C_0(\mathbb{C}^2) \rtimes_{\beta^\wedge} \mathbb{C}$, where the action $\beta^\wedge$ is defined by $\beta^\wedge_{z_1}(w_3, w_2) = (w_3, w_2 + z_1 w_3) \in \mathbb{C}^2$ via $\mathbb{C}^2 \cong (\mathbb{C}^2)^\wedge$ (the dual group of $\mathbb{C}^2$). Since $\{0\} \times \mathbb{C}$ is fixed under the action of $\mathbb{C}$, we have the following short exact sequence:

$$0 \to C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes \mathbb{C} \to C_0(\mathbb{C}^2) \rtimes \mathbb{C} \to C_0(\mathbb{C}) \otimes C^*(\mathbb{C}) \to 0,$$

and $C_0(\mathbb{C}) \otimes C^*(\mathbb{C}) \cong C_0(\mathbb{C}^2)$. Moreover, since the above quotient C^*-algebra is invariant under the dual action $\alpha^\wedge$ of $\mathbb{Z}_n$, we have

$$0 \to C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes \mathbb{C} \rtimes \mathbb{Z}_n \to C_0(\mathbb{C}^2) \rtimes \mathbb{C} \rtimes \mathbb{Z}_n \to C_0(\mathbb{C}^2) \rtimes \mathbb{Z}_n \to 0.$$

Furthermore, since the action of $\mathbb{C}$ on each $\{z\} \times \mathbb{C}$ for $z \in \mathbb{C} \setminus \{0\}$ is the shift, we have

$$C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes \mathbb{C} \cong C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K},$$

where $\mathbb{K}$ is the C^*-algebra of all compact operators. Since the action of $\mathbb{Z}_n$ on $\mathbb{K}$ is implemented by the adjoint action of unitaries,

$$C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K} \rtimes \mathbb{Z}_n \cong C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K} \otimes C^*(\mathbb{Z}_n) \cong \oplus^n (C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}),$$

where $C^*(\mathbb{Z}_n) \cong \oplus^n \mathbb{C}$ by the Fourier transform. For the above quotient C^*-algebra $C_0(\mathbb{C}^2) \rtimes \mathbb{Z}_n$, from the orbit structure for $\alpha^\wedge$ on $\mathbb{C}^2$ we have the following decompositions:

$$0 \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes \mathbb{Z}_n \to C_0(\mathbb{C}^2) \rtimes \mathbb{Z}_n \to \mathbb{C} \rtimes \mathbb{Z}_n \to 0,$$

$$0 \to C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{Z}_n \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes \mathbb{Z}_n \to \oplus^2 (C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{Z}_n) \to 0,$$

and $\mathbb{C} \rtimes \mathbb{Z}_n = C^*(\mathbb{Z}_n) \cong \mathbb{C}^n$. Furthermore, by the same method as before,

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{Z}_n \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z}_n) \cong C_0(\mathbb{R}_+) \otimes C(\mathbb{T}) \otimes M_n(\mathbb{C}),$$

$$C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{Z}_n \cong C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^2) \rtimes \mathbb{Z}_n),$$

and $C(\mathbb{T}^2) \rtimes \mathbb{Z}_n \cong C(\mathbb{T}^2/\mathbb{Z}_n) \otimes (\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong C(\mathbb{T}^2) \otimes M_n(\mathbb{C})$.

Therefore we obtain

Theorem 2.1. *Let $S_{3,n} = H_3^{\mathbb{C}} \rtimes_\alpha \mathbb{Z}_n$ be the complex 3-dimensional split oscillator group involving the n-cyclic symmetry α. For the group C^*-algebra $C^*(S_{3,n})$ of $S_{3,n}$, we have the following exact sequences*:

$$0 \to \oplus^n (C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \to C^*(S_{3,n}) \to C_0(\mathbb{C}^2) \rtimes \mathbb{Z}_n \to 0,$$

$$0 \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes \mathbb{Z}_n \to C_0(\mathbb{C}^2) \rtimes \mathbb{Z}_n \to \mathbb{C}^n \to 0, \quad \textit{and}$$

$$\begin{aligned} 0 &\to C_0(\mathbb{R}_+^2 \times \mathbb{T}^2) \otimes M_n(\mathbb{C}) \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes \mathbb{Z}_n \\ &\to \oplus^2 (C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})) \to 0. \end{aligned}$$

Using the structure theorem above we obtain

Theorem 2.2. *The group C^*-algebra $C^*(S_{3,n})$ of $S_{3,n}$ has stable rank 2, and*

$$\mathrm{sr}(C^*(S_{3,n})) = 2 = \mathrm{sr}(C_0(\mathbb{R}_+^2 \times \mathbb{T}^2) \otimes M_n(\mathbb{C})), \quad \textit{but}$$

$$\neq \begin{cases} \dim C^*(S_{3,n})_n^\wedge = 4, \\ \lfloor \dim C^*(S_{3,n})_n^\wedge / 2 \rfloor + 1 = 3, \end{cases}$$

where $C^*(S_{3,n})^\wedge_n$ *means the space of all* n*-dimensional irreducible representations of* $C^*(S_{3,n})$ *up to unitary equivalence, and has an open subset homeomorphic to* $\mathbb{R}_+^2 \times \mathbb{T}^2$ *whose complement is the disjoint union of two closed subsets* $\mathbb{R}_+ \times \mathbb{T}$.

Moreover, we obtain $\mathrm{csr}(C^*(S_{3,n})) \leq 2$ *and* $K_1(C^*(S_{3,n})) \cong 0$ *and*

$$GL_k(C^*(S_{3,n}))/GL_k(C^*(S_{3,n}))_0 \cong 0 \quad (k \geq 2).$$

Proof. Using the structure of Theorem 2.1 and the formula (F1) we have

$$\begin{aligned}
&\max\{\mathrm{sr}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}), \mathrm{sr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n)\} \leq \mathrm{sr}(C^*(S_{3,n}))\\
&\leq \max\{\mathrm{sr}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}), \mathrm{sr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n), \mathrm{csr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n)\}, \quad \text{and}\\
&\max\{\mathrm{sr}(C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}_n), \mathrm{sr}(\mathbb{C}^n)\} \leq \mathrm{sr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n)\\
&\leq \max\{\mathrm{sr}(C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}_n), \mathrm{sr}(\mathbb{C}^n), \mathrm{csr}(\mathbb{C}^n)\}, \quad \text{and}\\
&\max\{\mathrm{sr}(C_0(\mathbb{R}_+^2\times\mathbb{T}^2)\otimes M_n(\mathbb{C})), \mathrm{sr}(C_0(\mathbb{R}_+\times\mathbb{T})\otimes M_n(\mathbb{C}))\}\\
&\leq \mathrm{sr}(C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}_n)\\
&\leq \max\{\mathrm{sr}(C_0(\mathbb{R}_+^2\times\mathbb{T}^2)\otimes M_n(\mathbb{C})), \mathrm{sr}(C_0(\mathbb{R}_+\times\mathbb{T})\otimes M_n(\mathbb{C})),\\
&\quad \mathrm{csr}(C_0(\mathbb{R}_+\times\mathbb{T})\otimes M_n(\mathbb{C}))\}.
\end{aligned}$$

By (F1) we have

$$\begin{aligned}
&\mathrm{csr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n) \leq \max\{\mathrm{csr}(C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}_n), \mathrm{csr}(\mathbb{C}^n)\}\\
&\leq \max\{\mathrm{csr}(C_0(\mathbb{R}_+^2\times\mathbb{T}^2)\otimes M_n(\mathbb{C})), \mathrm{csr}(C_0(\mathbb{R}_+\times\mathbb{T})\otimes M_n(\mathbb{C}))\}.
\end{aligned}$$

By (F4)

$$\mathrm{sr}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}) = \min\{2, \mathrm{sr}(C_0(\mathbb{C}\setminus\{0\}))\} = 2.$$

Therefore, we obtain

$$\begin{aligned}
&\mathrm{sr}(C_0(\mathbb{R}_+^2\times\mathbb{T}^2)\otimes M_n(\mathbb{C})) = \lceil\lfloor 4/2\rfloor/n\rceil + 1\\
&= 2 \leq \mathrm{sr}(C^*(S_{3,n})) \leq\\
&\mathrm{csr}(C_0(\mathbb{R}_+^2\times\mathbb{T}^2)\otimes M_n(\mathbb{C})) \leq \lceil\lfloor 5/2\rfloor/n\rceil + 1 = 2
\end{aligned}$$

by (F2) and (F3).

