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MINIMAL ZERO-SUM SEQUENCES IN FINITE CYCLIC GROUPS

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Abstract. Let C_n be the cyclic group of order n, $n \geq 20$, and let $S = \prod_{i=1}^{k} g_i$ be a minimal zero-sum sequence of elements in C_n. We say that S is insplitable if for any $g_i \in S$ and any two elements $x, y \in C_n$ satisfying $x+y = g_i$, $Sg_i^{-1}xy$ is not a minimal zero-sum sequence any more. We define Index(S) $= \min_{(m,n)=1}\{\sum_{i=1}^{k} |mg_i|\}$, where $|x|$ denotes the least positive inverse image under homomorphism from the additive group of integers $\mathbb{Z}$ onto C_n. In this paper we prove that for an insplitable minimal zero-sum sequence S, if Index(S) $= 2n$, then $|S| \leq \lfloor \frac{n}{2} \rfloor + 1$.

1. Introduction and Main Results

Let G be a finite abelian group (written additively). *A sequence in* G is a multi-set in G and will be written in the form $S = \prod_{i=1}^{k} g_i = \prod_{g\in G} g^{v_g(S)}$, where $v_g(S) \in \mathbb{N}_0$ is the *multiplicity of* g *in* S, and a sequence T is *a subsequence of* S if $v_g(T) \leq v_g(S)$ for every $g \in G$, denoted by $T|S$. Let ST^{-1} denote the sequence obtained by deleting the terms of T from S. We call $|S| = k$ *the length of* S. By $\sigma(S)$ we denote *the sum of* S, that is $\sigma(S) = \sum_{i=1}^{k} g_i = \sum_{g\in G} v_g(S)g \in G$.

Let S be a sequence in G, we call S a *zero-sum sequence* if $\sigma(S) = 0$; a *zero-sum free sequence* if for any subsequence W of S, $\sigma(W) \neq 0$. We call S a *minimal zero-sum sequence* if it is a zero-sum sequence and every proper subsequence is zero-sum free.

Let C_n be the cyclic group of order n. For every $x \in C_n$, we define $|x|$ to be the least positive inverse image under homomorphism from the additive group of

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integers $\mathbb{Z}$ onto C_n. Let $S = \prod_{i=1}^{k} g_i$ be a sequence in C_n, by $|S|_n$ we denote the sum $\sum_{i=1}^{k} |g_i|$. Define

$$\text{Index}(S) = \min_{(m,n)=1} \{|mS|_n\}$$

and

$$I(C_n) = \max_{S} \{\text{Index}(S)\}$$

where S runs over all minimal zero-sum sequences of elements in C_n.

The question of considering equivalence classes of minimal zero-sum sequences(see Chapter 5 in [3]) arose when the following problem was posed at Algebra Conference in Marseille, France:

Let p be a prime, whether we have Index$(S) = p$ for any minimal zero-sum sequence S in C_p?

The answer to this question is no (see Theorem 2 of [1]). In addition, Gao [2] began to consider the minimal integer t such that every minimal zero-sum sequence S of at least t elements in C_n satisfies Index$(S) = n$, which defined as $l(C_n)$. The papers [4, 5] separately got the final value of $l(C_n)$, that is $l(C_n) = \lfloor \frac{n}{2} \rfloor + 2$ if $n \notin \{2, 3, 4, 5, 7\}$, and $l(C_n) = 1$ if $n \in \{2, 3, 4, 5, 7\}$.

In [2], The author considered the following kind of sequences:

Definition 1.1. Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements in an abelian group G, we say S is *splitable* if there exists an element $g \in S$ and two elements $x, y \in G$ such that $x + y = g$ and $Sg^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well, otherwise we say S is *insplitable*.

For some real number $x \in \mathbb{R}$, let $\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leq x\}$ and $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$.

