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RIGHT GENERALIZED (α, β)-DERIVATIONS HAVING POWER CENTRAL VALUES

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Abstract. Let R be a prime ring with center Z and $f \neq 0$ a right generalized (α, β)-derivation of R. If $f(x)^n \in Z$ for all $x \in L$, a nonzero ideal of R, and for some fixed positive integer n, then R is either commutative or is an order in a 4-dimensional simple algebra.

1. Introduction

In [12], Herstein proved that if R is a prime ring with center Z and $d \neq 0$ a derivation of R such that $d(x)^n \in Z$ for all $x \in R$, where n is a fixed positive integer, then either R is commutative or is an order in a 4-dimensional simple algebra. In [3], the author extended this result to an (α, β)-derivation. It is quite natural to generalize this result to a more general case, say, right generalized (α, β)-derivations. The main result we obtain in this paper also generalizes two recent results on generalized derivations obtained by Lee [15] and Wang [18].

The theorem we shall prove is

Theorem A. *Let R be a prime ring with center Z and $f \neq 0$ a right generalized (α, β)-derivation of R such that $f(x)^n \in Z$ for all $x \in L$, a nonzero ideal of R, and for some fixed positive integer n, then R is either commutative or is an order in a 4-dimensional simple algebra.*

Theorem A is an immediate consequence of the following

Theorem B. *Let R be a prime ring with center Z and $f \neq 0$ a right generalized β-derivation of R such that $f(x)^n \in Z$ for all $x \in L$, a nonzero ideal of R, and for some fixed positive integer n, then R is either commutative or is an order in a 4-dimensional simple algebra.*

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In what follows, let R be a prime ring with center Z, α and β automorphisms of R and δ an (α, β)-derivation of R, that is, an additive mapping $\delta : R \to R$ satisfies

$$\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$$

for all $x, y \in R$. If $\alpha = 1$ ($\beta = 1$ resp.), an identity map of R, then we will say that δ is a β-derivation (α-derivation resp.). A β-derivation is also called a skew derivation. We say that δ is an inner (α, β)-derivation if $\delta(x) = a\alpha(x) - \beta(x)a$ for some $a \in R$. An additive mapping $f : R \to R$ is said to be a right generalized (α, β)-derivation if it satisfies

$$f(xy) = f(x)\alpha(y) + \beta(x)\delta(y)$$

for all $x, y \in R$, where δ is an (α, β)-derivation of R. If $\beta = 1$, then we say that f is a generalized β-derivation.

We let ${}_FR$ denote the right Martindale quotient ring, Q the two sided Martindale quotient ring and C the center of ${}_FR$. Note that all automorphisms and all (α, β)-derivations of R can be extended to Q and ${}_FR$. An (α, β)-derivation δ will be called X-inner if $\delta(x) = a\alpha(x) - \beta(x)a$ for some $a \in Q$. Also an automorphism g of R will be called X-inner if $g(x) = b^{-1}xb$ for some $b \in Q$. We also note that a right generalized (α, β)-derivation f of R can be extended to ${}_FR$ and $f(x) = s\alpha(x) + \delta(x)$ with $s = f(1) \in {}_FR$, where δ is an (α, β)-derivation associated to f (See [4]).

We begin with one of the crucial results

Lemma 1. *Let R be a prime ring and let $a, b, c \in R$ with a invertible in R. If $(a(bx - xc))^n = 0$ for all $x \in L$, where $L \neq 0$ is an ideal of R and n is a fixed positive integer, then $b = c \in Z$.*

Proof. By [5, Theorem 2].

$$(a(bx - xc))^n = 0$$

for all $x \in Q$, since Q is also the two sided Martindale quotient ring of L. If $b \in Z$, then $(ax(b - c))^n = 0$ for all $x \in Q$. Substitute $a^{-1}x$ for all x, we have $(x(b - c))^n = 0$ for all $x \in Q$ and hence $b - c = 0$ by [11, Lemma 1.1]. Therefore $b = c \in Z$. Similarly, if $c \in Z$, we also have $b = c \in Z$.

Now we may assume that $b \notin Z$ and $c \notin Z$. In this case, Q is a GPI ring. By a theorem of Martindale [17], Q is isomorphic to a dense subring of $\mathrm{End}({}_DV)$, where V is a left vector space over D, the associated division ring of Q. If $\dim_D V = 1$, then $Q \simeq D$ and $a(bx - xc) = 0$ for all $x \in Q$. Consequently, we have $bx = xc$ for all $x \in Q$. This implies that $b = c \in Z$, a contradiction. So we may assume that $\dim_D V \geq 2$, Suppose there exists $v \in V$ such that v and vb are D-independent.

