TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 2B, pp. 777-788, April 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

AN EXTENDED GAUSS-SEIDEL METHOD FOR MULTI-VALUED MIXED COMPLEMENTARITY PROBLEMS

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Abstract. The complementarity problem (CP) is one of the basic topics in nonlinear analysis. Since the constraint set of CP is a convex cone or a cone segment, weak order monotonicity properties can be utilized for its analysis instead of the usual norm monotonicity ones. Such nonlinear CPs with order monotonicity properties have a great number of applications, especially in economics and mathematical physics. Most solution methods were developed for the single-valued case, but this assumption seems too restrictive in many applications. In the paper, we consider extended concepts of multi-valued Z-mappings and examine a class of generalized mixed complementarity problems (MCPs) with box constraints, whose cost mapping is a general composition of multi-valued mappings possessing Z type properties. We develop a Gauss-Seidel algorithm for these MCPs. Some examples of computational experiments are also given.

1. Introduction and Preliminaries

Together with optimization and fixed point problems, the complementarity problem is one of the basic problems in Nonlinear Analysis and its theory, methods and applications are well documented in literature; see, e.g. [2]-[3] and references therein. We recall that the classical *complementarity problem* (CP) consists in finding a point $x^* \in R^n$ such that

$$x^* \geq 0, \quad f(x^*) \geq 0, \quad \langle x^*, f(x^*)\rangle = 0, \tag{1}$$

Received September 22, 2009.
2000 *Mathematics Subject Classification*: 90C90.
Key words and phrases: Mixed complementarity problem, Multi-valued mappings, Z-mappings, Gauss-Seidel algorithm.
In this work, the authors were supported by Brescia University grant PRIN - 2006: "Oligopolistic models and order monotonicity properties", the third author was also supported by the joint RFBR-NNSF grant, project No. 07-01-92101.

where $f : R^n \to R^n$ is a given single-valued mapping. Here and below, $\langle \cdot, \cdot \rangle$ denotes the scalar product in R^n and all the inequalities for vectors are componentwise, i.e., $x \geq y$ means that $x_i \geq y_i$ for $i = 1, \ldots, n$ if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Most results in theory and solution methods of complementarity problems are traditionally devoted to the classical ones with single-valued or even affine mappings. At the same time, many problems arising in applications involve multi-valued mappings and may contain additionally box type constraints instead of non-negativity of variables; see e.g. [2, 3] and references therein. The corresponding generalization of problem (1) can be defined as follows. We are given the box-constrained set

$$D = \{x \in R^n \mid -\infty < a_i \leq x_i \leq b_i \leq +\infty \quad i = 1, \ldots, n\}$$

and a multi-valued mapping $G : R^n \to \Pi(R^n)$, where $\Pi(S)$ denotes the family of all non-empty subsets of a set S. The *mixed complementarity problem* (MCP for short) is to find a point $x^* \in D$ such that

$$\exists g^* \in G(x^*), \quad g_i^* \begin{cases} \geq 0 & \text{if} \quad x_i^* = a_i, \\ = 0 & \text{if} \quad x_i^* \in (a_i, b_i), \\ \leq 0 & \text{if} \quad x_i^* = b_i, \end{cases} \quad \text{for} \quad i = 1, \ldots, n. \tag{2}$$

Obviously, if the feasible set D coincides with the non-negative orthant R^n_+ and $G = f$, then MCP(2) coincides with (1). At the same time, MCP(2) can be also equivalently rewritten as the variational inequality: Find $x^* \in D$ such that

$$\exists g^* \in G(x^*), \quad \langle g^*, x - x^* \rangle \geq 0 \quad \forall x \in D. \tag{3}$$

Usually, existence and uniqueness results of solutions for complementarity problems and variational inequalities are based upon certain monotonicity properties. As to the classical problem (1), one of the most useful and fruitful concepts is that of the Z-mapping (or off-diagonal antitone mapping). On the one hand, there are a lot of equilibrium type problems in applications which lead to appearance of Z-mappings, they being formulated as CP's or MCP's; see e.g. [2, 4, 14]. On the other hand, this concept allows one to develop efficient solution methods in the single-valued case; see e.g. [15]-[16].