Moreover, by (F1) we have

$$\mathrm{csr}(C^*(S_{3,n})) \leq \max\{\mathrm{csr}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}), \mathrm{csr}(C_0(\mathbb{C}^2)\rtimes\mathbb{Z}_n)\}.$$

By (F4) we obtain $\mathrm{csr}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}) \leq 2$. Thus, it follows that $\mathrm{csr}(C^*(S_{3,n})) \leq 2$.

We next compute the K_1-group of $C^*(S_{3,n})$. Namely,

$$K_1(C^*(S_{3,n})) \cong K_1(C^*(H_3^{\mathbb{C}}) \rtimes \mathbb{Z}_n) \cong K_1^{\mathbb{Z}_n}(C^*(H_3^{\mathbb{C}}))$$

([1, Theorem 11.7.1]). Since we have

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H_3^{\mathbb{C}}) \to C_0(\mathbb{C}^2) \to 0,$$

its six term exact sequence of equivariant K-groups by $\mathbb{Z}_n$ is given by

$$\begin{array}{ccccc} K_0^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) & \longrightarrow & K_0^{\mathbb{Z}_n}(C^*(H_3^{\mathbb{C}})) & \longrightarrow & K_0^{\mathbb{Z}_n}(C_0(\mathbb{C}^2)) \\ \uparrow & & & & \downarrow \partial_{\mathbb{Z}_n} \\ K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}^2)) & \longleftarrow & K_1^{\mathbb{Z}_n}(C^*(H_3^{\mathbb{C}})) & \longleftarrow & K_1^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \end{array}$$

since the closed ideal $C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}$ is invariant under the action of $\mathbb{Z}_n$, where $\partial_{\mathbb{Z}_n}$ means the index map (one of two) ([1, Theorem 11.9.6]). Furthermore,

$$K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}^2)) \cong K_1^{\mathbb{Z}_n}(\mathbb{C}) \cong K_1(C^*(\mathbb{Z}_n)) \cong 0,$$
$$K_0^{\mathbb{Z}_n}(C_0(\mathbb{C}^2)) \cong K_0^{\mathbb{Z}_n}(\mathbb{C}) \cong K_0(C^*(\mathbb{Z}_n)) \cong \oplus^n \mathbb{Z}$$

by the Bott periodicity ([1, Theorem 11.9.4]), and

$$\begin{aligned} & K_1^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \cong K_1^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\})) \\ \cong\ & K_1(C_0(\mathbb{R}) \otimes C(\mathbb{T}) \otimes C^*(\mathbb{Z}_n)) \cong K_0(C(\mathbb{T}) \otimes \mathbb{C}^n) \cong \oplus^n \mathbb{Z}, \\ & K_0^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \cong K_0^{\mathbb{Z}_n}(C_0(\mathbb{C} \setminus \{0\})) \\ \cong\ & K_0(C_0(\mathbb{R}) \otimes C(\mathbb{T}) \otimes C^*(\mathbb{Z}_n)) \cong K_1(C(\mathbb{T}) \otimes \mathbb{C}^n) \cong \oplus^n \mathbb{Z}. \end{aligned}$$

since $K_j(C(\mathbb{T})) \cong \mathbb{Z}$ for $j = 0, 1$, where $\mathbb{C} \setminus \{0\} \approx \mathbb{R}_+ \times \mathbb{T} \approx \mathbb{R} \times \mathbb{T}$ (homeomorphic) and the restriction of the action of $\mathbb{Z}_n$ to $\mathbb{C} \setminus \{0\}$ is trivial. On the other hand, the six term exact sequence of K-groups of the above exact sequence is given by

$$\begin{array}{ccccc} K_0(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) & \longrightarrow & K_0(C^*(H_3^{\mathbb{C}})) & \longrightarrow & K_0(C_0(\mathbb{C}^2)) \cong \mathbb{Z} \\ \uparrow & & & & \downarrow \partial \\ K_1(C_0(\mathbb{C}^2)) \cong 0 & \longleftarrow & K_1(C^*(H_3^{\mathbb{C}})) & \longleftarrow & K_1(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \end{array}$$

([1, Theorem 9.3.1]) and $K_j(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}) \cong \mathbb{Z}$ $(j = 0, 1)$ and $K_0(C^*(H_3^{\mathbb{C}})) \cong \mathbb{Z}$, $K_1(C^*(H_3^{\mathbb{C}})) \cong 0$ by using Connes' Thom isomorphism for crossed products of C^*-algebras by $\mathbb{R}$ ([1, Theorem 10.2.2]) since $C^*(H_3^{\mathbb{C}}) \cong C_0(\mathbb{C}^2) \rtimes \mathbb{C}^2 \cong C_0(\mathbb{C}^2) \rtimes \mathbb{R}^2 \rtimes \mathbb{R}^2$. Therefore, the index map ∂ is an isomorphism so that the index map $\partial_{\mathbb{Z}}$ is an isomorphism since $\partial_{\mathbb{Z}_n}$ is induced from ∂ in this setting. Hence the map from $K_1^{\mathbb{Z}^n}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K})$ to $K_1^{\mathbb{Z}_n}(C^*(H_3^{\mathbb{C}}))$ is zero. Therefore, it follows that $K_1^{\mathbb{Z}_n}(C^*(H_3^{\mathbb{C}}))\cong 0$. Thus, $K_1(C^*(S_{3,n}))\cong 0$.

As we showed $\mathrm{sr}(C^*(S_{3,n}))=2$ and $K_1(C^*(S_{3,n}))\cong 0$, the last isomorphism in the statement follows from (F5). ■

Remark. It is very likely that $\mathrm{csr}(C^*(S_{3,n}))=1$.

Its generalization. We define the complex $(2m+1)$-dimensional generalized split oscillator group $S_{2m+1,n}$ involving an n-cyclic symmetry to be the group of all the following matrices:

$$S_{2m+1,n}=\left\{\begin{pmatrix} 1 & z_1 & \cdots & z_m & z_{2m+1} \\ 0 & e^{2\pi it/n} & & & z_{m+1} \\ & & \ddots & & \vdots \\ & & & e^{2\pi it/n} & z_{2m} \\ 0 & & & 0 & 1 \end{pmatrix}=(z_{2m+1},\cdots,z_1,t)\right\}$$

for $t\in\mathbb{Z}_n, z_1,\cdots,z_{2m+1}\in\mathbb{C}$. Note that $S_{2m+1,n}$ is isomorphic to the semi-direct product $H_{2m+1}^{\mathbb{C}}\rtimes_\alpha\mathbb{Z}_n$, where $H_{2m+1}^{\mathbb{C}}$ is the complex $(2m+1)$-dimensional Heisenberg group defined by the matrices $(z_{2m+1},\cdots,z_1,0)$ for $z_1,\cdots,z_{2m+1}\in\mathbb{C}$, and the action α of $\mathbb{Z}_n$ is defined by

$$\alpha_t(z_{2m+1},(z_{m+j})_{j=1}^m,(z_j)_{j=1}^m)=(z_{2m+1},(e^{2\pi it/n}z_{m+j})_{j=1}^m,(e^{-2\pi it/n}z_j)_{j=1}^m)$$

for $t\in\mathbb{Z}_n$. Note also that $H_{2m+1}^{\mathbb{C}}$ is isomorphic to the semi-direct product $\mathbb{C}^{m+1}\rtimes_\beta\mathbb{C}^m$, where the action β of $\mathbb{C}^m$ is defined by

$$\beta_{(z_j)_{j=1}^m}(z_{2m+1},(z_j)_{j=m+1}^{2m})=(z_{2m+1}+\sum_{j=1}^m z_jz_{m+j},(z_j)_{j=m+1}^{2m})$$

for $(z_j)_{j=1}^m\in\mathbb{C}^m$, $z_{2m+1}\in\mathbb{C}$, and $(z_j)_{j=m+1}^{2m}\in\mathbb{C}^m$.

