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HYBRID PROXIMAL POINT ALGORITHMS FOR SOLVING CONSTRAINED MINIMIZATION PROBLEMS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce two hybrid proximal point algorithms to solve the constrained minimization problem for a convex functional in a uniformly convex and uniformly smooth Banach space. Using those iterative schemes, we establish the strong convergence theorems for relatively nonexpansive mappings which generalize the recent results in the literature.

1. INTRODUCTION

Let H be a real Hilbert space and let $T : H \to 2^H$ be a maximal monotone operator. The problem of finding an element $x \in H$ such that $0 \in Tx$ is very important in the area of optimization and related fields. One well-known method of solving $0 \in Tx$ is the *proximal point algorithm* which was first introduced by Martinet [12] and generally studied by Rockafellar [17] in the framework of a Hilbert space. This proximal point algorithm generates a sequence $\{x_n\}$ in H by the iterative scheme

$$\begin{aligned} &x_0 \in H, \\ &x_{n+1} = (I + r_n T)^{-1} x_n, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1}$$

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where $\{r_n\}$ is a sequence in the interval $(0,\infty)$ and I denotes the identity operator on H. Algorithm (1) is equivalent to

$$\begin{aligned} &x_0 \in H,\\ &0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1}-x_n), \quad n = 0,1,2,\ldots. \end{aligned}$$

Many results for the convergence of (1) in a Hilbert space or a Banach space have been extensively studied; see [5, 6, 9, 21] and the references therein. Rockafellar [17] proved that if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence generated by (1) converges weakly to an element of $T^{-1}0$. He also posed an open question of whether or not the sequence generated by (1) converges strongly to an element of $T^{-1}0$. This problem was solved by Güler [9], who presented an example for which the sequence generated by (1) converges weakly but not strongly. On the other hand, Kamimura and Takahashi [10] and Solodov and Svaiter [19] modified this proximal point algorithm to generate a strongly convergent sequence in a Hilbert space. Moreover, Kamimura and Takahashi [11] introduced a proximal-type algorithm in a uniformly convex and uniformly smooth Banach space E and derived a strong convergence theorem which extends Solodov and Svaiter's result [19] to the setting of Banach spaces.

Let E be a real Banach space with dual E^*. A multifunction $T : E \to 2^{E^*}$ is *monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ whenever $x_i \in E$ and $y_i \in Tx_i$, $i = 1,2$. A monotone operator T is *maximal* if its graph is not properly contained in the graph of any other monotone operator. An extended real-valued function $f : E \to (-\infty,\infty]$ is a *proper convex* function if it is not identically $+\infty$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in E$ and $\lambda \in (0,1)$. A *subdifferential* of f at x is the multifunction $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^*\rangle, \quad \text{for all } y \in E\}.$$

Then $0 \in \partial f(x)$ is equivalent to $f(x) = \min\{f(z) : z \in E\}$. Rockafellar [18] proved that if $f : E \to (-\infty,\infty]$ is a lower semicontinuous proper convex function, then ∂f is a maximal monotone operator. In [3], Alber and Yao considered the minimization problems for nonsmooth convex functionals in a Banach space E by using the following non-traditional algorithm:

$$\begin{aligned} &x_0 \in E,\\ &x_{n+1} = \Pi_C(x_n - \lambda_n J^{-1}\partial f(x_{n+1})), \quad n = 0,1,2,\ldots, \end{aligned} \tag{2}$$

where $\lambda_n \geq \bar{\lambda} = \text{const.} > 0$ and x_{n+1} is assumed to exist for every x_n. This algorithm can be written in an equivalent form:

$$\begin{aligned} &x_0 \in E, \\ &0 \in -\lambda_n J^{-1} \partial f(\Pi_C y_n) + y_n - x_n, \\ &x_{n+1} = \Pi_C y_n, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

In particular, suppose that E be a uniformly convex and uniformly smooth Banach space, the minimizer set of the convex functional $f(x)$ is nonempty, the operator ∂f is bounded and the set $\{x \in E : f(x) \leq f(x_1)\}$ is bounded. Then Alber and Yao [3, Theorem 2.3] proved that the sequence $\{f(x_n)\}$ converges to the minimum f^* of $f(x)$, where $\{x_n\}$ is generated by (2).

