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MULTIPLICATION OPERATORS ON ANALYTIC FUNCTIONAL SPACES

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Dedicated to Mola Ali

Abstract. Let X be a reflexive Banach space of functions analytic on a bounded plane domain G such that for every λ in G the functional of evaluation at λ is bounded. Assume further that X contains the constants and admits multiplication by the independent variable z, M_z, as a bounded operator. We give sufficient conditions for M_z to be reflexive.

1. Introduction

In this section we include some preparatory material which will be needed later. By a *domain* we understand a connected open subset of the plane. If Ω is a bounded domain in the plane, then as comes in Sarason ([11]), the *Carathéodory hull* (or $\mathbb{C}$-hull) of Ω is the complement of the closure of the unbounded component of the complement of the closure of Ω. The $\mathbb{C}$-hull of Ω is denoted by Ω^*. Intuitively, Ω^* can be described as the interior of the outer boundary of Ω, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leq \sup\{|p(z)| : z \in \Omega\}$ for all polynomials p. The components of Ω^* are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of Ω^* that contains Ω is denoted by Ω_1. Note that for all polynomials p, $\|p\|_\Omega = \|p\|_{\Omega_1}$. The domain Ω is called a *Carathéodory domain* if $\Omega^* = \Omega$. In this case the Farrell-Rubel-Shields Theorem holds: let f be a bounded analytic function on Ω. Then there is a sequence $\{p_n\}$ of polynomials such that $||p_n||_\Omega \leq c$ for a constant c and $p_n(z) \to f(z)$ for all $z \in \Omega$ ([7, Theorem 5.1, p.151]).

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Now let X be a reflexive Banach space. For the algebra $\mathcal{B}(X)$ of all bounded operators on the Banach space X, the weak operator topology is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in X$.

For the following definition one can see [3, 9].

Definition. If $A \in \mathcal{B}(X)$, then $Lat(A)$ is the lattice of all invariant subspaces of A, and $AlgLat(A)$ is the algebra of all operators B in $\mathcal{B}(X)$ such that $Lat(A) \subset Lat(B)$. An operator A in $\mathcal{B}(X)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(X)$ that contains A and the identity I and is closed in the weak operator topology.

Consider a Banach space X of functions analytic on a plane domain G, such that for each $\lambda \in G$ the linear functional, e_λ, of evaluation at λ is bounded on X. Assume further that X contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on X. A complex valued function φ on G for which $\varphi f \in X$ for every $f \in X$ is called a *multiplier* of X and the collection of all these multipliers is denoted by $\mathcal{M}(X)$. Each multiplier φ of X determines a multiplication operator M_φ on X by $M_\varphi f = \varphi f$, $f \in X$. It is well-known that each multiplier is a bounded analytic function ([13]). Indeed $|\varphi(\lambda)| \le ||M_\varphi||$ for each λ in G. Also $M_\varphi 1 = \varphi \in X$. But $X \subset H(G)$, thus φ is a bounded analytic function. We also point out that if φ is a multiplier and $\lambda \in G$ then

$$M_\varphi^* e_\lambda = \varphi(\lambda) e_\lambda,$$

since for all f in X we have

$$\begin{aligned} < f, M_\varphi^* e_\lambda > &= < M_\varphi f, e_\lambda >= \varphi(\lambda) f(\lambda) \\ &= \varphi(\lambda) < f, e_\lambda >=< f, \varphi(\lambda) e_\lambda > \end{aligned}$$

(here for simplicity we used the notation $< x, x^* >$ instead of $x^*(x)$ for $x \in x$ and $x^* \in X^*$).

In this paper we will investigate the reflexivity of the operator M_z acting on functional Banach spaces of analytic functions on an arbitrary bounded plane domain (for a source of functional Banach spaces one can see [4]).