In this paper, we are to prove the following two results:

Theorem 1.2. *For any k, $n \leq kn \leq I(C_n)$, there exists minimal zero-sum sequence S such that Index$(S) = kn$.*

Theorem 1.3. *Let S be a minimal zero-sum sequence in C_n, $n \geq 20$. If Index$(S) = 2n$ and S is insplitable, then $|S| \leq \lfloor \frac{n}{2} \rfloor + 1$.*

2. Proofs of the Main Results

Proof of Theorem 1.2. Let $S = \prod_{i=1}^{t} g_i$ be a minimal zero-sum sequence and Index$(S) = I(C_n) = ln$, without loss of generality, say $g_1 \leq g_2 \leq \cdots \leq g_t$ and $\sum_{i=1}^{t} g_i = ln$. Consider the sequence

$$S_1 = |g_1 + g_2| \prod_{i=3}^{t} g_i,$$

then S_1 is minimal and $\text{Index}(S_1) = \text{Index}(S) + \delta$, where $\delta = 0$ or $-n$. If $\delta = -n$, then $\text{Index}(S_1) = I(C_n) - n$; if $\delta = 0$, set

$$S_1 = |g_1 + g_2 + g_3| \prod_{i=4}^{t} g_i,$$

then $\text{Index}(S_1) = \text{Index}(S) + \delta$, where $\delta = 0$ or $-n$. If $\delta = -n$, then $\text{Index}(S_1) = I(C_n) - n$; otherwise, set

$$S_1 = |g_1 + g_2 + g_3 + g_4| \prod_{i=5}^{t} g_i,$$

and continue the discussion. Then, we can derive a minimal zero-sum sequence S_1, such that $\text{Index}(S_1) = I(C_n) - n$. Continue this process and we will get minimal zero-sum sequences $S_2, S_3, \cdots, S_{l-1}$, such that $\text{Index}(S_2) = I(C_n) - 2n$, $\text{Index}(S_3) = I(C_n) - 3n$, $\cdots$, $\text{Index}(S_{l-1}) = n$. This process can be got since we have the minimal zero-sum sequence $S_0 = |g_1 + g_2 + \cdots + g_t|$ with $\text{Index}(S_0) = n$. This completes the proof. ∎

The following two simple lemmas play an important part in our proof of Theorem 1.3.

Lemma 2.4. *Let $S = g^k \prod_{i=1}^{r} x_i$ be an insplitable minimal zero-sum sequence in C_n, $k \geq 1$. If $x_i = tg$, $t > 1$ a positive integer, then $t \geq k + 2$.*

Proof. Without loss of generality, say $x_1 = tg, t > 1$. Since S is an insplitable minimal zero-sum sequence, the sequence $S' = g^{k+1} \cdot (t-1)g \prod_{i=2}^{r} x_i$ contains a proper zero-sum subsequence W with $(t-1)g|W$ or $g^{k+1}|W$. If $(t-1)g|W$, we claim that $t - 1 \geq k + 1$, i.e. $t \geq k + 2$, otherwise, $t - 1 \leq k$, replace $(t-1)g$ in W by g^{t-1}, we get that $W((t-1)g)^{-1}g^{t-1}$ is a proper zero-sum subsequence of S, a contradiction. If $g^{k+1}|W$, we also get $t \geq k + 2$, otherwise, the subsequence $Wg^{-(k+1)}x_1 g^{k+1-t}$ of S has the same sum as W, which is a contradiction. ∎

Lemma 2.5. *Let $S = 3^t \prod_{i=1}^{r} x_i$, $x_i \neq 3$, be a minimal zero-sum sequence in C_n, if $\sigma(S) = \text{Index}(S) = 2n$, then $t < \lceil \frac{n}{3} \rceil$.*

Proof. If $n \equiv 0 (\mod 3)$, it is evident that $t < \lceil \frac{n}{3} \rceil$. Now we suppose that $n \equiv i (\mod 3)$, $i = 1$ or 2. If $r \geq 2$, then there exists a subsequence W of $S3^{-t}$ such that $\sigma(W) \equiv i (\mod 3)$. If $\sigma(W) > n$, then $t \leq \lfloor \frac{2n - \sigma(W)}{3} \rfloor < \lceil \frac{n}{3} \rceil$; otherwise there is a positive integer k satisfying $\sigma(W) + 3k = n$ and thus $t < k = \frac{n - \sigma(W)}{3} < \lceil \frac{n}{3} \rceil$. If $r = 1$, note that $(3, n) = 1$, there exists m such that $(m, n) = 1$ and $mS = 1^t |mx_1|$, then $\sigma(mS) = n < \text{Index}(S)$ since $t < \frac{2n}{3}$ and $|mx_1| < n$, which is a contradiction. ∎