Let $C^*(S_{2m+1,n})$ be the group C^*-algebra of $S_{2m+1,n}$. Then it is isomorphic to the crossed product C^*-algebra $C^*(H_{2m+1}^{\mathbb{C}})\rtimes_\alpha\mathbb{Z}_n$, where $C^*(H_{2m+1}^{\mathbb{C}})$ is the group C^*-algebra of $H_{2m+1}^{\mathbb{C}}$. Since $H_{2m+1}^{\mathbb{C}}\cong\mathbb{C}^{m+1}\rtimes_\beta\mathbb{C}^m$, we have $C^*(H_{2m+1}^{\mathbb{C}})\cong C^*(\mathbb{C}^{m+1})\rtimes\mathbb{C}^m$. By the Fourier transform, $C^*(\mathbb{C}^{m+1})\rtimes_{\beta^\wedge}\mathbb{C}^m$ is isomorphic to $C_0(\mathbb{C}^{m+1})\rtimes\mathbb{C}^m$, where the dual action $\beta^\wedge$ of β via the isomorphism $\mathbb{C}^{m+1}\cong(\mathbb{C}^{m+1})^\wedge$ (the dual group of $\mathbb{C}^{m+1}$) is defined by

$$\beta^\wedge_{(z_j)_{j=1}^m}(w_{2m+1},(w_{m+j})_{j=1}^m)=(w_{2m+1},(w_{m+j}+z_jw_{2m+1})_{j=1}^m)\in\mathbb{C}\times\mathbb{C}^m.$$

Since $\{0\}\times\mathbb{C}^m$ is fixed under the action of $\mathbb{C}^m$, we have the following short exact sequence:

$$0\to C_0((\mathbb{C}\setminus\{0\})\times\mathbb{C}^m)\rtimes\mathbb{C}^m\to C_0(\mathbb{C}^{m+1})\rtimes\mathbb{C}^m\to C_0(\mathbb{C}^m)\otimes C^*(\mathbb{C}^m)\to 0,$$

and $C_0(\mathbb{C}^m)\otimes C^*(\mathbb{C}^m) \cong C_0(\mathbb{C}^{2m})$. Moreover, since the above quotient C^*-algebra is invariant under the action $\alpha^\wedge$ of $\mathbb{Z}_n$, we have

$$0 \to C_0((\mathbb{C}\setminus\{0\})\times\mathbb{C}^m)\rtimes\mathbb{C}^m\rtimes\mathbb{Z}_n \to C_0(\mathbb{C}^{m+1})\rtimes\mathbb{C}^m\rtimes\mathbb{Z}_n \to C_0(\mathbb{C}^{2m})\rtimes\mathbb{Z}_n \to 0.$$

Furthermore, since the action of $\mathbb{C}^m$ on each $\{z\}\times\mathbb{C}^m$ for $z\in\mathbb{C}\setminus\{0\}$ is the shift, we have

$$C_0((\mathbb{C}\setminus\{0\})\times\mathbb{C}^m)\rtimes\mathbb{C}^m \cong C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}.$$

Since the action of $\mathbb{Z}_n$ on $\mathbb{K}$ is implemented by the adjoint action of unitaries,

$$C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}\rtimes\mathbb{Z}_n \cong C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}\otimes C^*(\mathbb{Z}_n) \cong \oplus^n(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}).$$

For the above quotient C^*-algebra $C_0(\mathbb{C}^{2m})\rtimes\mathbb{Z}_n$, from the orbit structure for $\alpha^\wedge$ on $\mathbb{C}^{2m}$ and by the same method as before we have the following decompositions:

$$0 \to C_0(\mathbb{C}^{2m}\setminus\{(0_m)\})\rtimes\mathbb{Z}_n \to C_0(\mathbb{C}^{2m})\rtimes\mathbb{Z}_n \to \mathbb{C}\rtimes\mathbb{Z}_n \to 0,$$
$$0 \to C_0(X_2)\rtimes\mathbb{Z}_n \to C_0(\mathbb{C}^{2m}\setminus\{(0_m)\})\rtimes\mathbb{Z}_n \to \oplus^{2m}(C_0(\mathbb{C}\setminus\{0\})\rtimes\mathbb{Z}_n) \to 0,$$
$$0 \to C_0(X_{k+1})\rtimes\mathbb{Z}_n \to C_0(X_k)\rtimes\mathbb{Z}_n \to \oplus_{j=1}^{{}_{2m}C_k}(C_0((\mathbb{C}\setminus\{0\})^k)\rtimes\mathbb{Z}_n) \to 0$$

for $2\le k\le 2m-1$, and $X_{2m}=(\mathbb{C}\setminus\{0\})^{2m}$. Furthermore, by the same method as before,

$$C_0((\mathbb{C}\setminus\{0\})^k)\rtimes\mathbb{Z}_n \cong C_0(\mathbb{R}_+^k)\otimes(C(\mathbb{T}^k)\rtimes\mathbb{Z}_n),$$

and $C(\mathbb{T}^k)\rtimes\mathbb{Z}_n \cong C(\mathbb{T}^k/\mathbb{Z}_n)\otimes(\mathbb{C}^n\rtimes\mathbb{Z}_n)\cong C(\mathbb{T}^k)\otimes M_n(\mathbb{C})$.

Therefore we obtain

Theorem 2.3. *Let $S_{2m+1,n}=H^{\mathbb{C}}_{2m+1}\rtimes_\alpha\mathbb{Z}_n$ be the complex $(2m+1)$-dimensional generalized split oscillator group involving the n-cyclic symmetry α. For the group C^*-algebra $C^*(S_{2m+1,n})$ of $S_{2m+1,n}$, we have the following exact sequence*:

$$0 \to \oplus^n(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}) \to C^*(S_{2m+1,n}) \to C_0(\mathbb{C}^{2m})\rtimes_{\alpha^\wedge}\mathbb{Z}_n \to 0,$$

and the above quotient C^*-algebra has the following exact sequences:

$$0\to\mathfrak{I}_{k+1}\to\mathfrak{I}_k\to\mathfrak{I}_k/\mathfrak{I}_{k+1}\to 0 \quad \textit{for } 0\le k\le 2m$$

of its closed ideals such that $\mathfrak{I}_0=C_0(\mathbb{C}^{2m})\rtimes_{\alpha^\wedge}\mathbb{Z}_n$, $\mathfrak{I}_{2m+1}=\{0\}$, $\mathfrak{I}_0/\mathfrak{I}_1\cong\mathbb{C}^n$ and

$$\mathfrak{I}_k/\mathfrak{I}_{k+1}\cong\oplus_{j=1}^{{}_{2m}\mathrm{C}_k}[C_0((\mathbb{R}_+)^k)\otimes C(\mathbb{T}^k)\otimes M_n(\mathbb{C})]$$

for $1\le k\le 2m$.

Using the structure theorem above we obtain

Theorem 2.4. *For the group C^*-algebra $C^*(S_{2m+1,n})$ of $S_{2m+1,n}$,*

$$\mathrm{sr}(C^*(S_{2m+1,n})) = \lceil 2m/n \rceil + 1 = \mathrm{sr}(C_0(\mathbb{R}_+^{2m} \times \mathbb{T}^{2m}) \otimes M_n(\mathbb{C})),$$

where $2m = \dim(\mathbb{R}_+^{2m} \times \mathbb{T}^{2m})/2$, and $4m$ is the (maximum) dimension of the spaces $\mathbb{R}_+^k \times \mathbb{T}^k$ $(1 \leq k \leq 2m)$ of n-dimensional irreducible representations of $C^*(S_{2m+1,n})$ up to unitary equivalence. If $n \geq 2m$, then $\mathrm{sr}(C^*(S_{2m+1,n})) = 2$.