By the density of Q, there exists $x \in Q$ such that $vx = 0$ and $vbx = va^{-1}$. Then $va^{-1}(a(bx - xc))^n = va^{-1} \neq 0$, a contradiction. Thus v and vb are D-dependent for all $v \in V$ and as usual, there exists $\lambda \in D$ such that $vb = \lambda v$ for all $v \in V$. So if $x \in Q$, then $v(a(bx - xc)) = (va)bx - vaxc = \lambda vax - vaxc = vaxb - vaxc = v(a(x(b - c)))$ and $v(a(bx - xc))^n = v(a(x(b - c)))^n = 0$ for all $v \in V$. Since Q acts faithfully on V, we have $(a(x(b-c)))^n = 0$ for all $x \in Q$. As in the beginning of the proof, again we have $b = c \in Z$, a contradiction. This last contradiction proves the lemma. ■

For the next crucial result, we need the following

Lemma 2. *Let R be a prime ring with center Z. Let $b \in R$. If $(bx)^n \in Z$ ($(xb)^n \in Z$ resp.) for all $x \in L$, where L is a nonzero ideal of R and $n \geq 1$ is a fixed integer, then either $b = 0$ or R is commutative.*

Proof. Assume that $(bx)^n \in Z$ for all $x \in L$. If $Z = 0$, then $(bx)^n = 0$ for all $x \in L$. By [11, Lemma 1.1], $b = 0$. Now assume that $Z \neq 0$ and $b \neq 0$. Then $(bx)^n y - y(bx)^n = 0$ for all $x, y \in L$ and hence for all $x, y \in Q$. Therefore $(bx)^n \in C$ for all $x \in Q$. Substitute $\lambda \neq 0 \in C$ for x, we have $b^n \lambda^n \in C$ and hence $b^n \in Z$. Substitute $b^{n-1}x$ for x into $(bx)^n \in C$, we have $b^{n^2} x^n \in C$. This implies either $b^{n^2} = 0$ or $x^n \in C$ for all $x \in Q$. If $b^{n^2} = 0$, then there exists $\ell > 0$ such that $b^\ell = 0$ but $b^{\ell-1} \neq 0$. Thus $b^{\ell-1}(bx)^n = 0$ and hence $(bx)^n = 0$ for all $x \in Q$. Again, by [11, Lemma 1.1], we have $b = 0$, a contradiction. So $x^n \in C$ for all $x \in Q$. Therefore R is commutative by a result of Herstein and Kaplansky [10]. ■

Lemma 3. *Let R be a prime ring with center Z. Let $a, b, c \in R$ with a invertible in R. If $(a(bx - xc))^n \in Z$ for all $x \in L$, where L is a nonzero ideal of R and n is a fixed positive integer, then $b = c \in Z$, or R is commutative or R is an order in a 4-dimensional simple algebra.*

Proof. If $Z = 0$, then $(a(bx - xc))^n = 0$ for all $x \in L$. But then $b = c \in Z$ by Lemma 1. So we may assume that $Z \neq 0$. If $b \in Z$, then $(a(bx - xc))^n = (ax(b - c))^n \in Z$ for all $x \in L$. Substitute $a^{-1}x$ for x, we have $(x(b - c))^n \in Z$ for all $x \in L$. Thus $b - c = 0$ or R is commutative by Lemma 2. If $b - c = 0$, then $b = c \in Z$. Similarly, if $c \in Z$, then $b = c \in Z$ or R is commutative. So from now on we assume that $b \notin Z$ and $c \notin Z$. In this case, L satisfies the nontrivial GPI $(a(bx - xc))^n y - y(a(bx - xc))^n = 0$ and Q also satisfies the same GPI by [5, Theorem 2]. Again by a theorem of Martindale [17], Q is isomorphic to a dense subring of $\mathrm{End}(_DV)$, where D is a finite dimensional division ring over C and V is a left D-vector space.

If $\dim_D V = \infty$, then $(a(bx - xc))^n = 0$ holds on H, the socle of Q and hence holds on Q. But again by Lemma 1, $b = c = Z$, a contradiction. So we must have $\dim_D V < \infty$ and hence $Q \simeq \mathrm{End}(_D V)$. That is, Q is isomorphic to D_m, the $m \times m$ matrix ring over D for some m.

