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SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract. In this paper, we prove a common fixed point theorem for weakly compatible mappings satisfying an implicit relation. Our theorem generalizes many fixed point theorems.

1. Introduction and Preliminaries

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, many fixed point results have been developed (see [15, 17, 22, 23]). In [5] Branciari proved the following interesting result for fixed point theory.

Theorem 1. *Let (X, d) be a complete metric space, $\lambda \in (0, 1)$ and $T : X \to X$ be mapping such that for each $x, y \in X$ one has*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \lambda \int_0^{d(x,y)} f(t)dt$$

where $f : [0, \infty) \to [0, \infty]$ is a Lebesgue integrable mapping which is finite integral on each compact subset of $[0, \infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Theorem 1 has been generalized in [4, 21] and [31]. Again in [2], Aliouche proved a fixed point theorem using a general contractive condition of integral type on symmetric spaces.

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Sessa [25] generalized the concept of commuting mappings by calling self-mappings A and S of metric space (X, d) a weakly commuting pair if and only if $d(ASx, SAx) \leq d(Ax, Sx)$ for all $x \in X$, and he and others gave some common fixed point theorems of weakly commuting mappings [24-27]. Then, Jungck [11] introduced the concept of compatibility and he and others proved some common fixed point theorems using this concept [9, 11, 12, 14, 30].

Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible; examples in [25] and [11] show that neither converse is true.

Recently, Jungck [10] gave the concept of weak compatibility the following way.

Definition 2. ([10, 13]). Two maps $A, S : X \to X$ are said to be weakly compatible if they commute at their coincidence points.

Again, it is obvious that compatible mappings are weakly compatible; giving examples in [13] and [28] shows that neither converse is true. Many fixed point results have been obtained using weakly compatible mappings (see [1, 4, 6, 7, 13, 18] and [28]).

2. Implicit Relation

Implicit relation on metric spaces have been used in many articles. (see [3, 8, 19, 20, 29]).

Let $\mathbb{R}_+$ denote the non-negative real numbers and let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

F_1 $F(t_1, ..., t_6)$ is non-increasing in variables t_5 and t_6.

F_2 there exists an upper semi-continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0, f(t) < t$ for $t > 0$, such that for $u, v \geq 0$,

$$F(u, v, v, u, 0, u + v) \leq 0$$

or

$$F(u, v, u, v, u + v, 0) \leq 0$$

implies $u \leq f(v)$.

F_3 $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

Example 1. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)[at_5 + bt_6]$, where $0 \leq \alpha < 1, 0 \leq a < \frac{1}{2}, 0 \leq b < \frac{1}{2}$.

F_1 Obviously.

F_2 Let $u > 0$ and $F(u,v,v,u,0,u+v) = u-\alpha \max\{u,v\}-(1-\alpha)b(u+v) \leq 0$. If $u \geq v$, then $u \leq [\alpha+2b(1-\alpha)]\,u < u$ which is a contradiction. Thus $u < v$ and so $u \leq [\alpha+2b(1-\alpha)]v$. Similarly, let $u > 0$ and $F(u,v,u,v,u+v,0) \leq 0$. If $u \geq v$, then $u \leq [\alpha + 2a(1-\alpha)]u < u$ which is a contradiction. Thus $u < v$ and so $u \leq [\alpha + 2a(1-\alpha)]v$. If $u = 0$ then $u \leq \max\{[\alpha + 2a(1-\alpha)], [\alpha + 2b(1-\alpha)]\}v$. Thus F_2 is satisfied with $f(t) = \max\{[\alpha + 2a(1-\alpha)], [\alpha + 2b(1-\alpha)]\}t$.

F_3 $F(u,u,0,0,u,u) = u(1-\alpha)(1-a-b) > 0, \forall u > 0$.

Thus $F \in \mathcal{F}$.

Example 2. $F(t_1,...,t_6) = t_1 - k\max\{t_2,t_3,t_4,\frac{1}{2}[t_5+t_6]\}$, where $k \in (0,1)$.

F_1 Obviously.