Moreover, we have

$$\mathrm{csr}(C^*(S_{2m+1,n})) \leq \lceil 2m/n \rceil + 1,$$

and $K_1(C^*(S_{2m+1,n})) \cong 0$, and if $n \geq 2m$, then

$$GL_k(C^*(S_{2m+1,n}))/GL_k(C^*(S_{2m+1,n}))_0 \cong 0 \quad (k \geq 2).$$

Proof. Using the structure of Theorem 2.3 and the formula (F1) we have

$$\begin{aligned} &\max\{\mathrm{sr}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}), \mathrm{sr}(C_0(\mathbb{C}^{2m}) \rtimes \mathbb{Z}_n)\} \leq \mathrm{sr}(C^*(S_{2m+1,n})) \\ &\leq \max\{\mathrm{sr}(C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}), \mathrm{sr}(C_0(\mathbb{C}^{2m}) \rtimes \mathbb{Z}_n), \mathrm{csr}(C_0(\mathbb{C}^{2m}) \rtimes \mathbb{Z}_n)\}, \end{aligned}$$

and

$$\begin{aligned} &\max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(\mathbb{C}^n)\} \leq \mathrm{sr}(C_0(\mathbb{C}^{2m}) \rtimes \mathbb{Z}_n) \\ &\leq \max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(\mathbb{C}^n), \mathrm{csr}(\mathbb{C}^n)\}, \quad \text{and} \\ &\max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\} \leq \mathrm{sr}(\mathfrak{I}_k) \\ &\leq \max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\} \quad (1 \leq k \leq 2m-1). \end{aligned}$$

Furthermore, using (F2) and (F3) we have

$$\begin{aligned} &\mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) = \mathrm{sr}(C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}^k) \otimes M_n(\mathbb{C})) \\ &= \lceil \lfloor 2k/2 \rfloor / n \rceil + 1 = \lceil k/n \rceil + 1, \\ &\mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) = \mathrm{csr}(C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}^k) \otimes M_n(\mathbb{C})) \\ &\leq \lceil \lfloor (2k+1)/2 \rfloor / n \rceil + 1 = \lceil k/n \rceil + 1 \end{aligned}$$

for $1 \leq k \leq 2m$. Therefore, we obtain

$$\mathrm{sr}(C^*(S_{2m+1,n})) = \mathrm{sr}(\mathfrak{I}_{2m}) = \lceil 2m/n \rceil + 1.$$

Moreover, by (F1) we obtain

$$\mathrm{csr}(\mathfrak{I}_k) \leq \max\{\mathrm{csr}(\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\}$$

for $0 \leq k \leq 2m$. Hence, it follows that

$$\mathrm{csr}(C_0(\mathbb{C}^{2m}) \rtimes \mathbb{Z}_n) = \mathrm{csr}(\mathfrak{I}_0) \leq \max_{0\leq k\leq m} \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) \leq \lceil 2m/n \rceil + 1$$

and $\mathrm{csr}(C^*(S_{2m+1,n})) \leq \max\{2, \lceil 2m/n\rceil + 1\} = \lceil 2m/n\rceil + 1$ by (F4).

We next compute the K_1-group of $C^*(S_{2m+1,n})$:

$$K_1(C^*(S_{2m+1,n})) \cong K_1(C^*(H^{\mathbb{C}}_{2m+1}) \rtimes \mathbb{Z}_n) \cong K_1^{\mathbb{Z}_n}(C^*(H^{\mathbb{C}}_{2m+1}))$$

([1, Theorem 11.7.1]). Since we have

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H^{\mathbb{C}}_{2m+1}) \to C_0(\mathbb{C}^{2m}) \to 0,$$

its six term exact sequence of equivariant K-groups by $\mathbb{Z}_n$ is given by

$$\begin{array}{ccccc}
K_0^{\mathbb{Z}_n}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}) & \longrightarrow & K_0^{\mathbb{Z}_n}(C^*(H^{\mathbb{C}}_{2m+1})) & \longrightarrow & K_0^{\mathbb{Z}_n}(C_0(\mathbb{C}^{2m})) \\
\uparrow & & & & \downarrow \partial_{\mathbb{Z}_n} \\
K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}^{2m})) & \longleftarrow & K_1^{\mathbb{Z}_n}(C^*(H^{\mathbb{C}}_{2m+1})) & \longleftarrow & K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K})
\end{array}$$

since the closed ideal $C_0(\mathbb{C} \setminus \{0\}) \otimes \mathbb{K}$ is invariant under the action of $\mathbb{Z}_n$ ([1, Theorem 11.9.6]). Furthermore,

$$K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}^{2m})) \cong K_1^{\mathbb{Z}_n}(\mathbb{C}) \cong K_1(C^*(\mathbb{Z}_n)) \cong 0,$$
$$K_0^{\mathbb{Z}_n}(C_0(\mathbb{C}^{2m})) \cong K_0^{\mathbb{Z}_n}(\mathbb{C}) \cong K_0(C^*(\mathbb{Z}_n)) \cong \oplus^n\mathbb{Z}$$

by the Bott periodicity ([1, Theorem 11.9.4]), and $K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K}) \cong \oplus^n\mathbb{Z}$ as shown in the proof of Theorem 2.2. On the other hand, the six term exact sequence of K-groups of the above exact sequence is given by

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & K_0(C^*(H^{\mathbb{C}}_{2m+1})) & \longrightarrow & K_0(C_0(\mathbb{C}^{2m})) \cong \mathbb{Z} \\
\uparrow & & & & \downarrow \partial \\
K_1(C_0(\mathbb{C}^{2m})) \cong 0 & \longleftarrow & K_1(C^*(H^{\mathbb{C}}_{2m+1})) & \longleftarrow & \mathbb{Z}
\end{array}$$

([1, Theorem 9.3.1]) and $K_0(C^*(H^{\mathbb{C}}_{2m+1})) \cong \mathbb{Z}$, $K_1(C^*(H^{\mathbb{C}}_{2m+1})) \cong 0$ by using Connes' Thom isomorphism for crossed products of C^*-algebras by $\mathbb{R}$ since $C^*(H^{\mathbb{C}}_{2m+1})) \cong C_0(\mathbb{C}^{m+1}) \rtimes \mathbb{C}^m \cong C_0(\mathbb{C}^{m+1}) \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$ ($2m$-times) ([1, Theorem 10.2.2]). Therefore, the index map ∂ is an isomorphism so that the index map

$\partial_{\mathbb{Z}}$ is an isomorphism since $\partial_{\mathbb{Z}_n}$ is induced from ∂ in this setting. Hence the map from $K_1^{\mathbb{Z}_n}(C_0(\mathbb{C}\setminus\{0\})\otimes\mathbb{K})$ to $K_1^{\mathbb{Z}_n}(C^*(H_{2m+1}^{\mathbb{Z}}))$ is zero. Therefore, it follows that $K_1^{\mathbb{Z}_n}(C^*(H_{2m+1}^{\mathbb{C}}))\cong 0$. Thus, $K_1(C^*(S_{2m+1,n}))\cong 0$.

As we showed, $\mathrm{sr}(C^*(S_{2m+1,n}))=2$ if $n\geq 2m$. Hence, the last isomorphism in the statement follows from (F5). ■

3. The Complex (Generalized) Mautner Groups Involving Cyclic (and Free) Symmetries

We define the complex 2-dimensional Mautner group $M_{2,n}$ involving an n-cyclic (and free) symmetry to be the group of the following matrices:

$$M_{2,n}=\left\{\begin{pmatrix} e^{2\pi i\theta t} & 0 & z_1 \\ 0 & e^{2\pi it/n} & z_2 \\ 0 & 0 & 1\end{pmatrix}=(z_1,z_2,t)\quad \text{for } t\in\mathbb{Z}, z_1,z_2\in\mathbb{C}\right\}$$

where θ is an irrational number. Note that $M_{2,n}$ is isomorphic to the semi-direct product $\mathbb{C}^2\rtimes_\alpha\mathbb{Z}$, where the action α of $\mathbb{Z}$ is defined by $\alpha_t(z_1,z_2)=(e^{2\pi i\theta t}z_1,e^{2\pi it/n}z_2)$ for $t\in\mathbb{Z}$. Note that this action is not cyclic but cyclic on $\{0\}\times\mathbb{C}$ and (almost) free on its complement in the sense as given below. We call α an n-cyclic (and free) symmetry.

Let $C^*(M_{2,n})$ be the group C^*-algebra of $M_{2,n}$. Then $C^*(M_{2,n})$ is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{C}^2)\rtimes_\alpha\mathbb{Z}$. By the Fourier transform, $C^*(\mathbb{C}^2)\rtimes_\alpha\mathbb{Z}$ is isomorphic to $C_0(\mathbb{C}^2)\rtimes_{\alpha^\wedge}\mathbb{Z}$, where the action $\alpha^\wedge$ is defined by $\alpha_t^\wedge(w_1,w_2)=(e^{-2\pi i\theta t}w_1,e^{-2\pi it/n}w_2)\in\mathbb{C}^2$ via $\mathbb{C}^2\cong(\mathbb{C}^2)^\wedge$.