If C is finite, then D, being finite dimensional over C, is a finite division ring and thus is a field by Wedderburn's theorem [9]. In this case, $Q = C_m$. On the other hand, if C is infinite, let F be the algebraic closure of C, then by the van der Monde determinant argument, we see that $Q \otimes_C F$ satisfies the same GPI $(a(bx-xc))^n y - y(a(bx-xc))^n = 0$. But $Q \otimes_C F \simeq D_m \otimes_C F = (D \otimes_C F)_m = F_k$ for some $k > 1$ since R is not commutative.

Suppose that $k \geq 3$. If $x \in Q$ is of rank 1, then bx and xc are of rank at most 1. Hence $a(bx - xc)$ and $(a(bx - xc))^n$ are of rank at most 2. Consequently, $(a(bx - xc))^n = 0$ for all $x \in Q$ with rank 1. Since $b \notin F$, there is a $v \in V$ such that v and vb are linearly independent over F. Then there exists $x \in Q$ of rank 1 such that $vx = 0$ and $vbx = va^{-1}$ and hence $va^{-1}(a(bx - xc))^n = va^{-1} \neq 0$, a contradiction. Therefore $k = 2$ and $Q \simeq F_2$. Hence R is an order in a 4-dimensional simple algebra. ■

Lemma 4. *Let R be a prime ring and let $b, c \in R$. Let β be an automorphism of R. Suppose that $(bx - \beta(x)c)^n = 0$ for all $x \in L$, where L is a nonzero ideal of R and n is a fixed positive integer, then $bx - \beta(x)c = 0$ for all $x \in R$.*

Proof. If $b = 0$ or $c = 0$, then we are done by [11, Lemma 1.1]. So we may assume that $b \neq 0$ and $c \neq 0$. Suppose that β is X-inner. Then $\beta(x) = axa^{-1}$ for all $x \in R$, where a is invertible in Q. Hence

$$(bx - \beta(x)c)^n = (bx - axa^{-1}c)^n = (a(a^{-1}bx - xa^{-1}c))^n = 0$$

for all $x \in L$ and also for all $x \in Q$ by [5, Theorem 2]. By Lemma 1 we have $a^{-1}b = a^{-1}c \in C$. In particular, $b = c$ and then $bx - \beta(x)c = bx - \beta(x)b$ is a β-derivation. By Lemma 2 in [2], $bx - \beta(x)c = 0$ for all $x \in R$ and we are done.

Now suppose that β is X-outer. Since L satisfies the identity $(bx - \beta(x)c)^n = 0$, by [5, Theorem 2], Q also satisfies the same identity. Moreover, by the Main Theorem of [8], Q satisfies a nontrivial GPI. By Martindale's theorem [17], Q is isomorphic to a dense subring of $\mathrm{End}(_D V)$, where D is the associated division ring of Q, and V is a vector space over D and Q contains nonzero linear transformations of finite rank. By [9, P.79], there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\beta(x) = TxT^{-1}$ for all $x \in Q$. Now $(bx - \beta(x)c)^n = (bx - TxT^{-1}c)^n = (T(T^{-1}bx - xT^{-1}c))^n = 0$ for all $x \in Q$.

If $\dim_D V = 1$, then $Q \simeq D$ and hence $bx - \beta(x)c = 0$ for all $x \in R$. So we may assume that $\dim_D V \geq 2$. If v and $T^{-1}bv$ are D-dependent for all $v \in V$, then as usual, there is $\lambda \in D$ such that $T^{-1}bv = \lambda v$ for all $v \in V$ and this implies

$$\begin{aligned} vT^{-1}(bx-\beta(x)c) &= vT^{-1}(bx-TxT^{-1}c) \\ &= vT^{-1}bx-vxT^{-1}c \\ &= \lambda vx-vxT^{-1}c \\ &= vxT^{-1}b-vxT^{-1}c \\ &= vT^{-1}(TxT^{-1}b-TxT^{-1}c) \\ &= vT^{-1}(\beta(x)(b-c)) \end{aligned}$$

for all $v \in V$ and all $x \in Q$. Since T is a semi-linear automorphism of V and Q acts faithfully on V, we have $bx-\beta(x)c = \beta(x)(b-c)$ for all $x \in Q$. Hence $(\beta(x)(b-c))^n = 0$ for all $x \in Q$. By [11, Lemma 1.1], $b = c$ and hence $bx-\beta(x)c = 0$ for all $x \in Q$ as asserted.