Motivated by the recent work in [3, 11], we introduce two hybrid proximal point algorithms (see (4) and (25) in §3) in a uniformly convex and uniformly smooth Banach space E. Suppose that C is a nonempty closed convex subset of E, $f : E \to [0, \infty)$ is a lower semicontinuous proper convex function and $S : E \to E$ is a *relatively nonexpansive mapping* (see §2 for the definition) such that $(\partial f)^{-1}0 \cap F(S) \neq \emptyset$, where $F(S)$ is the fixed point set of S. The purpose of this paper is to prove that under certain conditions, each of the sequences $\{x_n\}$ generated by (4) and (25) converges strongly to the point $\Pi_{(\partial f)^{-1}0 \cap F(S)} x_0$. These strong convergence theorems of finding a minimizer of a convex functional in a uniformly convex and uniformly smooth Banach space generalize the results in [3, 11, 15].

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2. PRELIMINARIES

Let E be a real Banach space with dual E^* and let $S_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

We say that E is *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for all $x, y \in S_E$; if the limit (3) exists and is attained uniformly in $x, y \in S_E$, E is said to be *uniformly smooth*. It is well known that if E is smooth, then the duality mapping J is single-valued. We still denote the single-valued duality mapping by J. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E.

A Banach space E is *strictly convex* if

$$\left\|\frac{x+y}{2}\right\| < 1, \quad \text{for } x, y \in S_E \text{ and } x \neq y.$$

It is said to be *uniformly convex* if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta, \quad \text{for } x, y \in S_E \text{ and } \|x-y\| \geq \epsilon.$$

If E is uniformly convex, then E has the *Kadec-Klee property* [8, 20], that is, for any sequence $\{x_n\}$ in E which converges weakly to $x \in E$ and $\|x_n\| \to \|x\|$, we have $\{x_n\}$ converges strongly to x. All uniformly smooth or uniformly convex Banach spaces are reflexive.

Recall that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact characterizes Hilbert spaces and it is not available in general Banach spaces.

Let E be a smooth Banach space and define a *Lyapunov* function $\phi : E \times E \to \mathbf{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in E.$$

Suppose that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. Then (see [2]) for each $x \in E$, there exists a unique element $x_0 \in C$, denoted by $\Pi_C x$, such that

$$\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}.$$

The mapping Π_C is called the *generalized projection* from E onto C which was introduced by Alber [1] in 1994. If E is a Hilbert space, then $\phi(x, y) = \|x-y\|^2$, for all $x, y \in E$, and so Π_C is coincident with the metric projection P_C.

Let C be a closed convex subset of a Banach space E and let S be a mapping from C into itself. A point p of C is called an *asymptotic fixed point* of S [16] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Let the set of asymptotic fixed points of S be denoted by $\hat{F}(S)$. Then S is said to be *relatively nonexpansive* [4, 7, 13] if $\hat{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$.

We will need the following lemmas.

Lemma 2.1. (Kamimura and Takahashi [11]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2. (Alber [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle y - x_0, Jx - Jx_0 \rangle \le 0$, for all $y \in C$.*

Lemma 2.3. (Alber [1]). *Let E be a smooth, strictly convex and reflexive Banach space, and let C be a nonempty closed convex subset of E. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \text{for all } x \in E, y \in C.$$

Lemma 2.4. (Matsushita and Takahashi [14]). *Let E be a smooth and strictly convex Banach space, let C be a closed convex subset of E, and let S a relatively nonexpansive mapping from C into itself. Then $F(S)$ is closed and convex.*

3. Strong Convergence Theorems

In this section we will establish the strong convergence theorems for relatively nonexpansive mappings by using our algorithms. The first hybrid proximal point algorithm is defined as follows:

$$\begin{aligned} x_0 &\in E, \\ \tilde{x}_n &= \Pi_C J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n)), \\ z_n &= J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n), \\ y_n &= J^{-1}(\alpha_n J\tilde{x}_n + (1-\alpha_n) JSz_n), \\ H_n &= \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, \tilde{x}_n) \\ &\qquad + (1-\alpha_n)\phi(v, z_n), \langle v - \tilde{x}_n, \partial f(\tilde{x}_n) \rangle \le 0\}, \\ W_n &= \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \le 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{aligned} \tag{4}$$

where $\{\lambda_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$, and $\tilde{x}_n$ is assumed to exist for every x_n.