Since $\{0\}\times\{0\}$ is fixed under the action of $\mathbb{Z}$, we have the following short exact sequence:

$$0\to C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}\to C_0(\mathbb{C}^2)\rtimes\mathbb{Z}\to C^*(\mathbb{Z})\to 0,$$

and $C^*(\mathbb{Z})\cong C(\mathbb{T})$. Since the spaces $\{0\}\times(\mathbb{C}\setminus\{0\})$, $(\mathbb{C}\setminus\{0\})\times\{0\}$ are closed in $\mathbb{C}^2\setminus\{(0,0)\}$ and invariant under the action of $\mathbb{Z}$, we have

$$\begin{aligned}&0\to C_0((\mathbb{C}\setminus\{0\})^2)\rtimes\mathbb{Z}\to C_0(\mathbb{C}^2\setminus\{(0,0)\})\rtimes\mathbb{Z}\\ &\quad\to (C_0(\mathbb{C}\setminus\{0\})\rtimes_{\alpha^{\wedge,2}}\mathbb{Z})\oplus(C_0(\mathbb{C}\setminus\{0\})\rtimes_{\alpha^{\wedge,1}}\mathbb{Z})\to 0,\end{aligned}$$

where $\alpha^{\wedge,2}$, $\alpha^{\wedge,1}$ are the restrictions of $\alpha^\wedge$ to the above invariant spaces respectively. Set $\mathfrak{D}_j=C_0(\mathbb{C}\setminus\{0\})\rtimes_{\alpha^{\wedge,j}}\mathbb{Z}$ for $j=1,2$. Then $\mathfrak{D}_2$ has the following decomposition:

$$0\to C_0(\mathbb{R})\otimes(C_0(\mathbb{C}\setminus\{0\})\rtimes_{\alpha^{\wedge,2}}\mathbb{Z}_n)\to\mathfrak{D}_2\to C_0(\mathbb{C}\setminus\{0\})\rtimes_{\alpha^{\wedge,2}}\mathbb{Z}_n\to 0$$

because $\mathfrak{D}_2$ can be viewed as the mapping torus on $C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\alpha^{\wedge,2}} \mathbb{Z}_n$ since the action $\alpha^{\wedge,2}$ is n-cyclic (see [1, Section 10.3]). By the analysis for $C^*(A_{1,n})$ in Section 1, we have

$$C_0((\mathbb{C} \setminus \{0\}) \rtimes_{\alpha^{\wedge,2}} \mathbb{Z}_n \cong C_0(\mathbb{R}_+) \otimes C(\mathbb{T}) \otimes M_n(\mathbb{C}).$$

Since the action of $\mathbb{Z}$ on the radius direction in $\mathbb{C} \setminus \{0\}$ is trivial,

$$C_0((\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{Z} \cong C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^2) \rtimes_\theta \mathbb{Z}),$$
$$C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\alpha^{\wedge,1}} \mathbb{Z} \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z}),$$

where the actions θ mean the restrictions of $\alpha^\wedge$ to $\mathbb{T}^2$, and to $\mathbb{T}$ respectively, and the crossed product C^*-algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is the simple irrational rotation C^*-algebra corresponding to θ. For the crossed product C^*-algebra $C(\mathbb{T}^2) \rtimes_\theta \mathbb{Z}$, we have

$$C(\mathbb{T}^2) \rtimes_\theta \mathbb{Z} \cong C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}) \cong C(\mathbb{T}) \otimes ((\oplus^n C(\mathbb{T})) \rtimes_\theta \mathbb{Z})$$

since each subspace $\mathbb{T} \times \{p_j\}_{j=1}^n$ for an orbit $\{p_j\}_{j=1}^n$ (as the n-th roots of unity) in $\mathbb{T} = \{0\} \times \mathbb{T} \subset \mathbb{T}^2$ by $\mathbb{Z}$ is invariant under the action of $\mathbb{Z}$, and

$$C(\mathbb{T} \times \{p_j\}_{j=1}^n) \rtimes_\theta \mathbb{Z} \cong (C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z},$$

and note that the orbit space $\mathbb{T}/\mathbb{Z}$ for the second $\mathbb{T}$ in $\mathbb{T}^2$ is homeomorphic to $\mathbb{T}$, and the action by $\mathbb{Z}$ on each $\mathbb{T} \times \{p_j\}_{j=1}^n$ is free. Since each orbit by $\mathbb{Z}$ in $\mathbb{T} \times \{p_j\}_{j=1}^n$ is dense in it, the crossed product C^*-algebra $(C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}$ is simple, but not isomorphic to any noncommutative torus, and even not stably isomorphic to it. Indeed, the claims follow from considering their generators and relations (with or without orthogonality) and using universality (and also (in part) from computing their K-groups via the Pimsner-Voiculescu exact sequence for K-groups of crossed products by $\mathbb{Z}$ ([1, Theorem 10.2.1])). Note that a noncommutative k-torus is defined to be the universal C^*-algebra generated by k unitaries U_j $(1 \leq j \leq k)$ such that $U_j U_i = e^{2\pi i \theta_{ij}} U_i U_j$, where $(\theta_{ij})_{i,j=1}^k$ is a skew adjoint $k \times k$ matrix over $\mathbb{R}$, and its K_0 and K_1-groups are both isomorphic to $\mathbb{Z}^{2^{k-1}}$ (see Rieffel [16]).

Summing up the above argument we obtain

Theorem 3.1. *Let $M_{2,n} = \mathbb{C}^2 \rtimes_\alpha \mathbb{Z}$ be the complex 2-dimensional Mautner group involving the n-cyclic* (*and free*) *symmetry α corresponding to an irrational number θ. For the group C^*-algebra $C^*(M_{2,n})$ of $M_{2,n}$, we have the following exact sequences*:

$$0 \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z} \to C^*(M_{2,n}) \to C(\mathbb{T}) \to 0,$$
$$0 \to C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}) \to C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z}$$
$$\to \mathfrak{D}_2 \oplus (C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})) \to 0, \quad \textit{and}$$
$$0 \to C_0(\mathbb{R} \times \mathbb{R}_+) \otimes C(\mathbb{T}) \otimes M_n(\mathbb{C}) \to \mathfrak{D}_2 \to C_0(\mathbb{R}_+) \otimes C(\mathbb{T}) \otimes M_n(\mathbb{C}) \to 0,$$

where $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is the irrational rotation C^*-algebra corresponding to θ and $(C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}$ is the n-cyclic irrational rotation C^*-algebra corresponding to θ (where we call it so) and is simple but not isomorphic to any noncommutative torus, and even not stably isomorphic to it.

Using the structure theorem above we obtain

Theorem 3.2. *The group C^*-algebra $C^*(M_{2,n})$ of $M_{2,n}$ has stable rank 2, and*

$$\begin{aligned}\mathrm{sr}(C^*(M_{2,n})) = 2 = \lfloor \dim C^*(M_{2,n})^\wedge_n/2 \rfloor + 1, \quad &\textit{but} \\ \neq \dim C^*(M_{2,n})^\wedge_n = 3,&\end{aligned}$$

where $C^*(M_{2,n})^\wedge_n$ means the space of all n-dimensional irreducible representations of $C^*(M_{2,n})$ up to unitary equivalence, and has an open subset homeomorphic to $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$ whose complement is $\mathbb{R}_+ \times \mathbb{T}$.