However, generalization of this concept for multi-valued mappings meets considerable difficulties. In particular, the streamlined extension of this concept does not involve even the diagonal multi-valued mappings. In this talk, we consider some kinds of multi-valued Z-mappings and discuss their properties.

Recently, in [5], a Jacobi type algorithm for solving complementarity problems whose cost mappings are compositions of single-valued Z-mappings and multi-valued diagonal monotone mappings was proposed. A Gauss-Seidel algorithm for this class of CP's was proposed in [1]. An extended Jacobi algorithm for MCP (2) involving general compositions of multi-valued mappings was suggested in [6]. This approach was applied to multi-valued inclusions in [7].

In this paper, we describe and substantiate a Gauss-Seidel algorithm for a general class of MCP's of form (2).

2. Properties of Multi-valued Z-Mappings

We start our considerations from recalling several order monotonicity properties of single-valued mappings.

Definition 1. A mapping $F : D \to R^n$ is said to be

(a) *antitone* if the mapping $-F$ is isotone;

(b) *isotone* if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, it holds that $F(x') \geq F(x'')$;

(c) *inverse isotone* if for each pair of points $x', x'' \in D$ such that $F(x') \geq F(x'')$, it holds that $x' \geq x''$;

(d) a *Z-mapping* if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, it holds that $F_k(x') \leq F_k(x'')$ for each index k with $x'_k = x''_k$;

(e) an *M-mapping*, if it is an inverse isotone Z-mapping.

These properties have been investigated rather well, especially, in the affine case, then they are strongly related with the corresponding classes of matrices; see e.g. [2]. In particular, if F is of the form

$$F(x) = Ax + b,$$

F is a Z-mapping (respectively, an M-mapping) if and only if A is a Z-matrix (respectively, an M-matrix). In the general nonlinear case, these properties are strongly related to those of the Jacobian of F. Also, there exist many useful relationships among these concepts. For example, if F is of the form

$$F(x) = x - V(x),$$

where V is an isotone mapping, then F is clearly a Z-mapping.

Observe that there exist several equivalent or slightly modified definitions of Z properties, they are also known as off-diagonal antitonicity and gross substitutability; see e.g. [14, 15].

We present some extensions of the concept of the Z-mapping for the multi-valued case.

Definition 2. A multi-valued mapping $G : D \to \Pi(\mathbb{R}^n)$ is said to be

(a) a Z*-mapping* if for each pair of points $x', x'' \in D$ such that $x' \geq x''$, $x' \neq x''$, it holds that $g'_k \leq g''_k$ for all $g' \in G(x')$, $g'' \in G(x'')$ and for each index k such that $x'_k = x''_k$;

(b) an *upper* (a *lower*) Z*-mapping* if for each pair of points $x', x'' \in D$ such that $x' \geq x''$ and for each $g' \in G(x')$ there exists $g'' \in G(x'')$ (respectively, for each $g'' \in G(x'')$ there exists $g' \in G(x')$) such that $g'_k \leq g''_k$ for every index k such that $x_k = y_k$;

(c) a *weak* Z*-mapping* if it is both an upper and a lower Z-mapping.

Note that the additional condition $x' \neq x''$ can not be dropped in (a) since otherwise the Z-mapping becomes single-valued. Hence, the streamlined extension (a) of the Z-mapping may appear too restrictive.

Definition 3. A mapping $G : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ is said to be

(a) *diagonal* if $G(x) = \prod_{i=1}^{n} G_i(x_i)$;

(b) *quasi-diagonal* [8] if $G(x) = \prod_{i=1}^{n} G_i(x)$.