We investigate the conditions under which the algorithm (4) is well defined and obtain the following result.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space, $f : E \to (-\infty, \infty]$ a lower semicontinuous proper convex function and $S : C \to C$ is a relatively nonexpansive mapping. Suppose that $\tilde{x}_n$ exists for any x_n in (4). If $(\partial f)^{-1}0 \cap F(S) \ne \emptyset$, then the sequence $\{x_n\}$ generated by (4) is well defined.*

Proof. It is seen that each W_n is closed and convex. To prove that for each $n \geq 0$, H_n is a closed convex set, let C_n and D_n be two subsets of C defined by

$$C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, \tilde{x}_n) + (1 - \alpha_n)\phi(v, z_n)\}$$

and

$$D_n = \{v \in C : \langle v - \tilde{x}_n, \partial f(\tilde{x}_n)\rangle \leq 0\};$$

so that $H_n = C_n \cap D_n$, C_n is closed and D_n is closed and convex. Hence H_n is closed. Since

$$\phi(v, y_n) \leq \alpha_n \phi(v, \tilde{x}_n) + (1 - \alpha_n)\phi(v, z_n)$$

is equivalent to

$$\begin{aligned} &2\alpha_n \langle v, J\tilde{x}_n\rangle + 2(1 - \alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \\ &\leq \alpha_n \|\tilde{x}_n\|^2 + (1 - \alpha_n)\|z_n\|^2 - \|y_n\|^2, \end{aligned}$$

it follows that C_n is convex and so is H_n. Therefore H_n is closed and convex.

Next we claim that $F(S) \subset C_n$ for all $n \geq 0$. Let $w \in F(S)$. Since S is relatively nonexpansive, we have

$$\phi(w, Sz_n) \leq \phi(w, z_n), \quad \text{for all } n \geq 0,$$

and so it follows from (4) that

$$\begin{aligned} \phi(w, y_n) &= \|w\|^2 - 2\langle w, \alpha_n J\tilde{x}_n + (1 - \alpha_n)JSz_n\rangle \\ &\quad + \|\alpha_n J\tilde{x}_n + (1 - \alpha_n)JSz_n\|^2 \\ &\leq \|w\|^2 - 2\alpha_n\langle w, J\tilde{x}_n\rangle - 2(1 - \alpha_n)\langle w, JSz_n\rangle \\ &\quad + \alpha_n\|\tilde{x}_n\|^2 + (1 - \alpha_n)\|Sz_n\|^2 \\ &= \alpha_n \phi(w, \tilde{x}_n) + (1 - \alpha_n)\phi(w, Sz_n) \\ &\leq \alpha_n \phi(w, \tilde{x}_n) + (1 - \alpha_n)\phi(w, z_n), \end{aligned}$$

for all $n \geq 0$. So $w \in C_n$ for all $n \geq 0$. This asserts that

$$F(S) \subset C_n, \quad \text{for all } n \geq 0. \tag{5}$$

We will use the mathematical induction to verify that

$$(\partial f)^{-1}0 \cap F(S) \subset H_n \cap W_n, \quad \text{for all } n \geq 0; \tag{6}$$

hence $\{x_n\}$ generated by (4) is well defined. Take $u \in (\partial f)^{-1}0 \cap F(S)$ arbitrarily. Then (5) implies that $u \in C_n$ for all $n \geq 0$. By hypothesis given any $x_0 \in E$ there exists $\tilde{x}_0 \in C$ such that $\tilde{x}_0 = \Pi_C J^{-1}(Jx_0 - \lambda_0 \partial f(\tilde{x}_0))$. Since ∂f is monotone, we obtain

