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A CHARACTERIZATION OF NONLINEAR OPEN MAPPINGS BETWEEN PSEUDOMETRIZABLE TOPOLOGICAL VECTOR SPACES

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Abstract. We prove a new characterization of arbitrary (including nonlinear) open maps between pseudometrizable topological vector spaces. We use it to give a new proof that nonconstant analytic functions are open.

1. Introduction

As a basic principle of functional analysis, the open mapping theorem has obtained a series of important improvements since the Banach-Schauder theorem ([1], p. 166) was established. In 1958, V.Pták [2] pointed out that every continuous linear operator from a fully complete space onto a barrelled space must be open (see also [3], p. 195). In 1965, T.Husain [4] characterized those spaces from which continuous linear operators onto barrelled spaces are open, and then N.Adasch [5] characterized those spaces for which closed linear operators onto barrelled spaces are open. In 1998, Qiu Jinghui [6] obtained a series of open mapping theorems for closed linear operators, weakly singular linear operators, etc. In 2004, Edward Beckenstein and Lawrence Narici [7] obtained an open mapping theorem for basis separating linear operators.

Recently, Li Ronglu [8] improved the typical open mapping theorem by relaxing the linearity requirement forced on the mappings concerned.

In this paper, we obtain the characteristic of a mapping is an open mapping. As its application, we shows that each nonconstant analytic function $f : \mathbb{C} \to \mathbb{C}$ is an open mapping.

2. Open Mappings between Paranormed Spaces

Let X be a vector space over the scalar field $\mathbb{K}$. A function $\|\cdot\| : X \to [\,0, +\infty)$

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is called a paranorm on X if $\|0\| = 0$, $\|-x\| = \|x\|$, $\|x+z\| \le \|x\| + \|z\|$ and $\|t_n x_n - tx\| \to 0$ whenever $\|x_n - x\| \to 0$ and $t_n \to t$ in $\mathbb{K}$([3], p. 56).

Note that a first countable topological vector space is a paranormed space ([3], p. 52), so a topological vector space is a pseudometric space iff it is a paranormed space.

Our main results are as following:

Theorem 2.1. *Let X and Y be paranormed spaces and $f : X \to Y$ a mapping. Then the following (I) and (II) are equivalent.*

(I) *f is an open mapping, $i.e.$, $f(G)$ is an open set for each open set $G \subseteq X$.*

(II) *For each $x \in X$ and $y_n \to f(x)$ in Y there exists a sequence $(x_i) \subset X$ such that $x_i \to x$ and $(f(x_i))$ is just a subsequence of (y_n).*

Proof. $(I) \Rightarrow (II)$. Let $x \in X$ and $y_n \to f(x) = y$ in Y. Let $\{U_n\}$ be a neighborhood base of $0 \in X$ such that $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$.

Since $f : X \to Y$ is open, $f(x) = f(x+0) \in f(x+U_n)$ and $f(x+U_n)$ is a neighborhood of $y = f(x)$, $\forall n \in N$. It follows from $y_n \to y$ that there is a strictly increasing sequence $(n_i) \subset \mathbb{N}$ such that $y_{n_i} \in f(x+U_i)$ and so $y_{n_i} = f(x_i)$ for some $x_i \in x + U_i$, $i = 1, 2, 3, \cdots$.

Let U be a neighborhood of $0 \in X$. There is an $i_0 \in \mathbb{N}$ such that $U_i \subset U_{i_0} \subset U$ for all $i \ge i_0$. Then $x_i \in x + U_i \subset x + U, \forall\ i \ge i_0$. Thus, $x_i \to x$ and (II) holds for f.

$(II) \Rightarrow (I)$. Let G be an open subset of X and $x \in G$, $y = f(x) \in f(G)$. If $(Y \backslash f(G)) \cap \{z \in Y : \|z-y\| < 1/n\} \neq \emptyset$ for each $n \in \mathbb{N}$, then there is a sequence $(y_n) \subset Y \backslash f(G)$ such that $\|y_n - y\| < 1/n$ for all n and so $y_n \to y = f(x)$. By (II), there exist subsequences $(y_{n_i}) \subset (y_n)$ and $(x_i) \subset X$ such that $x_i \to x$ and $f(x_i) = y_{n_i}$ for each $i \in \mathbb{N}$. Since G is open and $x \in G$, $x_i \in G$ eventually and so $y_{n_i} = f(x_i) \in f(G)$ eventually. This contradicts $(y_n) \subset Y \backslash f(G)$ and so

$$(Y \backslash f(G)) \cap \{z \in Y : \|z - y\| < 1/n_0\} = \emptyset$$

for some $n_0 \in \mathbb{N}$, $i.e.$, $\{z \in Y : \|z - f(x)\| < 1/n_0\} \subseteq f(G)$ for some $n_0 \in \mathbb{N}$. Thus, $f(G)$ is open and $(II) \Rightarrow (I)$ holds for f. ∎

For a paranormed space $(X, \|\cdot\|)$ and $r > 0$ let $U_r = \{x \in X : \|x\| < r\}$ and $c_0(X) = \{(x_n)_1^\infty \in X^{\mathbb{N}} : \|x_n\| \to 0\}$.