F_2 Let $u > 0$ and $F(u,v,v,u,0,u+v) = u - k\max\{u,v\} \leq 0$. If $u \geq v$, then $u \leq ku$, which is a contradiction. Thus $u < v$ and so $u \leq kv$. Similarly, let $u > 0$ and $F(u,v,u,v,u+v,0) \leq 0$ then we have $u \leq kv$. If $u = 0$, then $u \leq kv$. Thus F_2 is satisfied with $f(t) = kt$.

F_3 $F(u,u,0,0,u,u) = u - ku > 0$, $\forall u > 0$.

Thus $F \in \mathcal{F}$.

Example 3. $F(t_1,...,t_6) = t_1 - \psi(\max\{t_2,t_3,t_4,\frac{1}{2}[t_5+t_6]\})$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ right continuous and $\psi(0) = 0, \psi(t) < t$ for $t > 0$.

F_1 Obviously.

F_2 Let $u > 0$ and $F(u,v,v,u,0,u+v) = u - \psi(\max\{u,v\}) \leq 0$. If $u \geq v$, then $u - \psi(u) \leq 0$, which is a contradiction. Thus $u < v$ and so $u \leq \psi(v)$. Similarly, let $u > 0$ and $F(u,v,u,v,u+v,0) \leq 0$ then we have $u \leq \psi(v)$. If $u = 0$ then $u \leq \psi(v)$. Thus F_2 is satisfied with $f = \psi$.

F_3 $F(u,u,0,0,u,u) = u - \psi(u) > 0, \forall u > 0$.

Thus $F \in \mathcal{F}$.

Example 4. $F(t_1,...,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a > 0$, $b,c,d \geq 0$, $a+b+c < 1$ and $a+b+d < 1$.

F_1 Obviously.

F_2 Let $u > 0$ and $F(u,v,v,u,0,u+v) = u^2 - u(av+bv+cu) \leq 0$. Then $u \leq (\frac{a+b}{1-c})v$. Similarly, let $u > 0$ and $F(u,v,u,v,u+v,0) \leq 0$ then we have $u \leq (\frac{a+c}{1-b})v$. If $u = 0$, then $u \leq (\frac{a+c}{1-b})v$. Thus F_2 is satisfied with $f(t) = \max\{(\frac{a+b}{1-c}),(\frac{a+c}{1-b})\}t$.

F_3 $F(u,u,0,0,u,u) = u^2(1-a-d) > 0, \forall u > 0$.

Thus $F \in \mathcal{F}$.

Example 5. $F(t_1, ..., t_6) = t_1^3 - \alpha \dfrac{t_3^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4 + 1}$, where $\alpha \in (0, 1)$.

F_1 Obviously.

F_2 Let $u > 0$ and $F(u, v, v, u, 0, u+v) = u^3 - \dfrac{\alpha v^2 u^2}{u + 2v + 1} \leq 0$, which implies $u \leq \dfrac{\alpha v^2}{u + 2v + 1}$. But $\dfrac{\alpha v^2}{u + 2v + 1} \leq \alpha v$, thus $u \leq \alpha v$. Similarly, let $u > 0$ and $F(u, v, u, v, u+v, 0) \leq 0$, then we have $u \leq \alpha v$. If $u = 0$, then $u \leq \alpha v$. Thus F_2 is satisfied with $f(t) = \alpha t$.

F_3 $F(u, u, 0, 0, u, u) = \dfrac{u^4(1 - \alpha) + u^3}{u + 1} > 0, \forall u > 0.$

Thus $F \in \mathcal{F}$.

3. Common Fixed Point Theorems

We need the following lemma for the proof of our main theorem.

Lemma 1. ([16]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semi-continuous function such that $f(t) < t$ for every $t > 0$, then $\lim_{n\to\infty} f^n(t) = 0$, where f^n denotes the composition of f, n-times with itself.*

Now we give our main theorem.