$$\langle \tilde{x}_0 - u, \partial f(\tilde{x}_0)\rangle \geq 0,$$

which implies that $u \in D_0$ and hence $u \in H_0$. It is of course $u \in W_0 = C$. So $u \in H_0 \cap W_0$ and $x_1 = \Pi_{H_0 \cap W_0} x_0$ is well defined. Suppose that $u \in H_{n-1} \cap W_{n-1}$ for some $n \geq 2$. Then $x_n = \Pi_{H_{n-1} \cap W_{n-1}} x_0$ is well defined. Again, by hypothesis there exists $\tilde{x}_n \in C$ such that $\tilde{x}_n = \Pi_C J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n))$. The monotonicity of ∂f implies that

$$\langle \tilde{x}_n - u, \partial f(\tilde{x}_n)\rangle \geq 0;$$

hence $u \in D_n$ and so $u \in H_n$. Since $x_n = \Pi_{H_{n-1} \cap W_{n-1}} x_0$, it follows from Lemma 2.2 that

$$\langle u - x_n, Jx_0 - Jx_n\rangle \leq 0,$$

and therefore $u \in W_n$. We conclude that $u \in H_n \cap W_n$ and $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ is well defined. ∎

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, $f : E \to (-\infty, \infty]$ a lower semicontinuous proper convex function and $S : C \to C$ a relatively nonexpansive and uniformly continuous mapping such that $(\partial f)^{-1}0 \cap F(S) \neq \emptyset$. Suppose that $\tilde{x}_n$ exists for any x_n in (4), $\{\lambda_n\}_{n=0}^{\infty} \subset [a, \infty)$ for some $a > 0$, and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \beta_n = 1$. Then the following hold:*

(i) *The sequence $\{x_n\}$ generated by (4) converges strongly to $\Pi_{(\partial f)^{-1}0 \cap F(S)} x_0$.*

(ii) *If ∂f is bounded, then $\{f(x_n)\}$ converges to the minimum f^* of f.*

Proof. The proof of conclusion (i) is divided into five steps.

Step 1. We first prove that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0, \tag{7}$$

and therefore by Lemma 2.1,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{8}$$

It follows from the definition of W_n and Lemma 2.2 that

$$x_n = \Pi_{W_n} x_0, \quad \text{for all } n \geq 0. \tag{9}$$

Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in W_n$, it follows that

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \text{for all } n \ge 0.$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. ¿From $x_n = \Pi_{W_n} x_0$ and Lemma 2.3 we have

$$\phi(x_n, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0),$$

for $w \in (\partial f)^{-1}0 \cap F(S) \subset W_n$ and for $n \ge 0$. This shows that $\{\phi(x_n, x_0)\}$ is bounded and so converges. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \le \phi(x_n, x_0) \le (\|x_n\| + \|x_0\|)^2,$$

the sequence $\{x_n\}$ is bounded. We use Lemma 2.3 and (9) to derive that

$$\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x_0) - \phi(x_n, x_0), \quad \text{for all } n \ge 0.$$

Since $\{\phi(x_n, x_0)\}$ converges, we have $\phi(x_{n+1}, x_n) \to 0$.

Step 2. Claim that

$$\lim_{n\to\infty} \phi(\tilde{x}_n, x_n) = 0$$

and therefore by Lemma 2.1

$$\lim_{n\to\infty} \|\tilde{x}_n - x_n\| = 0. \tag{10}$$

It suffices to prove that

$$\phi(x_{n+1}, x_n) \ge \phi(\Pi_{H_n} x_n, x_n) \ge \phi(\tilde{x}_n, x_n), \tag{11}$$

which implies that $\phi(\tilde{x}_n, x_n) \to 0$ by (7). The first inequality in (11) holds because $x_{n+1} \in H_n$. We now prove the second inequality in (11). Since $\|\tilde{x}_n\|^2 = \langle \tilde{x}_n, J\tilde{x}_n \rangle$, it follows that

$$\begin{aligned} \|\Pi_{H_n} x_n\|^2 - \|\tilde{x}_n\|^2 &\ge 2\langle \Pi_{H_n} x_n, J\tilde{x}_n \rangle - 2\|\tilde{x}_n\|^2 \\ &\ge -2\langle \tilde{x}_n - \Pi_{H_n} x_n, J\tilde{x}_n \rangle, \end{aligned}$$