Clearly, (a)$\Longrightarrow$(b). Moreover, each single-valued mapping is quasi - diagonal. Next, observe that each diagonal single-valued mapping is Z, but this is not the case if it is multi-valued. Hence, various compositions of multi-valued diagonal and Z-mappings may not possess the Z property as well. For this reason, it seems more suitable to utilize weaker concepts of multi-valued Z-mappings given in Definition 2, (b)-(c), which contain arbitrary diagonal multi-valued mappings.

We recall also the known continuity and monotonicity type properties for multi-valued mappings.

Definition 4. A mapping $G : \mathbb{R}^n \to \Pi(R^n)$ is said to be

(a) *monotone*, if for each pair of points $x', x'' \in \mathbb{R}^n$ and for all $g' \in G(x'), g'' \in G(x'')$, it holds that

$$\langle g' - g'', x' - x'' \rangle \geq 0;$$

(b) a *Kakutani-mapping* (K-mapping) if it is upper semicontinuons and has nonempty, convex, and compact image sets.

3. Extended Gauss-seidel Algorithm for MCP

We consider MCP (2), where $G : D \to \Pi(R^n)$ is of the form

$$G(x) = \sum_{s=1}^{l} F^{(s)} \circ H^{(s)}(x), \tag{4}$$

where $F^{(s)} : R^n \to \Pi(R^n)$ *is a quasi-diagonal, an upper* Z*- and a* K*-mapping on some rectangle containing* $H^{(s)}(D)$, $H^{(s)} : D \to \Pi(R^n)$ *is a diagonal monotone* K*-mapping for each* $s = 1, \dots, l$.

Recently, a Jacobi type algorithm for solving such MCP's was proposed in [6]. In [7], it was adjusted to the multi-valued inclusion

$$0 \in G(x^*),$$

where G satisfies the above assumptions with replacing the upper Z property with the weak Z one; see Definition 2. Various examples of applications of MCP (2), (4) satisfying the above assumptions can be found e.g. in [11, 12, 10, 9] and [6]. We now describe a Gauss-Seidel algorithm for this problem.

Let us first introduce the auxiliary set for MCP (2), (4) as follows:

$$Q = \{x \in D \mid \exists g \in G(x), x_i < b_i \Rightarrow g_i \geq 0 \quad \forall i = 1, \dots, n\}.$$

Algorithm (Gauss-Seidel). Choose a point $\tilde{x} \in Q$ and, beginning from the point $x^0 = \tilde{x}$, construct a sequence $\{x^k\}$ in conformity with the following rules.

At the k-th iteration, $k = 0, 1, \dots$, we have a point $x^k \in Q$ such that $x^k \leq x^0$ and that there exists $g^k \in \sum_{s=1}^{l} F^{(s)}(h^{(s),k})$ for some $h^{(s),k} \in H^{(s)}(x^k), s = 1, \dots, l$, satisfying conditions:

$$x_i^k < b_i \Rightarrow g_i^k \geq 0 \quad \text{for} \quad i = 1, \dots, n.$$

In the sequel we will use the notation:

$$\left(x_{-i}^{k+1,k}, y_i\right) = \left(x_1^{k+1}, \dots, x_{i-1}^{k+1}, y_i, x_{i+1}^k, \dots, x_n^k\right),$$

and

$$\left(h_{-i}^{(s),k+1,k}, p_i^{(s)}\right) = \left(h_1^{(s),k+1}, \dots, h_{i-1}^{(s),k+1}, p_i^{(s)}, h_{i+1}^{(s),k}, \dots, h_n^{(s),k}\right),$$

where $p_i^{(s)} \in \mathbb{R}$, so that $\left(h_{-0}^{(s),k+1,k}, p_0^{(s)}\right) = \left(h_1^{(s),k}, \dots, h_n^{(s),k}\right)$. Next, for each separate index $i = 1, \dots, n$, we determine numbers x_i^{k+1}, $p_i^{(1)}, \dots, p_i^{(l)}$ such that

$$a_i \leq x_i^{k+1} \leq x_i^k, p_i^{(s)} \in H_i^{(s)}(x_i^{k+1}), p_i^{(s)} \leq h_i^{(s),k} \quad \text{for} \quad s = 1, \dots, l, \tag{5}$$