Proof of Theorem 1.3. Note that for $n \geq 8$

$$S = \begin{cases} (\underbrace{1, \cdots, 1}_{\frac{n}{2}-2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+2}{2}), & \text{if } n \text{ is even,} \\ (\underbrace{1, \cdots, 1}_{\frac{n-5}{2}}, \frac{n+3}{2}, \frac{n+3}{2}, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

is an insplitable minimal zero-sum sequence with Index$(S) = 2n$, the length of which is $|S| = \lfloor \frac{n}{2} \rfloor + 1$.

Let S be the longest (in length) minimal zero-sum sequence in C_n satisfying the conditions in the theorem, then $|S| \geq \lfloor \frac{n}{2} \rfloor + 1$. Without loss of generality, set $S = 1^k \prod_{i=1}^{r} x_i$, where $\sigma(S) = k + \sum_{i=1}^{r} x_i = 2n$. By Lemma 2.1 and note that S is minimal zero-sum, we get

$$k + 2 \leq x_i \leq n - k - 1, \text{for all } i \in \{1, \cdots, r\},$$

and thus we derive $k + 2 \leq n - k - 1$, that is $k \leq \lfloor \frac{n-3}{2} \rfloor$. Note that $2n = \sigma(S) = k + \sum_{i=1}^{r} x_i \geq k + (k+2)(\lfloor \frac{n}{2} \rfloor + 1 - k)$, when n is big enough, say $n \geq 20$, we get $k \leq 2$ or $\lfloor \frac{n}{2} \rfloor - 2 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

Now we suppose $n \geq 20$, and distinguish the following cases:

Case 1. $\lfloor \frac{n}{2} \rfloor - 2 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

Set $S = 1^k \prod_{i=1}^{r} x_i$, by Lemma 2.1 we get $x_i \geq \lfloor \frac{n}{2} \rfloor - 2 + 2 = \lfloor \frac{n}{2} \rfloor$, thus $r \leq 3$ since $\sigma(S) = 2n$. If n is even, then $|S| \leq k + 3 \leq \lfloor \frac{n-3}{2} \rfloor + 3 = \lfloor \frac{n}{2} \rfloor + 1$. If n is odd, we have $|S| \leq k + 3 \leq \lfloor \frac{n-3}{2} \rfloor + 3 = \lfloor \frac{n}{2} \rfloor + 2$, if there exists S with $|S| = \lfloor \frac{n}{2} \rfloor + 2$, then $S = 1^{\frac{n-3}{2}} \cdot (\frac{n+1}{2})^3$ since $x_i \geq \lfloor \frac{n-3}{2} \rfloor + 2 = \frac{n+1}{2}$, it is evident that Index$(S) = n$, a contradiction. Therefore $|S| \neq \lfloor \frac{n}{2} \rfloor + 2$, that is $|S| \leq \lfloor \frac{n}{2} \rfloor + 1$.

Case 2. $k = 2$. Set $S = 1^2 \prod_{i=1}^{r} x_i$, where $x_i \geq k + 2 = 4$ according to Lemma 2.1. If n is even, then $|S| \leq 2 + \lfloor \frac{2n-2}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. If $n \equiv 1 (\mod 4)$, the sequence $S^* = 1^2 4^{\frac{2n-2}{4}}$ contains a zero-sum subsequence; and if $n \equiv 3 (\mod 4)$, set $n = 4l + 3$, then the sequence $S^* = 1^2 4^{\frac{2n-2}{4}}$ has Index$(S^*) = n$, since $|(l+1)S|_n = 2(l+1) + \frac{2n-2}{4} = n$. Therefore, if n is odd, $S^* = 1^2 4^{\frac{2n-2}{4}}$ is not a minimal zero-sum sequence with Index$(S) = 2n$, so there must exist some number $x_i > 4$ in S, and thus $|S| < 2 + \lfloor \frac{2n-2}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 2$, that is $|S| \leq \lfloor \frac{n}{2} \rfloor + 1$.