Now we may assume that there exists $v_0 \in V$ such that v_0 and $v_0T^{-1}b$ are D-independent. By the density of Q, there is a $x \in Q$ such that $v_0T^{-1}bx = v_0T^{-1}$ and $v_0x = 0$. This implies

$$v_0T^{-1}(T(T^{-1}bx-xT^{-1}c)) = v_0T^{-1}bx-v_0xT^{-1}c = v_0T^{-1}$$

and

$$v_0T^{-1}(T(T^{-1}bx-xT^{-1}c))^n = v_0T^{-1} \neq 0$$

a contradiction. The proof is complete. ■

Lemma 4 was proved in [16, Lemma 2.6] in a different way. As a corollary we have the following

Theorem 1. *Let R be a prime ring and f a right generalized β-derivation of R. If $f(x)^n = 0$ for all $x \in L$, where L is a nonzero ideal of R and n is a fixed positive integer, then $f = 0$.*

Proof. We can write $f(x) = sx+\delta(x)$ where $s \in {}_FR$ and where δ is the associated β-derivation of f. By [8, Theorem 2], we have

$$(sx+\delta(x))^n = 0 \tag{1}$$

for all $x \in {}_FR$. If δ is X-outer, then by [8, Theorem 1], we have $(sx+y)^n = 0$ for all $x, y \in R$. In particular, $y^n = 0$ for all $y \in R$. By [11, Lemma 1.1], this leads to a contradiction. Suppose now that δ is X-inner. Then $\delta(x) = bx-\beta(x)b$ for all $x \in R$, where $b \in Q$. We can rewrite (1) as

$$((s+b)x-\beta(x)b)^n = 0$$

for all $x \in R$ and hence for all $x \in {}_FR$ [8, Theorem 2]. By Lemma 4, $(s+b)x-\beta(x)b = 0$ for all $x \in {}_FR$. Thus $f = 0$ follows. This proves the theorem. ■

As a consequence of Theorem 1, we have

Corollary 1. *Let R be a prime ring and f a right generalized (α, β)-derivation of R. If $f(x)^n = 0$ for all $x \in L$, where L is a nonzero ideal of R and n is a fixed positive integer, then $f = 0$.*

Lemma 5. *Let R be a prime ring with center Z. Let $b, c \in R$ and let $f(x) = bx - \beta(x)c$ for all $x \in R$. Assume that $f \neq 0$. If $f(x)^n \in Z$ for all $x \in L$, where $L \neq 0$ is an ideal of R and $n \geq 1$ is a fixed integer, then either R is commutative or R is an order in a 4-dimensional simple algebra.*

Proof. If $Z = 0$, then $(bx - \beta(x)c)^n = 0$ for all $x \in L$ and hence $bx - \beta(x)c = 0$ for all $x \in R$ by Lemma 4, which is a contradiction. So we may assume that $Z \neq 0$. If β is X-inner, then there exists $a \in Q$ such that $\beta(x) = axa^{-1}$ for all $x \in R$. So by the hypothesis, we have

$$(bx - \beta(x)c)^n = (bx - axa^{-1}c)^n = (a(a^{-1}bx - xa^{-1}c))^n \in Z$$

for all $x \in L$. That is, L satisfies the identity $(a(a^{-1}bx - xa^{-1}c))^n y - y(a(a^{-1}bx - xa^{-1}c))^n = 0$. By [5, Theorem 2], Q also satisfies the same identity and hence

$$(a(a^{-1}bx - xa^{-1}c))^n \in C$$

for all $x \in Q$. By Lemma 3, we see that either $a^{-1}b = a^{-1}c \in C$ or R is commutative or R is an order in a 4-dimensional simple algebra. If $a^{-1}b = a^{-1}c \in C$, then $b = c$ and hence $bx - \beta(x)c = bx - axa^{-1}c = bx - cx = 0$ for all $x \in R$, which is not the case. Therefore R is commutative or R is an order in a 4-dimensional simple algebra as asserted.