and so we have

$$\begin{aligned} &\phi(\Pi_{H_n} x_n, x_n) - \phi(\tilde{x}_n, x_n) \\ &= \|\Pi_{H_n} x_n\|^2 - \|\tilde{x}_n\|^2 + 2\langle \tilde{x}_n - \Pi_{H_n} x_n, Jx_n \rangle \\ &\ge 2\langle \tilde{x}_n - \Pi_{H_n} x_n, Jx_n - J\tilde{x}_n \rangle \\ &= 2\lambda_n \langle \tilde{x}_n - \Pi_{H_n} x_n, \partial f(\tilde{x}_n) \rangle \\ &\quad + 2\langle \tilde{x}_n - \Pi_{H_n} x_n, Jx_n - J\tilde{x}_n - \lambda_n \partial f(\tilde{x}_n) \rangle. \end{aligned} \tag{12}$$

Since $\Pi_{H_n} x_n \in H_n$, we obtain

$$\langle \tilde{x}_n - \Pi_{H_n} x_n, \partial f(\tilde{x}_n) \rangle \geq 0,$$

and since $\tilde{x}_n = \Pi_C J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n))$, Lemma 2.2 asserts that

$$\langle \tilde{x}_n - \Pi_{H_n} x_n, Jx_n - \lambda_n \partial f(\tilde{x}_n) - J\tilde{x}_n \rangle \geq 0.$$

Therefore (12) yields $\phi(\Pi_{H_n} x_n, x_n) \geq \phi(\tilde{x}_n, x_n)$ and (11) holds. Further, it follows immediately from (10) and the boundedness of $\{x_n\}$ that $\{\tilde{x}_n\}$ is also bounded.

Step 3. Observe that

$$\lim_{n\to\infty} \phi(x_{n+1}, \tilde{x}_n) = \lim_{n\to\infty} \phi(x_{n+1}, z_n) = \lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0 \tag{13}$$

and therefore

$$\lim_{n\to\infty} \|x_{n+1} - \tilde{x}_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{14}$$

In fact, we have

$$\begin{aligned} \phi(x_{n+1}, \tilde{x}_n) &= \langle x_{n+1}, Jx_{n+1} - J\tilde{x}_n \rangle + \langle \tilde{x}_n - x_{n+1}, J\tilde{x}_n \rangle \\ &\leq \|x_{n+1}\| \|Jx_{n+1} - J\tilde{x}_n\| + \|\tilde{x}_n - x_{n+1}\| \|\tilde{x}_n\| \\ &\leq \|x_{n+1}\| (\|Jx_{n+1} - Jx_n\| + \|Jx_n - J\tilde{x}_n\|) \\ &\quad + (\|\tilde{x}_n - x_n\| + \|x_n - x_{n+1}\|) \|\tilde{x}_n\|. \end{aligned} \tag{15}$$

By (8), (10) and the uniform norm-to-norm continuity of J on bounded subsets of E, it follows that $J\tilde{x}_n - Jx_n \to 0$ and $Jx_{n+1} - Jx_n \to 0$. Since $\{x_n\}$ and $\{\tilde{x}_n\}$ are bounded, (15) shows that $\phi(x_{n+1}, \tilde{x}_n) \to 0$.

Again, since $\{\tilde{x}_n\}$ is bounded and $\phi(p, S\tilde{x}_n) \leq \phi(p, \tilde{x}_n)$ where $p \in F(S)$, we obtain that $\{S\tilde{x}_n\}$ is also bounded. It follows that

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n \rangle \\ &\quad + \|\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\beta_n \langle x_{n+1}, J\tilde{x}_n \rangle - 2(1-\beta_n)\langle x_{n+1}, JS\tilde{x}_n \rangle \\ &\quad + \beta_n \|\tilde{x}_n\|^2 + (1-\beta_n)\|S\tilde{x}_n\|^2 \\ &= \beta_n \phi(x_{n+1}, \tilde{x}_n) + (1-\beta_n)\phi(x_{n+1}, S\tilde{x}_n) \to 0, \quad \text{as } \beta_n \to 1. \end{aligned}$$

Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, \tilde{x}_n) + (1-\alpha_n)\phi(x_{n+1}, z_n) \to 0 \quad \text{as } n \to \infty.$$

Consequently (13) holds.