and

$$\exists \tilde{g}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)}), \quad \tilde{g}_i^k \begin{cases} \geq 0 & \text{if} \quad x_i^{k+1} = a_i, \\ = 0 & \text{if} \quad x_i^{k+1} \in (a_i, b_i), \\ \leq 0 & \text{if} \quad x_i^{k+1} = b_i, \end{cases} \tag{6}$$

with the help of the bisection procedure below. Afterwards, set $h_i^{(s),k+1} = p_i^{(s)}$ for $s = 1, \ldots, l$. If $i = n$, the k-th iteration is complete.

Procedure (Bisection). It is applied when the indices k and i are fixed and consists of the following sequence of steps.

Step 1. If there exist elements $p_i^{(s)} \in H_i^{(s)}(a_i)$ for $s = 1, \ldots, l$ and an element $\tilde{g}_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)})$ such that $\tilde{g}_i^k \geq 0$, then set $x_i^{k+1} = a_i$ and stop. Otherwise set $x_i' = a_i, \alpha_i^{(s)} = p_i^{(s)} \in H_i^{(s)}(a_i)$ for $s = 1, \ldots, l$ and go to Step 2.

Step 2. If $x_i^k = b_i$ and there exist elements $p_i^{(s)} \in H_i^{(s)}(b_i)$ for $s = 1, \ldots, l$ and an element $\tilde{g}_i^k \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)})$ such that $\tilde{g}_i^k \leq 0$, then set $x_i^{k+1} = b_i$ and stop. Otherwise set $x_i'' = x_i^k, \beta_i^{(s)} = p_i^{(s)} = h_i^{(s),k}$ for $s = 1, \ldots, l$ and go to Step 3.

Step 3. Generate a sequence of inscribed segments $[x_i', x_i'']$ contracting to a point z_i by choosing $y_i = \frac{1}{2}(x_i' + x_i'')$ and setting either $x_i'' = y_i$ if there exist numbers $\beta_i^{(s)} \in H_i^{(s)}(y_i)$ for $s = 1, \ldots, l$ such that $\tilde{g}_i \geq 0$ for some $\tilde{g}_i \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, \beta_i^{(s)})$ or $x_i' = y_i$ otherwise, i.e. when $\tilde{g}_i < 0$ for any element $\tilde{g}_i \in \sum\limits_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, \alpha_i^{(s)})$ for arbitrary numbers $\alpha_i^{(s)} \in H_i^{(s)}(y_i), s = 1, \ldots, l$.

Step 4. Set $x_i^{k+1} = z_i$ and compute numbers $p_i^{(s)} \in H_i^{(s)}(z_i)$ for $s = 1, \ldots, l$ such that conditions (5), (6) are satisfied.

We establish a convergence result for the Gauss-Seidel algorithm.

Theorem 1. *Suppose that the set Q is nonempty. Then the Gauss-Seidel algorithm with the bisection procedure is well defined and generates a sequence $\{x^k\}$ converging to a solution x^* of MCP (2), (4) such that $a \leq x^* \leq \tilde{x}$.*

Proof. First we show that the localization of the initial segment is right in Step 3 of the bisection procedure when it holds that $g_i < 0$ for all $g_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, \alpha_i^{(s)})$ and $\tilde{g}_i \geq 0$ for some $\tilde{g}_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, \beta_i^{(s)})$ with $\alpha_i^{(s)} \leq \beta_i^{(s)}$ for $s = 1, \ldots, l$. In fact, at the point z_i, we define a multi-valued mapping $\Phi : R^l \to \Pi(R)$ on the rectangle $[\alpha_i^{(1)}, \beta_i^{(1)}] \times \cdots \times [\alpha_i^{(l)}, \beta_i^{(l)}]$ as follows

$$\Phi(p_i) = \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)}) \quad \text{with} \quad p_i = (p_i^{(1)}, \ldots, p_i^{(l)}) \in R^l.$$