Now suppose that β is X-outer. By the hypothesis, L satisfies $(bx - \beta(x)c)^n y - y(bx - \beta(x)c)^n = 0$. By Theorem 1 of [6] Q also satisfies the same identity. Moreover, Q satisfies a nontrivial GPI by the Main Theorem of [6]. By a Martindale's result cited before, Q is a primitive ring having nonzero socle and its associated division ring D is finite dimensional over C. Hence Q is isomorphic to a dense subring of $\mathrm{End}({}_DV)$. If $\dim_D V = \infty$, then $(bx - \beta(x)c)^n = 0$ for all $x \in H$, the socle of Q and hence for all $x \in Q$. Again by Lemma 4, we have $bx - \beta(x)c = 0$ for all $x \in R$ and we are done in this case. So we may assume that $\dim_D V < \infty$. Thus $Q = \mathrm{End}({}_DV)$ and is isomorphic to D_m, the $m \times m$ matrix ring over D for some m.

We claim that $m \leq 2$. Suppose on the contrary that $m > 2$. By [9, P.79] there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\beta(x) = TxT^{-1}$ for all $x \in Q$. Hence we have $(bx - \beta(x)c)^n = (bx - TxT^{-1}c)^n = (T(T^{-1}bx - xT^{-1}c))^n \in C$ for all $x \in Q$. If v and $vT^{-1}b$ are D-dependent for all $v \in V$, then as before, there exists a $\lambda \in D$ such that $vT^{-1}b = \lambda v$ for all $v \in V$. This implies

$$\begin{aligned} vT^{-1}(bx - \beta(x)c) &= vT^{-1}(T(T^{-1}bx - xT^{-1}c)) \\ &= vT^{-1}bx - vxT^{-1}c \\ &= \lambda vx - vxT^{-1}c \\ &= vxT^{-1}b - vxT^{-1}c \\ &= vT^{-1}(TxT^{-1}b - TxT^{-1}c) \\ &= vT^{-1}(\beta(x)(b-c)) \end{aligned}$$

for all $v \in V$ and for all $x \in Q$. Since Q acts on V faithfully and $VT^{-1} = V$, we have $bx - \beta(x)c = \beta(x)(b-c)$ for all $x \in Q$. Hence $(\beta(x)(b-c))^n = (bx - \beta(x)c)^n \in C$ for all $x \in Q$. In particular $(x(b-c))^n \in C$ for all $x \in Q$. By Lemma 2, $b = c$ or Q is commutative. But Q is not commutative, since $Q \simeq D_m$, $m \geq 3$. On the other hand, if $b = c$, then $bx - \beta(x)c = bx - \beta(x)b$ is a β-derivation. Proposition 2 in [3] implies that Q is commutative or Q is an order in a 4-dimensional simple algebra which is absurd since $Q \simeq D_m$, $m \geq 3$.

Now we may assume that there exists $v_0 \in V$ such that v_0 and $v_0T^{-1}b$ are D-independent. By the density of Q, there exists $x \in Q$ of rank 1 such that $v_0x = 0$ and $v_0T^{-1}bx = v_0T^{-1}$. Hence $v_0T^{-1}(T(T^{-1}bx - xT^{-1}c)) = v_0T^{-1}bx - v_0xT^{-1}c = v_0T^{-1}$ and $v_0T^{-1}(bx - \beta(x)c)^n = v_0T^{-1}(T(T^{-1}bx - xT^{-1}c))^n = v_0T^{-1}$. On the other hand, since x is of rank 1, $(bx - \beta(x)c)^n$ is of rank at most 2. Being in C, we have $(bx - \beta(x)c)^n = 0$ since $\dim_D V \geq 3$. Therefore $v_0T^{-1} = 0$ which is a contradiction. This proves our claim and hence $\dim_D V \leq 2$.

If C is finite, then $\dim_C D < \infty$ implies D is also finite. Thus D is a filed by Wedderburn's Theorem [9, P.183]. In this case, R is commutative or R is an order in a 4-dimensional simple algebra. So we may assume that C is infinite for the rest of the proof. If β is not Frobenius, then by the Main Theorem of [7], we have $(bx - yc)^n \in C$ for all $x, y \in Q$. This implies $(bx)^n \in Z$ for all $x \in R$. Again, by Lemma 2, $b = 0$ or R is commutative. If R is not commutative, then $b = 0$ and this implies $(-cy)^n \in Z$ for all $y \in R$. Again this leads to $c = 0$ since R is not commutative. But if $b = 0$ and $c = 0$ then $f = 0$, a contradiction. Hence R is commutative in this case.