Step 4. Claim that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{16}$$

We first prove that

$$\lim_{n\to\infty} \|x_{n+1} - Sz_n\| = 0. \tag{17}$$

Since

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|,$$

it follows from (8) and (14) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{18}$$

Using

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|\alpha_n(Jx_{n+1} - J\tilde{x}_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\ &\ge (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n\|J\tilde{x}_n - Jx_{n+1}\|, \end{aligned}$$

we have

$$\|Jx_{n+1} - JSz_n\| \le \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|J\tilde{x}_n - Jx_{n+1}\|). \tag{19}$$

Since $\limsup_{n\to\infty} \alpha_n < 1$ and J is uniformly norm-to-norm continuous on bounded subsets of E, it follows from (14) and (19) that $\lim_{n\to\infty} \|Jx_{n+1} - JSz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain $\lim_{n\to\infty} \|x_{n+1} - Sz_n\| = 0$. Since S is uniformly continuous, by (8), (17) and (18)

$$\|x_n - Sx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 5. We now prove that $\{x_n\}$ converges strongly to $\Pi_{(\partial f)^{-1}0\cap F(S)}x_0$. Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ which converges weakly to $\hat{x} \in C$. Then by (16), $\hat{x}$ is an asymptotic fixed point of S. Since S is relatively nonexpansive, $\hat{x} \in F(S)$.

To prove that $\hat{x} \in (\partial f)^{-1}0$, let

$$u_n = J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n))$$

so that $\tilde{x}_n = \Pi_C u_n$ and

$$\partial f(\tilde{x}_n) = -\frac{1}{\lambda_n}(Jy_n - Jx_n).$$

Since $(\partial f)^{-1}0\cap F(S)\neq\emptyset$ and f^* is the minimum of f, there exists $w\in(\partial f)^{-1}0\cap F(S)$ such that $f(w)=f^*$ and hence

$$\begin{aligned} f(\tilde{x}_n)-f(w) &\leq \langle \tilde{x}_n-w,\partial f(\tilde{x}_n)\rangle \\ &= -\frac{1}{\lambda_n}\langle \tilde{x}_n-w, Jy_n-Jx_n\rangle \\ &= -\frac{1}{\lambda_n}\langle \tilde{x}_n-w, Jy_n-J\tilde{x}_n\rangle-\frac{1}{\lambda_n}\langle \tilde{x}_n-w, J\tilde{x}_n-Jx_n\rangle. \end{aligned} \tag{20}$$

Since $\tilde{x}_n=\Pi_C y_n$ and $w\in(\partial f)^{-1}0\cap F(S)\subset C$, by Lemma 2.2 we have

$$\langle \tilde{x}_n-w, Jy_n-J\tilde{x}_n\rangle\geq 0.$$

It follows from (20) that

$$\begin{aligned} f(\tilde{x}_n)-f^* &\leq -\frac{1}{\lambda_n}\langle \tilde{x}_n-w, J\tilde{x}_n-Jx_n\rangle \\ &\leq \frac{1}{a}\|\tilde{x}_n-w\|\|J\tilde{x}_n-Jx_n\|. \end{aligned}$$

Since f is convex and lower semicontinuous, it is weakly lower semicontinuous. Thus from $\tilde{x}_{n_i}\rightharpoonup\hat{x}$ and $J\tilde{x}_n-Jx_n\to 0$ we have

$$\begin{aligned} 0 &\leq f(\hat{x})-f^* \\ &\leq \liminf_{i\to\infty}[f(\tilde{x}_{n_i})-f(w)] \\ &\leq \limsup_{i\to\infty}[f(\tilde{x}_{n_i})-f(w)] \\ &\leq \limsup_{i\to\infty}\frac{1}{a}\|\tilde{x}_{n_i}-w\|\|J\tilde{x}_{n_i}-Jx_{n_i}\|=0. \end{aligned}$$

This implies that

$$\lim_{i\to\infty} f(\tilde{x}_{n_i})=f(\hat{x})=f^*, \tag{21}$$

and so $\hat{x}\in(\partial f)^{-1}0$. Therefore $\hat{x}\in(\partial f)^{-1}0\cap F(S)$.