Case 3. $k = 1$.

By Lemma 2.1, we can set $S = 1 \cdot 3^s \prod_{i=1}^{r} x_i$. If $s = 0$, then $|S| \leq 1 + \lfloor \frac{2n-1}{4} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$ and we are done, so we assume that $s \geq 1$. Also we have $s < \lceil \frac{n}{3} \rceil$ according to Lemma 2.2.

By Lemma 2.1, if $x_i \equiv 0(\mod 3)$, then $x_i \geq 3(s+2) \geq 9$, so 6 can't occur in S. Since S is insplitable, that is, if we split 3 into $1+2$, there exist two subsequences U and V of $S(1,3)^{-1}$ such that $\sigma(U) = \sigma(V) = n-2$. Set $v_3(U) = u \geq \lceil \frac{s-1}{2} \rceil$.

Now we consider the following subcases.

Subcase 1. $s - u \geq 2$.

Then $u \geq 1$ since $u \geq \lceil \frac{s-1}{2} \rceil$. There exist subsequences of U such that the sums of which are $n-2$ and $n-5$ respectively. Therefore V contains no $4, 5$ otherwise we can get a proper zero-sum subsequence of S. Now we consider U, if $4|U$, then $n-2-4 = n-6$ can be expressed as a sum of a subsequence of U, now we take $(3,3)$ from SU^{-1} since $s-u \geq 2$, and get a zero-sum subsequence of S, a contradiction; if $5|U$, then $n-7$ can be expressed as a sum of a subsequence of U, note $(1,3,3)|SU^{-1}$, and also we derive a zero-sum subsequence of S, a contradiction. Therefore, each term in S is bigger than or equal to 7 except 1 and 3, and thus $|S| < 1 + \lceil \frac{n}{3} \rceil + \lfloor \frac{n-1}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

Subcase 2. $s - u = 1$.

If s=1, then $|S| \leq 1 + 1 + \lfloor \frac{2n-4}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$.

Now we assume $s \geq 2$, $u = s-1 \geq 1$. There exist subsequences of U such that the sums of which are $n-2-3i, i = 0, 1, \cdots, s-1$ respectively. Therefore the numbers $3i+1, i = 1, 2, \cdots, s-1$ can't occur in V, since $1|SU^{-1}$ and $1+3i+1+n-2-3i = n$, and $3i+2, i = 1, 2, \cdots, s-1$ either since $n-2-3i+3i+2 = n$. Also for any numbers of the form $3i, i > 1$, we have $3i \geq 3(s+2)$. Therefore each term in V is not smaller than $3s+1 \geq 7$, and thus $|S| \leq \lfloor \frac{n-2}{3} \rfloor + 1 + 1 + \lfloor \frac{n-2}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

Case 4. $k = 0$.

Set $S = 2^s 3^t \prod_{i=1}^{r} x_i$, where s, t, r are nonnegative integers. If $s + t \leq 2$, we clearly have $|S| \leq 2 + \lfloor \frac{2n-4}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. Now suppose $s + t \geq 3$, we distinguish three subcases.

Subcase 1. $s = 0$.

$S = 3^t \prod_{i=1}^{r} x_i$, $t \geq 3$. By Lemma 2.1 we get $x_i \neq 6$. Since S is insplitable, there exist subsequences U, V of $S3^{-1}$, such that $\sigma(U) = n-1$, $\sigma(V) = n-2$. Set $v_3(U) = u$, $v_3(V) = v$, $u + v = t - 1$. We have $t < \lceil \frac{n}{3} \rceil$ according to Lemma 2.2.

(i) If $u \geq \lceil \frac{t-1}{2} \rceil$.

(1). $t - u \geq 3$.

If $u \geq 3$, there are subsequences of U such that the sums of which are $n-1$, $n-4$, $n-7$, $n-10$ respectively. Therefore, there is no $4, 7$ in V, and 5 can occur at most one time since $n - 10 + 5 + 5 = n$. In U, there is no 5 and at most one 4 , since $n-1-5+3+3 = n$ and $n-1-4-4+3+3+3 = n$.