Moreover, we obtain $\mathrm{csr}(C^*(M_{2,n})) = 2$ and

$$K_1(C^*(M_{2,n})) \cong \mathbb{Z} \cong GL_k(C^*(M_{2,n})))/GL_k(C^*(M_{2,n})))_0 \quad (k \geq 2).$$

Proof. Using the structure of Theorem 3.1 and the formula (F1) we have

$$\begin{aligned}
&\max\{\mathrm{sr}(C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z}), \mathrm{sr}(C(\mathbb{T}))\} \leq \mathrm{sr}(C^*(M_{2,n})) \\
&\leq \max\{\mathrm{sr}(C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z}), \mathrm{sr}(C(\mathbb{T})), \mathrm{csr}(C(\mathbb{T}))\}, \quad \text{and} \\
&\max\{\mathrm{sr}(C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z})), \\
&\mathrm{sr}(\mathfrak{D}_2), \mathrm{sr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z}))\} \\
&\leq \mathrm{sr}(C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z}) \\
&\leq \max\{\mathrm{sr}(C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z})), \mathrm{sr}(\mathfrak{D}_2), \mathrm{csr}(\mathfrak{D}_2), \\
&\mathrm{sr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})), \mathrm{csr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z}))\}, \quad \text{and} \\
&\max\{\mathrm{sr}(C_0(\mathbb{R}^2 \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{sr}(C_0(\mathbb{R} \times \mathbb{T}) \otimes M_n(\mathbb{C}))\} \\
&\leq \mathrm{sr}(\mathfrak{D}_2) \leq \max\{\mathrm{sr}(C_0(\mathbb{R}^2 \times \mathbb{T}) \otimes M_n(\mathbb{C})), \\
&\mathrm{sr}(C_0(\mathbb{R} \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{csr}(C_0(\mathbb{R} \times \mathbb{T}) \otimes M_n(\mathbb{C}))\}
\end{aligned}$$

where $\mathbb{R}^2 \approx \mathbb{R} \times \mathbb{R}_+$ (homeomorphic). Note that $\mathrm{sr}(C(\mathbb{T})) = 1$ by (F2) and $\mathrm{csr}(C(\mathbb{T})) = 2$ by [18, p. 381]. By (F2) and (F3)

$$\begin{aligned}
&\mathrm{sr}(C_0(\mathbb{R}^s \times \mathbb{T}) \otimes M_n(\mathbb{C})) = \lceil \lfloor (s+1)/2 \rfloor / n \rceil + 1 = 2, \quad (s = 1, 2) \\
&\mathrm{csr}(C_0(\mathbb{R} \times \mathbb{T}) \otimes M_n(\mathbb{C})) \leq \lceil \lfloor (2+1)/2 \rfloor / n \rceil + 1 = 2.
\end{aligned}$$

It is known that the irrational rotation C^*-algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is an AT-algebra, i.e., an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$ (see Elliott and Evans [7]) so that we have $C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})$ is an inductive limit of finite direct sums of matrix algebras over $C_0(\mathbb{R}_+ \times \mathbb{T})$. Hence, we obtain $\mathrm{sr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})) \le 2$ and $\mathrm{csr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})) \le 2$. Furthermore, we can show that $(C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}$ is also an AT-algebra since each C^*-subalgebra of the form $(C(\mathbb{T}) \otimes (\mathbb{C} \oplus 0 \cdots \oplus 0)) \rtimes_{\theta'} n\mathbb{Z}$ in $(C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z}$ is an AT-algebra, where the action θ' is the restriction of θ to $n\mathbb{Z}$ (cf. [8]). Thus, it follows that

$$\mathrm{sr}(C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z})) \le 2,$$
$$\mathrm{csr}(C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z})) \le 2$$

by [14, Theorem 5.1] for the stable rank of inductive limits of C^*-algebras and its connected stable rank version. Therefore, we obtain $\mathrm{sr}(C^*(M_{2,n})) = 2$.

Moreover, by (F1) we have

$$\begin{aligned}
&\mathrm{csr}(C^*(M_{2,n})) \le \max\{\mathrm{csr}(C_0(\mathbb{C}^2 \setminus \{(0,0)\}) \rtimes_{\alpha^\wedge} \mathbb{Z}), \mathrm{csr}(C(\mathbb{T}))\} \\
\le\ &\max\{\mathrm{csr}(C_0(\mathbb{R}_+^2) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}) \otimes \mathbb{C}^n) \rtimes_\theta \mathbb{Z})), \\
&\mathrm{csr}(C_0(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})), \mathrm{csr}(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes M_n(\mathbb{C})), \\
&\mathrm{csr}(C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})), \mathrm{csr}(C(\mathbb{T})) = 2\} \le 2.
\end{aligned}$$

To determine $\mathrm{csr}(C^*(M_{2,n}))$ we compute the K_1-group of $C^*(M_{2,n})$. Since $C^*(M_{2,n}) \cong C_0(\mathbb{C}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}$, we use Pimsner-Voiculescu six term exact sequence for crossed products of C^*-algebras by $\mathbb{Z}$ given by

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{C}^2)) & \xrightarrow{1-\alpha^\wedge_*} & K_0(C_0(\mathbb{C}^2)) & \longrightarrow & K_0(C_0(\mathbb{C}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(C_0(\mathbb{C}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}) & \longleftarrow & K_1(C_0(\mathbb{C}^2)) & \longleftarrow & K_1(C_0(\mathbb{C}^2))
\end{array}$$

([1, Theorem 10.2.1]) and $K_0(C_0(\mathbb{C}^2)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{C}^2)) \cong K_1(\mathbb{C}) \cong 0$ by the Bott periodicity ([1, Theorem 9.2.1]). Since the map $1-\alpha^\wedge_*$ on $K_0(C_0(\mathbb{C}^2))$ is trivial, it follows that $K_1(C_0(\mathbb{C}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}) \cong \mathbb{Z}$. Thus, we have $\mathrm{csr}(C^*(M_{2,n})) \ge 2$. Hence, $\mathrm{csr}(C^*(M_{2,n})) = 2$. Use also (F5). ■

Its generalization. We define the complex m-dimensional generalized Mautner group $M_{m,n}$ involving an n-cyclic (and free) symmetry to be the group of all the following matrices:

$$M_{m,n} = \left\{ \begin{pmatrix} e^{2\pi i\theta_1 t} & & & & z_1 \\ & \ddots & & & \vdots \\ & & e^{2\pi i\theta_{m-1}t} & 0 & z_{m-1} \\ & & & e^{2\pi it/n} & z_m \\ 0 & & & 0 & 1 \end{pmatrix} = (z_1, \cdots, z_m, t) \right\}$$

for $t \in \mathbb{Z}, z_1, \cdots, z_m \in \mathbb{C}$, where $\theta_1, \cdots, \theta_{m-1}$ are mutually rationally independent irrational numbers. Note that $M_{m,n}$ is isomorphic to the semi-direct product $\mathbb{C}^m \rtimes_\alpha \mathbb{Z}$, where the action α of $\mathbb{Z}$ is defined by $\alpha_t(z_j)_{j=1}^m = ((e^{2\pi i\theta_j t} z_j)_{j=1}^{m-1}, e^{2\pi it/n} z_m)$ for $t \in \mathbb{Z}$. Note that this action is not cyclic but cyclic on $\{0\} \times \cdots \times \{0\} \times \mathbb{C}$ and (almost) free on its complement in the sense as given below. We call α an n-cyclic (and free) symmetry.

Let $C^*(M_{m,n})$ be the group C^*-algebra of $M_{m,n}$. Then it is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{C}^m) \rtimes_\alpha \mathbb{Z}$. By the Fourier transform, $C^*(\mathbb{C}^m) \rtimes_\alpha \mathbb{Z}$ is isomorphic to $C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}$, where the action $\alpha^\wedge$ is defined by $\alpha_t^\wedge(w_j)_{j=1}^m = ((e^{-2\pi i\theta_j t} w_j)_{j=1}^{m-1}, e^{-2\pi it/n} w_m) \in \mathbb{C}^m$ via $\mathbb{C}^m \cong (\mathbb{C}^m)^\wedge$.

Since $0_m \equiv \{0\} \times \cdots \times \{0\}$ is fixed under the action of $\mathbb{Z}$, we have the following short exact sequence:

$$0 \to C_0(\mathbb{C}^m \setminus \{0_m\}) \rtimes \mathbb{Z} \to C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z} \to C^*(\mathbb{Z}) \to 0,$$

and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$. Since the following m subspaces X_{1j} $(1 \le j \le m)$:

$$X_{11} = (\mathbb{C} \setminus \{0\}) \times \{0\} \times \cdots \times \{0\}, \cdots, X_{1m} = \{0\} \times \cdots \times \{0\} \times (\mathbb{C} \setminus \{0\})$$

are invariant under $\alpha^\wedge$ and disjoint and closed in $\mathbb{C}^m \setminus \{0_m\}$, we have

$$0 \to C_0(X_2) \rtimes \mathbb{Z} \to C_0(\mathbb{C}^m \setminus \{0_m\}) \rtimes \mathbb{Z} \to \oplus_{j=1}^m (C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{Z}) \to 0,$$

where X_2 is the complement of $\sqcup_{j=1}^m X_{1j}$ in $\mathbb{C}^m \setminus \{0_m\}$. Inductively, we can construct the following:

$$0 \to C_0(X_{k+1}) \rtimes_{\alpha^\wedge} \mathbb{Z} \to C_0(X_k) \rtimes_{\alpha^\wedge} \mathbb{Z} \to \oplus_{j=1}^{{}_m\mathrm{C}_k} (C_0(X_{kj}) \rtimes_{\alpha^\wedge} \mathbb{Z}) \to 0$$

for $2 \le k \le m-1$, where the subspaces X_{kj} are defined by

$$X_{k1} = (\mathbb{C} \setminus \{0\})^k \times (\{0\})^{m-k}, \cdots, X_{k({}_m\mathrm{C}_k)} = (\{0\})^{m-k} \times (\mathbb{C} \setminus \{0\})^k.$$

Indeed, note that the subspaces X_{kj} are invariant under $\alpha^\wedge$ and disjoint and closed in X_k so that X_{k+1} is defined to be the complement of $\sqcup_{j=1}^{{}_m\mathrm{C}_k} X_{kj}$ in X_k. In particular, we have $X_m = X_{m1} = (\mathbb{C} \setminus \{0\})^m$.