Next we shall prove that $\{x_{n_i}\}$ converges strongly to $\hat{x}$ and $\hat{x}=\Pi_{(\partial f)^{-1}0\cap F(S)}x_0$. Let $\bar{x}=\Pi_{(\partial f)^{-1}0\cap F(S)}x_0$. Since $x_{n+1}=\Pi_{H_n\cap W_n}x_0$ and $\bar{x}\in(\partial f)^{-1}0\cap F(S)\subset H_n\cap W_n$, we have $\phi(x_{n+1},x_0)\leq\phi(\bar{x},x_0)$. On the other hand from the weak lower semicontinuity of the norm, we have

$$\begin{aligned} \phi(\hat{x},x_0) &= \|\hat{x}\|^2-2\langle\hat{x},Jx_0\rangle+\|x_0\|^2 \\ &\leq \liminf_{i\to\infty}(\|x_{n_i}\|^2-2\langle x_{n_i},Jx_0\rangle+\|x_0\|^2) \\ &= \liminf_{i\to\infty}\phi(x_{n_i},x_0) \\ &\leq \limsup_{i\to\infty}\phi(x_{n_i},x_0) \\ &\leq \phi(\bar{x},x_0) \end{aligned} \tag{22}$$

which shows that $\hat{x} = \bar{x}$ and hence

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\bar{x}, x_0).$$

Therefore $\lim_{i\to\infty} \|x_{n_i}\| = \|\bar{x}\|$. Applying the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof of conclusion (i).

To prove conclusion (ii), suppose that the operator ∂f is bounded. Then $\{\partial f(x_n)\}$ is bounded. Since $\{x_n\}$ converges strongly to $\bar{x}$, it follows from (21) that

$$\lim_{n\to\infty} f(\tilde{x}_n) = f(\bar{x}) = f^*. \tag{23}$$

Observe that for all $n \geq 0$

$$\begin{aligned} f(x_n) &\leq f(\tilde{x}_n) + \langle x_n - \tilde{x}_n, \partial f(x_n)\rangle \\ &\leq f(\tilde{x}_n) + \|x_n - \tilde{x}_n\|\|\partial f(x_n)\|. \end{aligned} \tag{24}$$

Since f is lower semicontinuous, by (10) and (23) we obtain

$$f^* = f(\bar{x}) \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} f(\tilde{x}_n) = f^*.$$

Hence

$$\lim_{n\to\infty} f(x_n) = f(\bar{x}) = f^*. \qquad \blacksquare$$

In Theorem 3.2, if $f(x) = 0$ for all $x \in E$, then $\partial f(x) = 0$ for all $x \in E$ and hence $F(S) = (\partial f)^{-1}0 \cap F(S) \neq \emptyset$. So we have

$$\tilde{x}_n = \Pi_C J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n)) = \Pi_C x_n = \begin{cases} \Pi_C x_0, & \text{if } n = 0, \\ x_n, & \text{if } n \geq 1. \end{cases}$$

This is the case of Theorem 2.1 in [15].

Next we use the second algorithm to establish our main result which includes [15, Theorem 2.2] as a special case.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, $f : E \to (-\infty, \infty]$ a lower semicontinuous proper convex function and $S : C \to C$ a relatively nonexpansive and uniformly continuous mapping such that*

$(\partial f)^{-1}0 \cap F(S) \neq \emptyset$. *Define a sequence* $\{x_n\}$ *by the following algorithm*

$$\begin{aligned} x_0 &\in E, \\ \tilde{x}_n &= \Pi_C J^{-1}(Jx_n - \lambda_n \partial f(\tilde{x}_n)), \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\tilde{x}_n), \\ H_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) \\ &\qquad +(1-\alpha_n)\phi(v, \tilde{x}_n), \langle v - \tilde{x}_n, \partial f(\tilde{x}_n)\rangle \leq 0\}, \\ W_n &= \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{25}$$

where $\{\lambda_n\}_{n=0}^{\infty} \subset [a, \infty)$ *for some* $a > 0$, *and* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$. *Suppose that* $\tilde{x}_n$ *exists for any* x_n *in (25) and* $\lim_{n\to\infty} \alpha_n = 0$. *Then the following hold:*

(i) *The sequence* $\{x_n\}$ *generated by* (25) *converges strongly to* $\Pi_{(\partial f)^{-1}0 \cap F(S)} x_0$.