2. Main Results

The operator M_z has been the focus of attention for several decades and many of its properties have been studied ([2]). The study of reflexive operators has been playing a key role in the theory of invariant subspaces and harmonic analysis of operators on Hilbert spaces. It is well-known, due to the work by Sarason in

[10], that normal operators are reflexive and one of the intensively studied models of non-normal reflexive operators is the so-called shift on Hilbert spaces, whose concrete realization is basically the multiplication by z on the Hardy space of "square integrable" $\mathcal{K}$ -valued holomorphic functions for some Hilbert space $\mathcal{K}$. It was shown by J. Deddens ([5]) that every isometry is reflexive. Also, R. Olin and J. Thomson ([8]) have shown that subnormal operators are reflexive. H. Bercovici, C. Foias, J. Langsam, and C. Pearcy ([1]) have shown that (BCP)-operators are reflexive. The reflexive operators on a finite dimensional space were characterized by J. Deddens and P. A. Fillmore ([6]). Reflexivity of certain bilateral weighted shifts are also studied in [12, 14]. In this article we would like to give some sufficient conditions so that the operator M_z becomes reflexive on certain functional Banach spaces of analytic functions on a bounded plane domain (for a good source of reflexivity see [9]).

Main Theorem. *Let X be a reflexive functional Banach space of analytic functions on a bounded plane domain G such that X contains the constant functions and $M_z \in \mathcal{B}(X)$. If $||M_p|| \leq c||p||_G$ for every polynomial p and $||f||_G \leq d||f||_X$ for all f in $X \cap H^\infty(G)$, then M_z is reflexive.*

Proof. Let R be the Riemann mapping from the open unit disk U onto G_1. Set $\Omega = R^{-1}(G)$ and $E = \{f \circ R : f \in X\}$. It is easy to see that E turns out to be a reflexive Banach space with the norm $||f \circ R||_E = ||f||_X$, $f \in X$. Furthermore, the functions in E are analytic on Ω, $\Omega \subset U$, and E contains the constants. Consider the Banach space (E, Ω) and denote the functional of point evaluation at $\lambda \in \Omega$ by e'_λ. Since $e'_\lambda(f \circ R) = e_{R(\lambda)}(f)$, the functional of point evaluations are bounded on (E, Ω). The operator $S : X \to E$ given by $Sf = f \circ R$ is clearly an isomorphism. Also, observe that the map $\varphi \to \varphi \circ R$ is an isometric isomorphism of

$$\mathcal{M}(X) \to \mathcal{M}(E) = \{\varphi \circ R : \varphi \in \mathcal{M}(X)\}$$

just as the map $A \to SAS^{-1}$ of $\mathcal{B}(X) \to \mathcal{B}(E)$ is, since if $||f||_X \leq 1$, then

$$||Sf||_E = ||f \circ R||_E = ||f||_X \leq 1$$

and also we have

$$||SAS^{-1}(f \circ R)||_E = ||SAf||_E = ||Af \circ R||_E = ||Af||_X.$$

Hence indeed $||SAS^{-1}|| = ||A||$. Let the multiplication operator by $\phi \in \mathcal{M}(E)$ be denoted by M'_ϕ. Therefore for every polynomial p we get

$$c||p \circ R||_\Omega = c||p||_G \geq ||M_p|| = ||SM_pS^{-1}|| = ||M'_{p \circ R}||,$$

since

$$\begin{aligned} SM_pS^{-1}(f \circ R) &= SM_pf = S(pf) = (pf) \circ R \\ &= (p \circ R) \cdot (f \circ R) = M'_{p \circ R}(f \circ R). \end{aligned}$$

Hence $c||p \circ R||_\Omega \geq ||M'_{p\circ R}||$ for every polynomial p. Note that

$$M'_R(f \circ R) = R \cdot (f \circ R) = (zf) \circ R$$

for every $f \in X$ and since $zf \in X$ whenever $f \in X$, we see that this definition make sense. Also, note that

$$SM_zf = (zf) \circ R = R \cdot (f \circ R) = M'_RSf,$$

hence $SM_z = M'_RS$. The boundedness of M'_R follows directly from it's definition. Now let $L \in Lat(M_z)$ and define