Furthermore, since the action of $\mathbb{Z}$ is trivial on the radius direction in $\mathbb{C} \setminus \{0\}$ of each X_{kj},

$$C_0(X_{kj}) \rtimes_{\alpha^\wedge} \mathbb{Z} \cong C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\alpha^\wedge} \mathbb{Z} \cong C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z}),$$

where $C(\mathbb{T}^k)\rtimes\mathbb{Z}$ is the simple noncommutative $(k+1)$-torus corresponding to $\Theta_k = (\theta_j)_{j=s_1}^{s_k}$ if $1 \le s_1 < \cdots < s_k \le m-1$ and is not isomorphic to a noncommutative $(k+1)$-torus if $s_k = m$. Indeed, this follows from considering their generators and relations, and using universality. For the former case, let $C(\mathbb{T}^k) \rtimes \mathbb{Z} = \mathbb{T}_{\Theta_k}^{k+1}$. For the latter case, we have

$$\begin{aligned} C(\mathbb{T}^k) \rtimes_{\alpha^\wedge} \mathbb{Z} &\cong C(\mathbb{T}) \otimes ((C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}) \\ &\cong C(\mathbb{T}) \otimes ((\oplus^n C(\mathbb{T}^{k-1})) \rtimes_{\Theta_{k-1}} \mathbb{Z}), \end{aligned}$$

where $\Theta_{k-1} = (\theta_j)_{j=s_1}^{s_{k-1}}$ for $1 \le s_1 < \cdots < s_{k-1} \le m-1$. Then the crossed product C^*-algebra $(C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}$ is simple but not isomorphic to any noncommutative torus, and even not stably isomorphic to it as shown before Theorem 3.1.

Summing up the above argument we obtain

Theorem 3.3. *Let $M_{m,n} = \mathbb{C}^m \rtimes_\alpha \mathbb{Z}$ be the complex m-dimensional generalized Mautner group involving the n-cyclic* (*and free*) *symmetry α. For the group C^*-algebra $C^*(M_{m,n})$ of $M_{m,n}$, we have the following exact sequences*:

$$0 \to \mathfrak{I}_{k+1} \to \mathfrak{I}_k \to \mathfrak{I}_k/\mathfrak{I}_{k+1} \to 0 \quad \text{for } 0 \le k \le m$$

of its closed ideals such that $\mathfrak{I}_0 = C^*(M_{m,n})$, $\mathfrak{I}_{m+1} = \{0\}$, $\mathfrak{I}_0/\mathfrak{I}_1 \cong C(\mathbb{T})$, and

$$\mathfrak{I}_k/\mathfrak{I}_{k+1} \cong \oplus_{j=1}^{{}_m C_k} [C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z})]$$

for $1 \le k \le m$, where for $k = 1$ and each $1 \le j \le {}_m C_1 - 1 = m-1$, $C(\mathbb{T}) \rtimes \mathbb{Z}$ is the simple irrational rotation C^*-algebra corresponding to given θ_j $(1 \le j \le m-1)$, and for $j = m$, $C(\mathbb{T}) \rtimes \mathbb{Z}$ is the mapping torus on $C(\mathbb{T}) \rtimes \mathbb{Z}_n$, i.e., we have the decomposition:

$$\begin{aligned} &0 \to C_0(\mathbb{R}) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z}_n) \to C(\mathbb{T}) \rtimes \mathbb{Z} \to C(\mathbb{T}) \rtimes \mathbb{Z}_n \to 0, \quad \textit{and} \\ &C(\mathbb{T}) \rtimes \mathbb{Z}_n \cong C(\mathbb{T}/\mathbb{Z}_n) \otimes (\mathbb{C}^n \rtimes \mathbb{Z}_n) \cong C(\mathbb{T}) \otimes M_n(\mathbb{C}), \end{aligned}$$

and for $k \ge 2$, $C(\mathbb{T}^k) \rtimes \mathbb{Z} = \mathbb{T}_{\Theta_k}^{k+1}$ is the simple noncommutative $(k+1)$-torus corresponding to $\Theta_k = (\theta_j)_{j=s_1}^{s_k}$ if $1 \le s_1 < \cdots < s_k \le m-1$ and is not isomorphic to a noncommutative $(k+1)$-torus if $s_k = m$, and then

$$\begin{aligned} C(\mathbb{T}^k) \rtimes_{\alpha^\wedge} \mathbb{Z} &\cong C(\mathbb{T}) \otimes ((C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}) \\ &\cong C(\mathbb{T}) \otimes ((\oplus^n C(\mathbb{T}^{k-1})) \rtimes_{\Theta_{k-1}} \mathbb{Z}), \end{aligned}$$

where $\Theta_{k-1} = (\theta_j)_{j=s_1}^{s_{k-1}}$ for $1 \le s_1 < \cdots < s_{k-1} \le m-1$, and $(C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}$ is the n-cyclic noncommutative k-torus (*where we call it so*) *and* *is simple but not isomorphic to any noncommutative torus, and even not stably isomorphic to it.*

Using the structure theorem above we obtain

Theorem 3.4. *The group C^*-algebra $C^*(M_{m,n})$ of $M_{m,n}$ has stable rank 2, and*

$$\begin{aligned}\mathrm{sr}(C^*(M_{m,n})) = 2 &= \lfloor \dim C^*(M_{m,n})^\wedge_n/2\rfloor + 1, \quad \textit{but}\\ &\neq \dim C^*(M_{m,n})^\wedge_n = 3,\end{aligned}$$

where $C^*(M_{m,n})^\wedge_n$ means the space of all n-dimensional irreducible representations of $C^*(M_{m,n})$ up to unitary equivalence and has an open subset homeomorphic to $\mathbb{R}\times\mathbb{R}_+\times\mathbb{T}$ whose complement is $\mathbb{R}_+\times\mathbb{T}$.

Moreover, we obtain $\mathrm{csr}(C^*(M_{m,n})) = 2$ and

$$K_1(C^*(M_{m,n})) \cong \mathbb{Z} \cong GL_k(C^*(M_{m,n}))/GL_k(C^*(M_{m,n}))_0 \quad (k \geq 2).$$

Proof. Using the structure of Theorem 3.3 and (F1) we have

$$\begin{aligned}&\max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(C(\mathbb{T}))\} \leq \mathrm{sr}(C^*(M_{m,n})) \leq \max\{\mathrm{sr}(\mathfrak{I}_1), \mathrm{sr}(C(\mathbb{T})), \mathrm{csr}(C(\mathbb{T}))\},\\ &\max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\} \leq \mathrm{sr}(\mathfrak{I}_k)\\ \leq\ &\max\{\mathrm{sr}(\mathfrak{I}_{k+1}), \mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\}.\end{aligned}$$