Theorem 2. *Let A, B, S and T be self-maps defined on a metric space (X, d) satisfying the following conditions:*

(i) $S(X) \subseteq B(X)$, $T(X) \subseteq A(X)$,

(ii) *for all* $x, y \in X$,

$$F\left(\int_0^{d(Sx,Ty)} \varphi(t)dt, \int_0^{d(Ax,By)} \varphi(t)dt, \int_0^{d(Sx,Ax)} \varphi(t)dt, \int_0^{d(Ty,By)} \varphi(t)dt, \int_0^{d(Sx,By)} \varphi(t)dt, \int_0^{d(Ty,Ax)} \varphi(t)dt\right) \leq 0$$

where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesque integrable mapping which is summable,

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt \tag{3.1}$$

for all $a, b \in \mathbb{R}_+$ *and such that*

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for each } \varepsilon > 0. \tag{3.2}$$

If one of $A(X), B(X), S(X)$ *or* $T(X)$ *is a complete subspace of* X*, then*

(1) A *and* S *have a coincidence point,or*

(2) B *and* T *have a coincidence point.*

Further, if S *and* A *as well as* T *and* B *are weakly compatible, then*

(3) A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point of X. From (i) we can construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1} \text{ and } y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}$$

for all $n = 0, 1, ...$ Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ for some n. Then $y_{2n} = y_{2n+1}$; that is, $Tx_{2n-1} = Ax_{2n} = Sx_{2n} = Bx_{2n+1}$, and A and S have a coincidence point. Similarly, if $d_{2n+1} = 0$, then B and T have a coincidence point. Assume that $d_n \neq 0$ for each n. Then by (ii), we have

$$F\left(\int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt, \int_0^{d(Ax_{2n},Bx_{2n+1})} \varphi(t)dt, \int_0^{d(Sx_{2n},Ax_{2n})} \varphi(t)dt,\right.$$
$$\left.\int_0^{d(Tx_{2n+1},Bx_{2n+1})} \varphi(t)dt, \int_0^{d(Sx_{2n},Bx_{2n+1})} \varphi(t)dt, \int_0^{d(Tx_{2n+1},Ax_{2n})} \varphi(t)dt\right) \leq 0.$$

Thus we have

$$\begin{aligned} &F\left(\int_0^{d_{2n+1}} \varphi(t)dt, \int_0^{d_{2n}} \varphi(t)dt, \int_0^{d_{2n}} \varphi(t)dt,\right. \\ &\qquad \left.\int_0^{d_{2n+1}} \varphi(t)dt, 0, \int_0^{d_{2n}+d_{2n+1}} \varphi(t)dt\right) \leq 0. \end{aligned} \tag{3.3}$$

On the other hand, from (3.1) we have

$$\int_0^{d_{2n}+d_{2n+1}} \varphi(t)dt \leq \int_0^{d_{2n}} \varphi(t)dt + \int_0^{d_{2n+1}} \varphi(t)dt. \tag{3.4}$$

Now from (3.3), (3.4) and F_1, we have

$$F\left(\int_0^{d_{2n+1}} \varphi(t)dt, \int_0^{d_{2n}} \varphi(t)dt, \int_0^{d_{2n}} \varphi(t)dt, \int_0^{d_{2n+1}} \varphi(t)dt, 0,\right.$$
$$\left.\int_0^{d_{2n}} \varphi(t)dt + \int_0^{d_{2n+1}} \varphi(t)dt\right) \leq 0.$$

From F_2, there exists an upper semi-continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that

$$\int_0^{d_{2n+1}} \varphi(t)dt \leq f\left(\int_0^{d_{2n}} \varphi(t)dt\right)$$

Similarly we can have

$$\int_0^{d_{2n}} \varphi(t)dt \leq f\left(\int_0^{d_{2n-1}} \varphi(t)dt\right).$$

In general, we have for all $n = 1, 2, ...,$

$$\int_0^{d_n} \varphi(t)dt \leq f\left(\int_0^{d_{n-1}} \varphi(t)dt\right). \tag{3.5}$$

From (3.5), we have

$$\begin{aligned}\int_0^{d_n} \varphi(t)dt &\leq f\left(\int_0^{d_{n-1}} \varphi(t)dt\right)\\ &\leq f^2\left(\int_0^{d_{n-2}} \varphi(t)dt\right)\\ &\;\;\vdots\\ &\leq f^n\left(\int_0^{d_0} \varphi(t)dt\right)\end{aligned}$$

and taking the limit as $n \to \infty$ we have, from Lemma 1, for $d_0 > 0$,

$$\lim_{n\to\infty} \int_0^{d_n} \varphi(t)dt \leq \lim_{n\to\infty} f^n\left(\int_0^{d_0} \varphi(t)dt\right) = 0,$$

which from (3.2) implies that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