(ii) *If* ∂f *is bounded, then* $\{f(x_n)\}$ *converges to the minimum* f^* *of* f.

Proof. By applying the same arguments as in the proof of Lemma 3.1, all the sets H_n and W_n, $n \geq 0$, are closed and convex and

$$(\partial f)^{-1}0 \cap F(S) \subset H_n \cap W_n, \quad \text{for all } n \geq 0;$$

hence $\{x_n\}$ generated by (25) is well defined. The rest of the proof are the same as that of Theorem 3.2. First observe that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = \lim_{n\to\infty} \phi(x_n, \tilde{x}_n) = 0 = \lim_{n\to\infty} \phi(x_{n+1}, \tilde{x}_n) = 0 \tag{26}$$

and the sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ are bounded. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \tilde{x}_n).$$

It follows from (26) and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{27}$$

Therefore (26) and (27) imply that

$$\begin{aligned} &\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - \tilde{x}_n\| \\ &= \lim_{n\to\infty} \|x_{n+1} - \tilde{x}_n\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \end{aligned} \tag{28}$$

The inequality

$$\|Jx_{n+1} - Jy_n\| = \|Jx_{n+1} - [\alpha_n Jx_0 + (1-\alpha_n)JS\tilde{x}_n]\|$$
$$\geq (1-\alpha_n)\|Jx_{n+1} - JS\tilde{x}_n]\| - \alpha_n\|Jx_{n+1} - Jx_0\|$$

yields

$$\|Jx_{n+1} - JS\tilde{x}_n\| \leq \frac{1}{1-\alpha_n}\left[\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_{n+1} - Jx_0\|\right].$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and J is uniformly norm-to-norm continuous on bounded subsets of E, the previous inequality and (28) imply that

$$\lim_{n\to\infty}\|Jx_{n+1} - JS\tilde{x}_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - S\tilde{x}_n\| = 0. \tag{29}$$

Now

$$\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - S\tilde{x}_n\| + \|S\tilde{x}_n - Sx_n\|.$$

Therefore by uniform continuity of S, (28) and (29),

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{30}$$

To prove $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{(\partial f)^{-1}0\cap F(S)}x_0$, we see that if $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ which converges weakly to a point $\hat{x}\in E$, then $\hat{x}\in F(S)$ by (30). Since $(\partial f)^{-1}0\cap F(S)\neq\emptyset$ and f^* is the minimum of f, there exists $w\in(\partial f)^{-1}0\cap F(S)$ such that $f(w) = f^*$. Hence (20) still holds in this case. Since f is convex and lower semicontinuous, f is weakly lower semicontinuous. Therefore we have the inequality

$$0 \leq f(\hat{x}) - f^* \leq \limsup_{i\to\infty}\frac{1}{a}\|\tilde{x}_{n_i} - w\|\|J\tilde{x}_{n_i} - Jx_{n_i}\| = 0,$$

because $\lim_{n\to\infty}\|J\tilde{x}_n - Jx_n\| = 0$. This shows that

$$\lim_{i\to\infty} f(\tilde{x}_{n_i}) = f(\hat{x}) = f^*, \tag{31}$$

and so $\hat{x}\in(\partial f)^{-1}0$. Hence $\hat{x}\in(\partial f)^{-1}0\cap F(S)$. By (22) and the Kadec-Klee property of E, $\hat{x} = \bar{x}$ and $\{x_n\}$ converges strongly to $\bar{x}$. The proof of (i) is complete.

Next, suppose that ∂f is bounded. Since $\{x_n\}$ converges strongly to $\bar{x}$, it follows from (31) that

$$\lim_{n\to\infty} f(\tilde{x}_n) = f(\bar{x}) = f^*.$$

Since f is lower semicontinuous, by (23), () and (28) we then have

$$f^* = f(\bar{x}) \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} f(\tilde{x}_n) = f^*.$$

Hence

$$\lim_{n\to\infty} f(x_n) = f(\bar{x}) = f^*.$$

Conclusion (ii) holds. ■

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