$$L_R = \{f \circ R : f \in L\}.$$

Then the correspondence $L \to L_R$ is clearly a bijection of $Lat(M_z) \to Lat(M'_R)$. The map $A \to SAS^{-1}$ establishes a bijection of $AlgLat(M_z) \to AlgLat(M'_R)$ and also of $W(M_z) \to W(M'_R)$. Hence M_z is reflexive on (X, G) if and only if M'_R is reflexive on (E, Ω). So we could reduce the problem of reflexivity concerning the operator of multiplication on (X, G) into a problem of reflexivity of an operator on (E, Ω) where Ω is a subset of the open unit disk U. Now we show that M'_R is reflexive on (E, Ω). For this let $A \in AlgLat(M'_R)$. Note that

$$\begin{aligned} < f \circ R, (M'_R)^*e'_z > &= < M'_R(f \circ R), e'_z > \\ &= R(z) \cdot (f \circ R)(z) \\ &= R(z) < f \circ R, e'_z > \\ &= < f \circ R, R(z)e'_z > \end{aligned}$$

for all f in X. Hence

$$(M'_R)^*e'_z = R(z)e'_z$$

(here for simplicity we used the notation $< y, y^* >$ instead of $y^*(y)$ whenever $y \in E$ and $y^* \in E^*$). So we conclude that the one dimensional span of $\{e'_z\}$ is invariant under $(M'_R)^*$ and so $A^*e'_z = \varphi(z)e'_z$. Using the Hahn-Banach Theorem and the reflexivity of E, we see that the linear span of $\{e'_\lambda\}_{\lambda\in\Omega}$ is weak star dense in E^* . Hence $\varphi \in \mathcal{M}(E)$ and $A = M'_\varphi$, and so $\varphi \in H^\infty(\Omega)$. Note that $M'_\varphi \in \mathcal{B}(E)$. Now we show that $E_0 = E \cap H^\infty(\Omega_1)$ is a closed subspace of E that is invariant under M'_R. For this let $\{f_n \circ R\}_n$ be a sequence in E_0 converging to $f \circ R$ in E. Then $||f_n \circ R||_E < b$ for some constant b. Note that

$$||f_n \circ R||_\Omega = ||f_n||_G \leq d||f_n||_X = d||f_n \circ R||_E < bd.$$

We have $||f_n \circ R||_\Omega = ||f_n \circ R||_{\Omega_1}$. This implies that $f \circ R \in E_0$. Clearly E_0 is invariant under M'_R. Thus $E_0 \in Lat(M'_R)$ and so $E_0 \in Lat(A)$. But $A = M'_\varphi$, this implies that $\varphi E_0 \subset E_0$. Hence $\varphi \in E_0 \subset H^\infty(\Omega_1)$ since E_0 contains the constants. But Ω_1 is a Carathéodory domain and so by the Farrell-Rubel-Shields Theorem there is a sequence $\{p_n\}_n$ of polynomials converging to φ such that for all n, $\|p_n\|_{\Omega_1} \leq a$ for some $a > 0$. So we obtain

$$\|M'_{p_n}\| \leq c\|p_n\|_{\Omega_1} \leq ca$$

for all n. But ball $\mathcal{B}(E)$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $B \in \mathcal{B}(E)$, $M'_{p_n} \longrightarrow B$ in the weak operator topology. Using the fact that $(M'_{p_n})^* \longrightarrow B^*$ in the weak operator topology and acting these operators on e'_λ we obtain that

$$\overline{p_n(\lambda)}e'_\lambda = (M'_{p_n})^* e'_\lambda \longrightarrow B^* e'_\lambda$$

weakly. Since $p_n(\lambda) \longrightarrow \varphi(\lambda)$ we see that $B^* e'_\lambda = \varphi(\lambda) e'_\lambda$. Because the closed linear span of $\{e'_\lambda : \lambda \in \Omega\}$ is dense in E^*, we conclude that $B = M'_\varphi = A$. This implies that $A \in W(M'_z)$. We now show that $W(M'_z) = W(M'_R)$ which implies that $AlgLat(M'_R) = W(M'_R)$ and this completes the proof. Because R can be approximated pointwise boundedly by polynomials, we see that $W(M'_R) \subseteq W(M'_z)$. Conversely, by proposition 2 of [11], $R : U \to G_1$ is a generator of H^∞, so there exists a sequence of polynomials $\{p_n\}_n$ such that $p_n(R) \to z$ weak star, i.e., $\{p_n \circ R\}_n$ is uniformly bounded and converges to z at every point of U (see [11, Lemma 1]). But it is clear that $M'_{p_n \circ R} = p_n(M'_R)$ is in $W(M'_R)$. Hence $M'_z \in W(M'_R)$ and so indeed $W(M'_z) = W(M'_R)$. The proof is now complete. ■