Therefore, the terms in S are not smaller than 7 except 3 and one 4 and one 5, and thus $|S| < \lceil \frac{n}{3} \rceil + 2 + \lfloor \frac{n-4-5}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

If $u = 2$, then $t = 5$. Note that there is no 4 in V, $|S| \leq 5 + \lfloor \frac{n-1-3-3}{4} \rfloor + \lfloor \frac{n-2-3-3}{5} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

(2). $t - u \leq 2$.

If $u \geq 3$, according to the discussion above, there is no $4, 6, 7$ in V, and 5 exists at most one time. So, $|S| \leq 2 + \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n+1-6-5}{8} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

If $u \leq 2$, then $t \leq 4$, and $|S| \leq 4 + \lfloor \frac{2n-12}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$.

(ii) If $v \geq \lceil \frac{t-1}{2} \rceil$.

(1). $t - v \geq 4$.

Since $v \geq \lceil \frac{t-1}{2} \rceil \geq 3$, there are subsequences of V such that the sum of which are $n-2$, $n-5$, $n-8$, $n-11$ respectively. Therefore 5 can't occur in U, and 4 occurs at most one time since $n - 8 + 4 + 4 = n$. In V, there is no 4, 7 since $n - 2 - 4 + 3 + 3 = n$ and $n - 2 - 7 + 3 + 3 + 3 = n$, and 5 can only occur one time since $n - 2 - 5 - 5 + 3 + 3 + 3 + 3 = n$. Therefore, $|S| \leq \lceil \frac{n}{3} \rceil + 2 + \lfloor \frac{n-4-5}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

(2). $t - v \leq 3$.

If $v \geq 3$, using the same methods as above, there is no $5, 6$ in U, and 4 exists at most one time. So, $|S| \leq 3 + 1 + \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{n+2-4-9}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

If $v \leq 2$, and $t \leq 4$, then$|S| \leq 4 + \lfloor \frac{2n-12}{4} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. Otherwise we have $v = 2$ and $t = 5$, then there is no $5, 6$ in U, and 4 occurs at most one time, and thus $|S| \leq 5 + 1 + \lfloor \frac{n-2-6}{4} \rfloor + \lfloor \frac{n+2-4-9}{7} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

Subcase 2. $s = 1$.

$S = 2 \cdot 3^t \prod_{i=1}^{r} x_i$, $t \geq 2$. Just as the discussion in the subcase $s = 0$, we have $x_i \neq 4, 6$, and $t < \lceil \frac{n}{3} \rceil$. Since S is insplitable, there exists subsequence U of $S2^{-1}$, such that $\sigma(U) = n - 1$ and $v_3(U) = u \geq \lceil \frac{t}{2} \rceil \geq 1$.

(i) $t - u \geq 2$. Then $u \geq 2$.

Using the same methods as in subcase $s = 0$ (i), we derive that each term in S is not smaller than 8 except 2 and 3, and thus $|S| \leq \lceil \frac{n}{3} \rceil + 1 + \lfloor \frac{n-2}{8} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

(ii) $t - u \leq 1$.

If $t \geq 3$, from the discussion above, we get that $4, 5, 6, 7$ can't occur in SU^{-1}, so $|S| \leq 1 + 1 + \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n-1-3}{8} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

If $t = 2$, and note that $x_i \neq 4$, therefore, $|S| \leq 1 + 2 + \lfloor \frac{2n-2-6}{5} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$, and we are done.

Subcase 3. $s \geq 2$.

Let $S = 2^s \prod_{i=1}^{r} x_i$. There exist subsequences U, V such that $\sigma(U) = \sigma(V) = n-1$, suppose $u = v_2(U) \geq v_2(V)$, that is $u \geq \lceil \frac{s-1}{2} \rceil$.

By Lemma 2.1 and note that S is minimal zero-sum, just as the discussion above, we derive the following conclusions:

(a) If x_i is even, then $x_i \geq 2(s+2)$;

(b) If x_i is odd in U, then $x_i \geq 2(s-u)+1$;

(c) If x_i is odd in V, then $x_i \geq 2u+3$;

(d) If n is odd, and x_i is odd, then $x_i \leq n-2s-2$;

(e) If n is even, and x_i is even, then $x_i \leq n-2s-2$.