We now show that $\{y_n\}$ is Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. suppose that $\{y_{2n}\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ such that for an even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon. \tag{3.6}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (3.6) such that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon. \tag{3.7}$$

Now

$$\begin{aligned} 0 &< \delta := \int_0^{\varepsilon} \varphi(t)dt \\ &\leq \int_0^{d(y_{2n(k)},y_{2m(k)})} \varphi(t)dt \\ &\leq \int_0^{d(y_{2n(k)},y_{2m(k)-2})+d_{2m(k)-2}+d_{2m(k)-1}} \varphi(t)dt. \end{aligned}$$

Then by (3.6) and (3.7) it follows that

$$\lim_{k\to\infty} \int_0^{d(y_{2n(k)},y_{2m(k)})} \varphi(t)dt = \delta. \tag{3.8}$$

Also, by the triangular inequality, we have

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leq d_{2m(k)-1}$$

and

$$\left|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leq d_{2m(k)-1} + d_{2n(k)}.$$

Thus we have

$$\int_0^{\left|d(y_{2n(k)},y_{2m(k)-1})-d(y_{2n(k)},y_{2m(k)})\right|} \varphi(t)dt \leq \int_0^{d_{2m(k)-1}} \varphi(t)dt$$

and

$$\int_0^{\left|d(y_{2n(k)+1},y_{2m(k)-1})-d(y_{2n(k)},y_{2m(k)})\right|} \varphi(t)dt \leq \int_0^{d_{2m(k)-1}+d_{2n(k)}} \varphi(t)dt.$$

By using (3.8) we get

$$\int_0^{d(y_{2n(k)},y_{2m(k)-1})} \varphi(t)dt \to \delta \tag{3.9}$$

and

$$\int_0^{d(y_{2n(k)+1},y_{2m(k)-1})} \varphi(t)dt \to \delta \tag{3.10}$$

as $k \to \infty$. Now we get

$$\begin{aligned} d(y_{2n(k)}, y_{2m(k)}) &\leq d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq d_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1}) \end{aligned}$$

and so

$$\int_0^{d(y_{2n(k)},y_{2m(k)})}\varphi(t)dt \leq \int_0^{d_{2n(k)}+d(Sx_{2n(k)},Tx_{2m(k)-1})}\varphi(t)dt.$$

Letting $k \to \infty$ both of the last inequality, we have

$$\begin{aligned}\delta &\leq \lim_{k\to\infty}\int_0^{d(Sx_{2n(k)},Tx_{2m(k)-1})}\varphi(t)dt\\ &= \lim_{k\to\infty}\int_0^{d(y_{2n(k)+1},y_{2m(k)})}\varphi(t)dt\\ &\leq \lim_{k\to\infty}\int_0^{d(y_{2n(k)+1},y_{2m(k)-1})+d_{2m(k)-1}}\varphi(t)dt\\ &= \delta.\end{aligned} \tag{3.11}$$

On the other hand, from (ii), we have

$$F\left(\int_0^{d(Sx_{2n(k)},Tx_{2m(k)-1})}\varphi(t)dt, \int_0^{d(Ax_{2n(k)},Bx_{2m(k)-1})}\varphi(t)dt,\right.$$
$$\int_0^{d(Sx_{2n(k)},Ax_{2n(k)})}\varphi(t)dt, \int_0^{d(Tx_{2m(k)-1},Bx_{2m(k)-1})}\varphi(t)dt,$$
$$\left.\int_0^{d(Sx_{2n(k)},Bx_{2m(k)-1})}\varphi(t)dt, \int_0^{d(Tx_{2m(k)-1},Ax_{2n(k)})}\varphi(t)dt\right) \leq 0$$

and so

$$\begin{aligned}&F\left(\int_0^{d(y_{2n(k)+1},y_{2m(k)})}\varphi(t)dt, \int_0^{d(y_{2n(k)},y_{2m(k)-1})}\varphi(t)dt,\right.\\ &\int_0^{d_{2n(k)}}\varphi(t)dt, \int_0^{d_{2m(k)-1}}\varphi(t)dt, \int_0^{d(y_{2n(k)+1},y_{2m(k)-1})}\varphi(t)dt,\\ &\left.\int_0^{d(y_{2n(k)},y_{2m(k)-2})}\varphi(t)dt\right) \leq 0\end{aligned} \tag{3.12}$$