By construction, $-\Phi(\alpha_i) \subseteq R_+$ and $\Phi(\beta_i) \bigcap R_+ \neq \emptyset$ for $\alpha_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(l)})$ and $\beta_i = (\beta_i^{(1)}, \ldots, \beta_i^{(l)})$. Since Φ is a K-mapping, there exists a number $\lambda \in [0,1]$ such that $0 \in \Phi(p_i)$ for the point $p_i = \lambda\alpha_i + (1-\lambda)\beta_i \in R^l$. Since each $H_i^{(s)}$ has convex images, it follows that $p_i^{(s)} \in H_i^{(s)}(z_i)$ for $s = 1, \ldots, l$, then all the relations in (5), (6) are satisfied.

If $i = 1$, then termination in Step 1 or 2 clearly yields (5), (6). Otherwise, in Step 2 we must have $x_1' < x_1'' = x_1^k$ and $\beta_1^{(s)} = h_1^{(s),k}$, but $g_1'' \geq 0$ for some $g_1'' \in \sum_{s=1}^{l} F_1^{(s)}(h^{(s),k})$ and $h^{(s),k} = (h_{-1}^{(s),k+1,k}, \beta_1^{(s)})$ if $x_1^k < b_1$, i.e. the localization of the initial segment is right, which yields (5), (6). Suppose that this is the case for $1, 2, \ldots, i-1$. Then, by construction, $h_m^{(s),k+1} \leq h_m^{(s),k}$ for $m = 1, 2, \ldots, i-1$. Again, termination in Step 1 or 2 yields (5), (6). Otherwise, $x_i^k = b_i$ leads to the right localization of the initial segment. In case $x_i^k = x_i'' < b_i$ we have $h^{(s),k} \geq (h_{-i}^{(s),k+1,k}, h_i^{(s),k})$, but $g_i^k \geq 0$ for some $g_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h^{(s),k})$ by construction. By using the upper Z property of $F^{(s)}$, we see that there exists $\tilde{g}_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, h_i^{(s),k})$ such that $\tilde{g}_i \geq g_i^k \geq 0$, hence $x_i' < x_i'' = x_i^k$ and the localization of the initial segment is right. Therefore, (5), (6) hold true for each i.

So, the procedure is well-defined. We now proceed to show that there exists $g_i^{k+1} \in \sum_{s=1}^{l} F_i^{(s)}(h^{(s),k+1})$ such that

$$x_i^{k+1} < b_i \Rightarrow g_i^{k+1} \geq 0 \quad \text{for} \quad i = 1, \ldots, n.$$

By construction, $(x_{-i}^{k+1,k}, x_i^{k+1}) \geq x^{k+1}$, besides, by (5), $(h_{-i}^{(s),k+1,k}, p_i^{(s)}) \geq p^{(s)}$ for $p^{(s)} = (p_1^{(s)}, \ldots, p_n^{(s)})$, $p_i^{(s)} \in H_i^{(s)}(x_i^{k+1})$, and, by the upper Z property of $F^{(s)}$, we see that for each $\tilde{f}_i^{(s)} \in F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)})$ there exists $f_i^{(s)} \in F_i^{(s)}(p^{(s)})$ such that $\tilde{f}_i^{(s)} \leq f_i^{(s)}$ for $i = 1, \dots, n$. By (6), we now conclude that $x_i^{k+1} < b_i$ implies

$$g_i^{k+1} = \sum_{s=1}^{l} f_i^{(s)} \geq \sum_{s=1}^{l} \tilde{f}_i^{(s)} = \tilde{g}_i^k \geq 0$$

for $i = 1, \dots, n$. But $g^{k+1} \in G(x^{k+1})$, therefore $x^{k+1} \in Q$. It means that the algorithm is also well-defined. On account of (5), the sequence $\{x^k\}$ is non-increasing and bounded from below. Therefore, it converges to a point x^* such that $a \leq x^* \leq \tilde{x}$. Analogously, on account of (5), we have $h^{(s),k+1} \leq h^{(s),k}$ and $h^{(s),k} \in H^{(s)}(x^k)$, but for each s the sequence $\{h^{(s),k}\}$ must be bounded, hence