Furthermore, we have $\mathrm{sr}(C(\mathbb{T})) = 1$, $\mathrm{csr}(C(\mathbb{T})) = 2$, and

$$\begin{aligned}\mathrm{sr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) &= \mathrm{sr}(C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z})) \leq 2,\\ \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) &= \mathrm{csr}(C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z})) \leq 2\end{aligned}$$

because if $C(\mathbb{T}^k) \rtimes \mathbb{Z} = \mathbb{T}^{k+1}_{\Theta_k}$ is a simple noncommutative $(k+1)$-torus $(k \geq 1)$, then it is an AT-algebra (see Elliott and Q. Lin [8]) so that $C_0((\mathbb{R}_+)^k)\otimes(C(\mathbb{T}^k)\rtimes\mathbb{Z})$ is an inductive limit of finite direct sums of matrix algebras over $C_0(\mathbb{R}^k_+\times\mathbb{T})$, from which the tensor product C^*-algebra has stable rank ≤ 2 and connected stable rank ≤ 2 ([14, Theorem 5.1] for the stable rank of inductive limits of C^*-algebras and use its connected stable rank version), and if

$$C(\mathbb{T}^k) \rtimes_{\alpha^\wedge} \mathbb{Z} \cong C(\mathbb{T}) \otimes ((C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}),$$

then we can show that $(C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}$ is an AT-algebra since each C^*-subalgebra of the form $(C(\mathbb{T}^{k-1}) \otimes (\mathbb{C}\times\{0\}\cdots\times\{0\})) \rtimes_{\Theta'_{k-1}} n\mathbb{Z}$ in $(C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z}$ is an AT-algebra, where the action Θ'_{k-1} is the restriction of Θ_{k-1} to $n\mathbb{Z}$ (cf. [8]). Hence, $C_0((\mathbb{R}_+)^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{Z})$ is an inductive limit of finite direct sums of matrix algebras over $C_0(\mathbb{R}_+^k \times \mathbb{T}^2)$, from which the tensor product C^*-algebra also has stable rank ≤ 2 and connected stable rank ≤ 2 ([14, Theorem 5.1]). Furthermore, by [10, Proposition 5.3] we obtain $\mathrm{sr}(C_0((\mathbb{R}_+)^k) \otimes \mathbb{T}_{\Theta_k}^{k+1}) \geq 2$ for $k \geq 2$ and

$$\mathrm{sr}(C_0((\mathbb{R}_+)^k) \otimes C(\mathbb{T}) \otimes ((C(\mathbb{T}^{k-1}) \otimes \mathbb{C}^n) \rtimes_{\Theta_{k-1}} \mathbb{Z})) \geq 2$$

for $k \geq 1$. As shown in the proof of Theorem 3.2, the mapping torus on $C(\mathbb{T}) \rtimes \mathbb{Z}_n$ has stable rank 2 (and connected stable rank ≤ 2). Therefore, it follows that $\mathrm{sr}(C^*(M_{m,n})) = 2$.

On the other hand, using Theorem 3.3 we deduce that the space $C^*(M_{m,n})_n^\wedge$ has an open subset homeomorphic to $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$ whose complement is $\mathbb{R}_+ \times \mathbb{T}$.

Moreover, by (F1) we obtain

$$\mathrm{csr}(\mathfrak{I}_k) \leq \max\{\mathrm{csr}(\mathfrak{I}_{k+1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1})\}$$

for $0 \leq k \leq m$. Hence, it follows that

$$\mathrm{csr}(C^*(M_{m,n})) = \mathrm{csr}(\mathfrak{I}_0) \leq \max_{0 \leq k \leq m} \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k+1}) \leq 2.$$

To determine $\mathrm{csr}(C^*(M_{m,n}))$ we compute the K_1-group of $C^*(M_{m,n})$. Since $C^*(M_{m,n}) \cong C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}$, we use Pimsner-Voiculescu six term exact sequence for crossed products of C^*-algebras by $\mathbb{Z}$ given by

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{C}^m)) & \xrightarrow{1-\alpha_*^\wedge} & K_0(C_0(\mathbb{C}^m)) & \longrightarrow & K_0(C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}) & \longleftarrow & K_1(C_0(\mathbb{C}^m)) & \longleftarrow & K_1(C_0(\mathbb{C}^m))
\end{array}$$

([1, Theorem 10.2.1]) and $K_0(C_0(\mathbb{C}^m)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{C}^m)) \cong K_1(\mathbb{C}) \cong 0$ by the Bott periodicity ([1, Theorem 9.2.1]). Since the map $1 - \alpha_*^\wedge$ on $K_0(C_0(\mathbb{C}^m))$ is trivial, it follows that $K_1(C_0(\mathbb{C}^m) \rtimes_{\alpha^\wedge} \mathbb{Z}) \cong \mathbb{Z}$. Thus, $\mathrm{csr}(C^*(M_{m,n})) \geq 2$. Hence, $\mathrm{csr}(C^*(M_{m,n})) = 2$. Use also (F5). ■

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References

1. B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

2. B. Blackadar, Symmetries of the CAR algebra, *Ann. of Math.*, **131** (1990), 589-623.
3. O. Bratteli, G. Elliott, D. Evans and A. Kishimoto, Non-commutative spheres I, *Internat. J. Math.*, **2** (1991), 139-166.
4. O. Bratteli, G. Elliott, D. Evans and A. Kishimoto, Non-commutative spheres II: Rational rotations, *J. Operator Theory*, **27** (1992), 53-85.
5. O. Bratteli and A. Kishimoto, Non-commutative spheres III. Irrational rotations, *Commun. Math. Phys.*, **147** (1992), 605-624.
6. J. Dixmier, *C^*-algebras*, North-Holland (1962).
7. G. A. Elliott and D. E. Evans, The structure of irrational rotation C^*-algebras, *Ann. of Math.*, **138** (1993), 477-501.
8. G. A. Elliott and Q. Lin, Cut-down method in the inductive limit decomposition of noncommutative tori, II: The degenerate case, Operator Algebras and Their Applications, *Fields Ints. Commun.*, **13** (1997), 91-123.
9. P. Green, The structure of imprimitivity algebras, *J. Funct. Anal.*, **36** (1980), 88-104.
10. M. Nagisa, H. Osaka and N. C. Phillips, Ranks of algebras of continuous C^*-algebra valued functions, *Canad. J. Math.*, **53** (2001), 979-1030.
11. V. Nistor, Stable range for tensor products of extensions of $\mathfrak{K}$ by $C(X)$, *J. Operator Theory*, **16** (1986), 387 396.
12. G. K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press (1979).
13. D. Poguntke, Simple quotients of group C^*-algebras for two step nilpotent groups and connected Lie groups, *Ann. Scient. Éc. Norm. Sup.*, **16** (1983), 151-172.
14. M. A. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc.*, **46** (1983), 301-333.
15. M. A. Rieffel, The homotopy groups of the unitary groups of non-commutative tori, *J. Operator Theory*, **17** (1987), 237-254.
16. M. A. Rieffel, Projective modules over higher-dimensional non-commutative tori, *Canad. J. Math.*, **XL** (40) (1988), 257-338.
17. M. Rørdam and E. Størmer, *Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras*, EMS **126**, Operator Algebras and Non-Commutative Geometry VII, Springer (2002).
18. A. J-L. Sheu, A cancellation theorem for projective modules over the group C^*-algebras of certain nilpotent Lie groups, *Canad. J. Math.*, **39** (1987), 365-427.
19. T. Sudo, Dimension theory of group C^*-algebras of connected Lie groups of type I, *J. Math. Soc. Japan*, **52** (2000), 583-590.
20. T. Sudo, Structure of group C^*-algebras of Lie semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}$, *J. Operator Theory*, **46** (2001), 25-38.

21. T. Sudo, The structure of group C^*-algebras of the generalized Dixmier groups, *Publ. Res. Inst. Math. Sci. Kyoto Univ*., **39** (2003), 205-225.

22. T. Sudo, Structure of group C^*-algebras of the generalized Mautner groups, *J. Math. Kyoto Univ*., **42** (2002), 393-402.

23. T. Sudo, Structure of group C^*-algebras of semi-direct products $\mathbb{C}^n$ by $\mathbb{Z}$, *Nihonkai Math. J*., **12(2)** (2001), 135-143.

24. T. Sudo, Structure of group C^*-algebras of the generalized disconnected Dixmier groups, *Sci. Math. Jpn*., **54(3)** (2001), 449-454, :e4, 861-866.

25. T. Sudo, Structure of group C^*-algebras of the generalized disconnected Mautner groups, *Linear Algebra Appl*., **341** (2002), 317-326.

26. T. Sudo and H. Takai, Stable rank of the C^*-algebras of nilpotent Lie groups, *Internat. J. Math*., **6** (1995), 439-446.

27. T. Sudo and H. Takai, Stable rank of the C^*-algebras of solvable Lie groups of type I, *J. Operator Theory*, **38** (1997), 67-86.

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