From (3.12), considering F_1, (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), letting $k \to \infty$ we have the following,

$$F(\delta, \delta, 0, 0, \delta, \delta) \leq 0$$

which is a contradiction with F_3. Thus $\{y_{2n}\}$ is a Cauchy sequence and so $\{y_n\}$ is a Cauchy sequence.

Now, suppose that $A(X)$ is complete. Note that the sequence $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(X)$. Call it u. Let $v \in A^{-1}u$. Then $Av = u$. We shall use the fact that the sequence $\{y_{2n-1}\}$ also converges to u. To prove that $Sv = u$, let $r = d(Sv, u) > 0$. Then taking $x = v$ and $y = x_{2n-1}$ in (ii),

$$F\left(\int_0^{d(Sv,Tx_{2n-1})} \varphi(t)dt, \int_0^{d(Av,Bx_{2n-1})} \varphi(t)dt, \int_0^{d(Sv,Av)} \varphi(t)dt,\right.$$

$$\left.\int_0^{d(Tx_{2n-1},Bx_{2n-1})} \varphi(t)dt, \int_0^{d(Sv,Bx_{2n-1})} \varphi(t)dt, \int_0^{d(Tx_{2n-1},Av)} \varphi(t)dt\right) \leq 0$$

and so

$$\begin{aligned}&F(\int_0^{d(Sv,y_{2n})} \varphi(t)dt, \int_0^{d(u,y_{2n-1})} \varphi(t)dt, \int_0^{d(Sv,u)} \varphi(t)dt,\\ &\int_0^{d(y_{2n},y_{2n-1})} \varphi(t)dt, \int_0^{d(Sv,y_{2n-1})} \varphi(t)dt, \int_0^{d(y_{2n},u)} \varphi(t)dt) \leq 0\end{aligned} \tag{3.13}$$

Since $\lim_{n\to\infty} d(Sv, y_{2n}) = \lim_{n\to\infty} d(Sv, y_{2n-1}) = r$ and $\lim_{n\to\infty} d(u, y_{2n-1}) = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = \lim_{n\to\infty} d(y_{2n}, u) = 0$, we have from (3.13)

$$F\left(\int_0^{r} \varphi(t)dt, 0, \int_0^{r} \varphi(t)dt, 0, \int_0^{r} \varphi(t)dt, 0\right) \leq 0$$

which is a contradiction with F_2. Hence from (3.2) we have $Sv = u$. This proves (1)

Since $S(X) \subseteq B(X)$, $Sv = u$ implies that $u \in B(X)$. Let $w \in B^{-1}u$. Then $Bw = u$. Hence by using the argument of the previous section, it can be easily verified that $Tw = u$. This proves (2).

The same result holds if we assume that $B(X)$ is complete instead of $A(X)$.

Now if $T(X)$ is complete, then by (i), $u \in T(X) \subseteq A(X)$. Similarly if $S(X)$ is complete, then $u \in S(X) \subseteq B(X)$. Thus (1) and (2) are completely established.

To prove (3), note that S, A and T, B are weakly compatible and

$$u = Sv = Av = Tw = Bw \tag{3.14}$$

then

$$Au = ASv = SAv = Su \tag{3.15}$$

$$Bu = BTw = TBw = Tu. \tag{3.16}$$

If $Tu \neq u$, then from (ii), (3.14), (3.15) and (3.16) we have

$$F\left(\int_0^{d(Sv,Tu)}\varphi(t)dt, \int_0^{d(Av,Bu)}\varphi(t)dt, \int_0^{d(Sv,Av)}\varphi(t)dt,\right.$$

$$\left.\int_0^{d(Tu,Bu)}\varphi(t)dt, \int_0^{d(Sv,Bu)}\varphi(t)dt, \int_0^{d(Tu,Av)}\varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(u,Tu)}\varphi(t)dt, \int_0^{d(u,Tu)}\varphi(t)dt, 0, 0, \int_0^{d(u,Tu)}\varphi(t)dt, \int_0^{d(Tu,u)}\varphi(t)dt\right) \leq 0$$

which is a contradiction with F_3. So $Tu = u$. Similarly $Su = u$. Then, evidently from (3.15) and (3.16), u is a common fixed point of A, B, S and T.