$$\lim_{k \to \infty} h^{(s),k} = h^{(s),*}$$

for some $h^{(s),*} \in H^{(s)}(x^*)$. Without loss of generality we can suppose that

$$\lim_{k \to \infty} \tilde{g}_i^k = g_i^* \in \sum_{s=1}^{l} F_i^{(s)}(h^{(s),*}),$$

i.e. $g^* \in G(x^*)$, and (6) now implies (2). Thus x^* is a solution of MCP (2), (4). The proof is complete. ■

The above theorem contains also the existence result.

Corollary 1. *If the set Q is nonempty, then MCP (2), (4) has a solution.*

In general, the Gauss-Seidel algorithm above is not so hard for implementation, however, verification of the relations for all the elements of the set $\sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)})$ where $p_i^{(s)} \in H_i^{(s)}(y_i)$ for $s = 1, \dots, m$ and for some y_i in Steps 1-3 may meet certain difficulties. Then we can simplify the procedure by replacing Steps 1-3 with the following.

Step 1'. Compute elements $p_i^{(s)} \in H_i^{(s)}(a_i)$ for $s = 1, \dots, l$ and an element $\tilde{g}_i^k \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, p_i^{(s)})$. If $\tilde{g}_i^k \geq 0$, then set $x_i^{k+1} = a_i$ and stop. Otherwise set $x_i' = a_i, \alpha_i^{(s)} = p_i^{(s)}$ for $s = 1, \dots, l$ and go to Step $2'$.

Step 2'. If $x_i^k = b_i$ and $g_i^k \leq 0$, then set $x_i^{k+1} = b_i$ and stop. Otherwise set $x_i'' = x_i^k, \beta_i^{(s)} = h_i^{(s),k}$ for $s = 1, \dots, l$. If $x_i' = x_i''$, set $z_i = x_i^k$ and go to Step 4.

Step 3'. Generate a sequence of inscribed segments $[x_i', x_i'']$ contracting to a point z_i by choosing $y_i = \frac{1}{2}(x_i' + x_i'')$, computing $\tilde{\beta}_i^{(s)} \in H_i^{(s)}(y_i)$ for $s = 1, \dots, l$

and $\tilde{g}_i \in \sum_{s=1}^{l} F_i^{(s)}(h_{-i}^{(s),k+1,k}, \tilde{\beta}_i^{(s)})$, and setting $x_i'' = y_i$, $\beta_i^{(s)} = \tilde{\beta}_i^{(s)}$ for $s = 1, \ldots, l$ if $\tilde{g}_i \geq 0$ and $x_i' = y_i, \alpha_i^{(s)} = \tilde{\beta}_i^{(s)}$ for $s = 1, \ldots, l$ if $\tilde{g}_i < 0$.

It is easy to see that the assertion of Theorem 1 remains true for this modified version of the algorithm.

It was also shown in [6] that the auxiliary set Q is a *meet semi-sublattice*, i.e., for each pair of points $x, y \in Q$ it contains their minimal point (meet) $z = \min\{x, y\}$ with $z_i = \min\{x_i, y_i\}$ for $i = 1, \ldots, n$; if (4) is replaced by

$$G(x) = F \circ H(x) + V(x), \tag{7}$$

where $V : D \to \Pi(R^n)$ *is a quasi-diagonal, an upper* Z*- and a* K*-mapping,* $H : D \to \Pi(R^n)$ *is a diagonal monotone* K*-mapping, and* $F : R^n \to \Pi(R^n)$ *is a quasi-diagonal, an upper* Z*- and a* K*-mapping on a rectangle containing* $H(D)$. Hence, the set Q has the least element $\min Q$ which is a solution of GCP.

4. Numerical Experiments

In this section we present some results of numerical experiments obtained under the following computer environment:

OS: Windows XP Pro; CPU: Pentium (R) M 1.6 GHz; Memory: 1.5 GB; Software: Matlab.