Note that if $T \in \mathcal{B}(X)$, then by definition a compact set K containing the spectrum of T, $\sigma(T)$, is a *spectral set* for T if $\|f(T)\| \leq \|f\|_K$ for all rationals f with poles off K.

In the proof of the main theorem we used the condition "$||M_p|| \leq c||p||_G$ for every polynomial p". In the following we show that there are other alternatives.

Corollary. *The conclusion in the main theorem also holds if "$||M_p|| \leq c||p||_G$ for every polynomial p" is replaced by any one of the followings*

(*i*) *The map $\varphi \longrightarrow M_\varphi$ of $\mathcal{M}(X) \longrightarrow \mathcal{B}(X)$ is an isometry,*

(*ii*) *$\overline{G}$ is a spectral set for M_z,*

(iii) $\|M_\varphi\| \leq c\|\varphi\|_G$ *for every multiplier* φ*,* $H^\infty(G_1) \subset \mathcal{M}(X)$.

Proof. First note that each $\varphi \in \mathcal{M}(X)$ determines a bounded multiplication operator M_φ on X by $M_\varphi f = \varphi f$, $f \in X$. If the condition (i) holds, then $\|M_\varphi\| = \|\varphi\|_G$ for all φ in $\mathcal{M}(X)$. Since each polynomial is a multiplier of X, hence $||M_p|| = ||p||_G$ for every polynomial p. If the condition (ii) holds, then clearly by the definition of the spectral set, we get $||M_p|| = ||p(M_z)|| \leq ||p||_{\overline{G}} = \|p\|_G$ for every polynomial p. Also, since each polynomial is a multiplier of X, hence the condition (iii) implies that $||M_p|| \leq c||p||_G$ for every polynomial p. Now we show that the condition (iv) implies "$||M_p|| \leq c||p||_G$ for every polynomial p". For this we show that $L : H^\infty(G_1) \longrightarrow \mathcal{B}(X)$ given by $L(\varphi) = M_\varphi$ is continuous. Note that by condition (iv) if $\varphi \in H^\infty(G_1)$, then $\varphi \in \mathcal{M}(X)$ and so the multiplication operator M_φ is defined on X and in fact $M_\varphi \in \mathcal{B}(X)$. Suppose that the sequence $\{\varphi_n\}_n$ converges to φ in $H^\infty(G_1)$ and $L(\varphi_n) = M_{\varphi_n}$ converges to A in $\mathcal{B}(X)$. Then for each f in X,

$$Af = \lim_n M_{\varphi_n} f = \lim_n \varphi_n f$$

and so $\{\varphi_n f\}_n$ is convergent in X. Note that by the continuity of point evaluations, $\varphi_n f$ converges pointwise to φf. Thus Af is analytic on G and agrees with φf on G. Hence $A = M_\varphi$ and so by the closed graph theorem L is continuous. This implies that there is a constant $c > 0$ such that

$$\begin{aligned} \|L\| &= sup\{\|L(\varphi)\| : \quad \|\varphi\|_{G_1} \leq 1\} \\ &= sup\{\|M_\varphi\| : \quad \|\varphi\|_{G_1} \leq 1\} \\ &\leq c. \end{aligned}$$

Hence $\|M_\varphi\| \leq c\|\varphi\|_{G_1}$ for all φ in $H^\infty(G_1)$. But $\|p\|_G = \|p\|_{G_1}$ for all polynomials p, hence $||M_p|| \leq c||p||_G$ for every polynomial p. This completes the proof. ■

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