On the other hand, if β is Frobenius, then $\operatorname{char} Q = p > 0$. Otherwise if $\operatorname{char} Q = 0$, then $\beta(\lambda) = \lambda$ for all $\lambda \in C$ and hence β must be X-inner by [1, Theorem 4.7.4], a contradiction. Also $\beta(\lambda) = \lambda^{p^k}$ for all $\lambda \in C$ and for some integer $k \neq 0$. Substitute λx for x into $(bx - \beta(x)c)^n$ with $\lambda \neq 0$, we have $(b(\lambda x) - \beta(\lambda x)c)^n = (\lambda bx - \lambda^{p^k}\beta(x)c)^n \in C$ for all $x \in Q$ and hence $(bx - \lambda^{p^k - 1}\beta(x)c)^n \in C$ for all $x \in Q$. Expanding this, we have

$$\sum_{i=0}^{n}\left(\sum_{(i,n-i)} y_1y_2\cdots y_n\right)\lambda^{(p^k-1)i}\in C, \tag{2}$$

where the inside summation are taken over all permutations of $n-i$ (bx)'s and i $(\beta(x)c)$'s, that is, each term has exactly $n-i$ (bx) and i $(\beta(x)c)$ but in some different order. Let $u=\lambda^{p^k-1}$ and

$$t_i=\sum_{(i,n-i)} y_1y_2\cdots y_n$$

for $i=0,1,2,\ldots,n$. Then we can rewrite (2) into the following

$$t_0+ut_1+\cdots+u^nt_n\in C \tag{3}$$

Replacing λ successively by $1,\lambda,\ldots,\lambda^n$, (3) gives the system of equations

$$\begin{aligned} t_0+t_1+\cdots+t_n &= \tau_0\\ t_0+ut_1+\cdots+u^nt_n &= \tau_1\\ &\vdots\\ t_0+u^nt_1+\cdots+u^{n^2}t_n &= \tau_n \end{aligned} \tag{4}$$

where $\tau_0,\tau_1,\ldots,\tau_n\in C$. Since C is infinite, there exists infinitely many $\lambda\in C$ such that $\lambda^{(p^k-1)\ell}\neq 1$ for $\ell=1,2,\ldots,n$ and so the van der Monde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1\\ 1 & u & \cdots & u^n\\ \vdots & \vdots & & \vdots\\ 1 & u^n & \cdots & u^{n^2}\end{vmatrix}=\prod_{\substack{i,j=0\\ i<j}}^{n}(u^i-u^j)=\prod_{\substack{i,j=0\\ i<j}}^{n}\left(\lambda^{i(p^k-1)}-\lambda^{j(p^k-1)}\right)$$

is not zero. Therefore we can solve from (4) and obtain $t_0\in C$. But $t_0=(bx)^n$ and so we have $(bx)^n\in C$ for all $x\in Q$. As before, we can conclude that R is commutative. The proof is complete. ∎

Now we are ready to give

Proof of Theorem B. If $Z=0$, then $f(x)^n=0$ for all $x\in L$, a nonzero ideal of R. By Theorem 1, $f=0$ which is not the case. Thus $Z\neq 0$. We can write $f(x)=sx+\delta(x)$, where $s\in {}_FR$ and where $\delta:R\to R$ is the associated β-derivation of f. By the hypothesis, we have $(sx+\delta(x))^ny-y(sx+\delta(x))^n=0$ for all $x,y\in L$. By [8, Theorem 2], we see that $(sx+\delta(x))^ny-y(sx+\delta(x))^n=0$ also holds for all $x,y\in {}_FR$. If δ is X-outer, then by [8, Theorem 1] we have $(sx+z)^ny-y(sx+z)^n=0$ for all x, y, $z\in {}_FR$. In particular, we have

$(sx)^n \in C$ for all $x \in {}_FR$. By Lemma 2, $s = 0$ or ${}_FR$ is commutative. If R is not commutative, then $s = 0$ and hence $f(x) = \delta(x)$ for all $x \in R$. We are done in this case by [3, Theorem B]. Hence we may assume that δ is X-inner and write $\delta(x) = bx - \beta(x)b$, where $b \in Q$. In this case, $f(x)^n = ((s+b)x - \beta(x)b)^n \in C$ for all $x \in {}_FR$. Hence we are done by Lemma 5. The proof is complete. ■

As a corollary, we have Theorem A immediately.

Example. Let F be a field of characteristic 2 and let $R = F_2$, the 2×2 matrix ring over F. Let

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $x \in R$, define $f(x) = ax - u^{-1}xub$. It is easy to see that $f(x)^2 \in Z$.

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