In order to get the upper bound of s, we consider the following two cases.

(i) n is odd.

If there is an odd number x_i in V, then $2\lceil \frac{s-1}{2} \rceil + 3 \leq 2u+3 \leq x_i \leq n-2s-2$, and we get $s \leq \lfloor \frac{n-4}{3} \rfloor$.

If there are two even numbers in V except 2, then $4(s+2) \leq n-1$, so $s \leq \lfloor \frac{n-9}{4} \rfloor$.

Now we assume that there is only one term x_1 in V except 2, and x_1 is an even number. In this case, if there are k odd numbers in U, then $k \geq 2$, and $|S| \leq s-u+1+k+\lfloor \frac{n-1-k(2(s-u)+1)}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$; otherwise, there are only even numbers in U, and $|S|$ is maximal when U contains only 2, that is $u = \frac{n-1}{2}$, and $n-1 = 2(s-u-1)+x_i \geq 2(s-u-1)+2(s+2)$, we get $s \leq \lfloor \frac{2n-4}{4} \rfloor$, and thus $|S| = s+1 < \lfloor \frac{n}{2} \rfloor + 1$.

(ii) n is even.

If there is an even number x_i in S except 2, then $2(s+2) \leq x_i \leq n-2s-2$, that is $s \leq \lfloor \frac{n-6}{4} \rfloor$. Now suppose each term in S is odd except 2, note that $\sigma(V) = \sigma(U) = n-1$, there are odd numbers in V. If V contains at least 3 odd numbers, then $6\lceil \frac{s-1}{2} \rceil + 9 \leq 3(2u+3) \leq n-1$, that is $s \leq \lfloor \frac{n-7}{3} \rfloor$, otherwise, set there are $k \geq 1$ odd numbers in U, then $|S| \leq s-u+1+k+\lfloor \frac{n-1-k(2(s-u)+1)}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

According to the discussion above, we only need to prove the theorem in the case of $s \leq \lfloor \frac{n-4}{3} \rfloor$.

(i) $s-u \geq 4$.

Then $u \geq 3$. By the conclusions a,b,c before, we derive that each term in S is bigger than or equal to 9 except 2, and thus $|S| \leq \lfloor \frac{n-4}{3} \rfloor + \lfloor \frac{2n-2\lfloor \frac{n-4}{3} \rfloor}{9} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1$.

(ii) $2 \le s - u \le 3$.

If $u \ge 5$, then except 2 the terms in U are not smaller than 5, and in V are not smaller than 13, so $|S| \le \lfloor \frac{n-4}{3} \rfloor + \lfloor \frac{n-1-2}{13} \rfloor + \lfloor \frac{n-1+4-2\lfloor \frac{n-4}{3} \rfloor}{5} \rfloor \le \lfloor \frac{n}{2} \rfloor + 1$.

If $3 \le u \le 4$, then $5 \le s \le 7$. Note that except 2 the terms in V are not smaller than 9, and in u are not smaller than 5, so $|S| \le 7 + \lfloor \frac{n-1-2}{9} \rfloor + \lfloor \frac{n-1-6}{5} \rfloor \le \lfloor \frac{n}{2} \rfloor + 1$.

If $u \le 2$, then $s \le 5$, and each term in S is bigger than or equal to 5, therefore $|S| \le 5 + \lfloor \frac{2n-2\times 5}{5} \rfloor \le \lfloor \frac{n}{2} \rfloor + 1$.

(iii) $s - u = 1$.

If $u \ge 8$, then except 2 the terms in V are not smaller than 19, and thus $|S| \le \lfloor \frac{n-4}{3} \rfloor + \lfloor \frac{n-1}{19} \rfloor + \lfloor \frac{n-1-2(\lfloor \frac{n-4}{3} \rfloor - 1)}{3} \rfloor \le \lfloor \frac{n}{2} \rfloor + 1$.

If $u \le 7$, then $s = u + 1 \le 8$, we can check that $|S| \le s + \lfloor \frac{n-1}{2u+3} \rfloor + \lfloor \frac{n-1-2u}{3} \rfloor \le \lfloor \frac{n}{2} \rfloor + 1$.

This completes the proof. ■

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