Now let u and v be two common fixed points of A, B, S and T. Then from (ii), we have

$$F\left(\int_0^{d(Su,Tv)}\varphi(t)dt, \int_0^{d(Au,Bv)}\varphi(t)dt, \int_0^{d(Su,Au)}\varphi(t)dt,\right.$$

$$\left.\int_0^{d(Tv,Bv)}\varphi(t)dt, \int_0^{d(Su,Bv)}\varphi(t)dt, \int_0^{d(Tv,Au)}\varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(u,v)}\varphi(t)dt, \int_0^{d(u,v)}\varphi(t)dt, 0, 0, \int_0^{d(u,v)}\varphi(t)dt, \int_0^{d(v,u)}\varphi(t)dt\right) \leq 0$$

which is a contradiction with F_3. Thus $u = v$. This completes the proof. ■

If $\varphi(t) = 1$ in Theorem 2, we obtain Theorem 2.1 of [8] and a generalization of Theorem 1 of [20].

If we combine Example 1 with Theorem 2 we obtain the following result.

Corollary 1. *Let A, B, S and T be self-maps defined on a metric space (X, d) satisfying the following conditions:*

(i) $S(X) \subseteq B(X)$, $T(X) \subseteq A(X)$,

(ii) *for all* $x, y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le \alpha \int_0^{\max\{d(Ax,By),d(Sx,Ax),d(Ty,By)\}} \varphi(t)dt$$

$$+(1-\alpha)\left[a \int_0^{d(Sx,By)} \varphi(t)dt + b \int_0^{d(Ty,Ax)} \varphi(t)dt\right]$$

where $0 \le \alpha < 1, 0 \le a < \frac{1}{2}, 0 \le b < \frac{1}{2}$ *and* φ *is as in Theorem 2.*

If one of $A(X), B(X), S(X)$ *or* $T(X)$ *is a complete subspace of* X*, then*

(1) A *and* S *have a coincidence point, or*

(2) B *and* T *have a coincidence point.*

Further, if S *and* A *as well as* T *and* B *are weakly compatible, then*

(3) A, B, S *and* T *have a unique common fixed point.*

If $\varphi(t) = 1$ in Corollary 1, we get Theorem 2.1 of [1] for single-valued mappings.

If we combine Example 2 with Theorem 2 we obtain the following result.

Corollary 2. *Let* A, B, S *and* T *be self-maps defined on a metric space* (X, d) *satisfying the following conditions:*

(i) $S(X) \subseteq B(X)$, $T(X) \subseteq A(X)$,

(ii) *for all* $x, y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le k \max\{\int_0^{\max\{d(Ax,By),d(Sx,Ax),d(Ty,By)\}} \varphi(t)dt,$$

$$\frac{1}{2}[\int_0^{d(Sx,By)} \varphi(t)dt + \int_0^{d(Ty,Ax)} \varphi(t)dt]\}$$

where $0 < k < 1$ *and* φ *is as in Theorem 2.*

If one of $A(X), B(X), S(X)$ *or* $T(X)$ *is a complete subspace of* X*, then*

(1) A *and* S *have a coincidence point, or*

(2) B *and* T *have a coincidence point.*

Further, if S *and* A *as well as* T *and* B *are weakly compatible, then*

(3) A, B, S *and* T *have a unique common fixed point.*

By Corollary 2, we have a generalized version of Theorem 1 in this paper.

If we combine Example 3 with Theorem 2 we obtain Theorem 2.1 of [4].

If $\varphi(t) = 1$ in Theorem 2 and combine with Example 3, we have Theorem 1 of [9] and Theorem 2.1 of [28]. Also by Theorem 2, we have a different version of Theorem 3.1 of [6].

Remark 1. We can have some new fixed point results if we combine Theorem 2 with some examples of F.

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