Corollary 2.2. *Let X, Y be paranormed spaces and $f : X \to Y$ a mapping such that $\forall\ x \in X$ and $\varepsilon > 0\ \exists\ \delta > 0$ satisfies that $f(x) + f(U_\delta) \subset f(x + U_\varepsilon)$. If for each $(y_n) \in c_0(Y)$ there exist positive integers $n_1 < n_2 < \cdots$ such that $y_{n_k} \in f(U_{1/k})$ for each k, then f is an open mapping.*

Proof. Let $y = f(x) \in f(X)$ and $y_n \to y$ in Y. Then $y_n - y \to 0$ and so there exist integers $n_1 < n_2 < \cdots$ such that $y_{n_k} - f(x) = y_{n_k} - y \in f(U_{1/k})$, $\forall k \in \mathbb{N}$, $i.e.$, $y_{n_k} \in f(x) + f(U_{1/k})$, $k = 1, 2, 3, \cdots$.

It follows from the conditions and $f(U_\gamma) \subset f(U_\delta)$ whenever $0 < \gamma < \delta$ that there exist integers $k_1 < k_2 < \cdots$ such that $f(x)+f(U_{1/k_i}) \subset f(x+U_{1/i})$ for each $i \in \mathbb{N}$, $i.e.$, $y_{n_{k_i}} \in f(x + U_{1/i})$, $\forall\ i \in \mathbb{N}$. Then $y_{n_{k_i}} = f(x + u_i)$ where $u_i \in U_{1/i}$, $\|u_i\| < 1/i$, $i = 1, 2, 3, \cdots$. Thus, $x + u_i \to x$ and $(f(x + u_i)) = (y_{n_{k_i}}) \subset (y_n)$.

Thus, by Theorem 2.1, f is an open mapping. ∎

3. An Application of Theorem 2.1

If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a nonconstant complex polynomial, then f is an open mapping from $\mathbb{C}$ into $\mathbb{C}$. This fact implies the fundamental theorem of algebra. The usual proof of this open mapping theorem is considerably complicated. However, this important fact is a convenient consequence of Theorem 2.1.

Theorem 3.1. *Every nonconstant analytic function $f : \mathbb{C} \to \mathbb{C}$ is an open mapping.*

Proof. Let $z_0 \in \mathbb{C}$, $y_0 = f(z_0)$. Suppose $y_n \to y_0$ in $\mathbb{C}$. There is an $r > 0$ such that $f(z) \neq y_0$ whenever $0 < |z - z_0| \leq r$.

Let $r_1 = \min_{|z|=r} |f(z) - y_0|$. Then $r_1 > 0$ and so there is an $n_0 \in \mathbb{N}$ for which $|y_n - y_0| < r_1$ whenever $n > n_0$. Fix an arbitrary $n > n_0$ and let $F(z) = f(z) - y_0$, $G(z) = f(z) - y_n$. Then

$$|G(z) - F(z)| = |y_0 - y_n| < r_1 \leq |F(z)|,\ \forall\ |z| = r.$$

By the Rouché theorem, there is a z_n for which $|z_n - z_0| < r$ and $G(z_n) = f(z_n) - y_n = 0$, that is, $y_n = f(z_n)$. Thus, we have a sequence $(z_n)_{n>n_0} \subset \{z \in \mathbb{C} : |z - z_0| < r\}$ such that $f(z_n) = y_n$ for each $n > n_0$. But $(z_n)_{n>n_0}$ has a subsequence (z_{n_k}) such that $z_{n_k} \to a \in \{z \in \mathbb{C} : |z - z_0| \leq r\}$ and so $y_{n_k} = f(z_{n_k}) \to f(a)$. Since $y_{n_k} \to y_0 = f(z_0)$, $f(a) = f(z_0) = y_0$ and, moreover, $a = z_0$ by the property of $r > 0$. Thus, $z_{n_k} \to z_0$.

By Theorem 2.1, $f : \mathbb{C} \to \mathbb{C}$ is open. ∎

References

1. S. Banach, *Théorie Des Opérations Linéaires.* Warsaw, 1932.
2. V. Ptak, Completeness and the Open Mapping Theorem. *Bull. Soc. Math. France.*, **86** (1958), 41-74.

3. A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, 1978.

4. T. Husain, *The Open Mapping and Closed Graph Theorems in Topological Vector Spaces*, Oxford, 1965.

5. N. Adasch, Tonnelierte Räume und zwei Sätze von Banach, *Math. Ann.*, **186** (1970), 209-214.

6. J. H. Qiu, General Open Mapping Theorems for Linear Topological Spaces. *Acta Mathematica Scientia*, **18** (1998), 241-248.

7. E. Beckenstein and L. Narici, An Open Mapping Theorem for Basis Separating Maps., *Topological and its Applications*, **137** (2004), 39-50.

8. R. L. Li, S. H. Zhong and C. R. Cui, New Basic Principles of Functional Analysis, *J. Yanbian University*, **30** (2004), 158-160.

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