For each numerical example we compare work of the Jacobi and Gauss-Seidel algorithms with the same input values and the same stopping criteria. We made all the calculations with double precision, the zero tolerance was chosen to be 10^{-10}. The stopping criteria were the following:
for the bisection procedure: $|x_i' - x_i''| < 10^{-4}$,
for the main process: $\|x^{(k+1)} - x^{(k)}\| < 10^{-2}$ and it was terminated if the number of iterations became equal to MAXITER.

In order to test the algorithms we took the following CP:

$$x^* \geq 0, \exists g(x^*) \in G(x^*), g^* \geq 0, \langle g^*, x^* \rangle = 0 \tag{8}$$

with the multi-valued mapping $G : R^n \to \Pi(R^n)$ of the form:

$$G(x) = Ax - b + \Phi(x) + \Psi(x), \tag{9}$$

where A is an $n \times n$ matrix with nonpositive off-diagonal entries, the mappings Φ and Ψ are diagonal. Note that G in (9) is a particular case of that in (4), where

Table 1: Test 1 - Average of CPU time (sec)

Algorithm	n = 3	n = 10	n = 50	n = 100
Jacobi	0.650	5.611	151.098	283.245
Gauss-Seidel	0.400	3.817	122.441	245.461
G-S/J	61.54%	68.03%	81.03%	86.66%

$l = 3$, $F^{(1)} = Ax - b$, $F^{(2)} = I$, $F^{(3)} = I$, $H^{(1)} = I$, $H^{(2)} = \Phi$, $H^{(3)} = \Psi$. Hence, we can apply the Gauss-Seidel algorithm to this problem and all the results of Section 2 remain true.

More precisely, Φ and Ψ were chosen as follows:

$$\Phi(x) = \prod_{i=1}^{n} \Phi_i(x_i), \quad \Phi_i(x_i) = \max\left\{x_i^2 - 1/\sin(i), 0\right\}, \quad i = 1, \dots, n;$$
$$\Psi(x) = \prod_{i=1}^{n} \Psi_i(x_i), \quad \Psi_i(x_i) = \partial\psi_i(x_i),$$
$$\psi_i(x_i) = \alpha_i|x_i - \beta_i|, \alpha_i = (1+i)/i, \beta_i = 1/\cos(i), \quad i = 1, \dots, n.$$

That is, Φ is nonsmooth and continuous, Ψ is a multi-valued K-mapping and all the assumptions are satisfied.

Test 1. The matrix A was defined as follows:

$$a_{ij} = \begin{cases} -|\sin(i)\cos(j)| & i \neq j; \\ 1 + \sum\limits_{j \neq i} |a_{ij}| & i = j; \end{cases} \quad i, j = 1, \dots, n;$$

the feasible starting point $\tilde{x} = (9, \dots, 9)^T$. Then we obtain the diagonally dominant matrix. We chose also

$$b_i = \sin(i)/i, \quad i = 1, \dots, n.$$

A comparison of the average CPU time for the Jacobi and Gauss-Seidel algorithms is shown in Table 1.

Test 2. The matrix A was defined as follows:

$$a_{ij} = \begin{cases} -|\sin(i)\cos(j)| & i \neq j; \\ \sin(i)\cos(i) & i = j; \end{cases} \quad i, j = 1, \dots, n;$$

Table 2: Test 2 - Average of CPU time (sec)

Algorithm	n = 3	n = 10	n = 50	n = 100
Jacobi	0.222	1.602	10.386	24.2362
Gauss-Seidel	0.1094	1.231	9.725	23.856
G-S/J	49.3%	76.9%	93.64%	98.43%

the feasible starting point $\tilde{x} = (100, \ldots, 100)^T$. Then A is not diagonally dominant and its diagonal is not positive. We chose also

$$b_i = \sin(i)/i, \quad i = 1, \ldots, n.$$

A comparison of the average CPU time for the Jacobi and Gauss-Seidel algorithms is shown in Table 2.

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