

WELL-POSEDNESS FOR VECTOR QUASIEQUILIBRIA

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Abstract. We consider well-posedness under perturbations of vector quasiequilibrium and bilevel-equilibrium problems. This kind of well-posedness relates Hadamard and Tikhonov well-posedness notions to sensitivity analysis and we apply largely techniques of the latter to establish sufficient conditions for well-posedness under perturbations. We also propose several new semicontinuity and quasiconvexity notions to weaken the imposed assumptions. Our results are new or include as special cases recent existing results. Many examples are provided for the illustration purpose.

1. INTRODUCTION

Well-posedness of optimization-related problems can be defined in two ways. The first and oldest is Hadamard well-posedness [15], which means existence, uniqueness and continuous dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [36], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Just after the Tikhonov's paper [36] dealing with unconstrained problems, Levitin and Polyak [26] extended the notion to constrained (scalar) optimization, allowing minimizing sequences $\{x_n\}$ to be outside of the feasible set X_0 and requiring $d(x_n, X_0)$ (the distance from x_n to X_0) to tend to zero. Since then, various definitions of well-posedness have been proposed, extending and blending the mentioned three notions, and intensively studied. In [6] Hadamard well-posedness is extended, using Attouch-Wets topology τ_{aw} (known also as the bounded Hausdorff topology or epi-distance topology) for perturbations and convergence. It is proved in [6] that, under a constraint qualification, Tikhonov

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well-posed convex programs are τ_{aw} -Hadamard well-posed. Revalski [34] establishes the opposite implication, without constraint qualification. The Levitin and Polyak well-posedness is generalized in [16, 23] for problems with explicit constraint $g(x) \in K$, where g is a continuous map between two metric spaces and K is a closed set. For minimizing sequences $\{x_n\}$, instead of $d(x_n, X_0)$, here the distance $d(g(x_n), K)$ is required to tend to zero. This generalization is appropriate for penalty-type methods (e.g., penalty function methods, augmented Lagrangian methods) with iteration processes terminating when $d(g(x_n), K)$ is small enough (but $d(x_n, X_0)$ may be large). [13] is the first paper to characterize well-posedness by the Kuratowski measure of noncompactness of sets of approximately feasible and approximately optimal points tending to zero. This approach is developed later for various optimization-related problems in [11, 16, 17].

Since the data of mathematical models are approximately obtained from measurements or statistical methods and since numerical algorithms for solving mathematical models produce minimizing sequences, only well-posed (in some sense) problems are significant. Fortunately, [18, 19, 20] show that almost every usual optimization problem (e.g. quadratic problem, convex program, semi-infinite program, etc) is well-posed. Here “almost every” is in the Baire category sense or, in the Euclidean space case, means having full Lebesgue measure. For such generic well-posedness considerations, see also [21, 35]. To show also that well-posed problems are very often met, [33] proves sufficient conditions for well-posedness under lower sequential pseudocontinuity assumptions (weaker than lower sequential semicontinuity).

The condition number of a unconstrained optimization problem is the Lipschitz modulus of the unique minimizer as a vector function of the perturbing parameter. Condition number theorems (known in the literature also as distance theorems) assert that the condition number is the reciprocal of the distance of the objective function (in a unconstrained problem) to the set of ill-posed problems. Condition number theorems are investigated in connection with well-posedness in [10, 39, 40, 41].

Well-posedness for various problems related to optimization has been recently intensively considered: for variational inequalities in [7, 9, 11, 27, 31], for Nash equilibria in [29, 32], for fixed-point problems in [11, 24, 25], for inclusion problems in [11, 24, 25] and for equilibrium problems in [22].

Well-posedness under perturbations (called also parametric well-posedness) is a blending of Hadamard and Tikhonov notions, gives also links to stability theory and seems well adapted to describe behaviors of solutions under perturbations. This issue is investigated in [22, 24, 37, 38, 41]. Well-posed bilevel problems are developed in [12, 22, 28, 29]. Recently, well-posedness for vector optimization-related problems have been attracted the attention of many researchers, see e.g., [9, 17, 22]. Note that for vector problems, both minimizing sequences and well-posedness can be

understood in many different meanings.

In this paper we consider well-posedness under perturbations for rather general vector quasiequilibrium problems. As far as we know, there is only reference [22] to deal with equilibrium (not quasiequilibrium) problems. Moreover, we investigate both problem settings: weak and strong (the latter is much less developed in the literature). We pay attention on weakening semicontinuity and generalized convexity assumptions for well-posedness. Hence in sections 2 and 4 we propose new cone semicontinuity and cone quasiconvexity notions and consider their properties for our use in establishing sufficient conditions for well-posedness in sections 3 and 5. However, we hope that these properties may be useful for other purposes as well. In section 3 we demonstrate sufficient conditions for well-posedness under perturbations for our two quasiequilibrium problems. These results are shown to include properly those of the only paper on well-posedness of equilibrium we know [22]. Since quasiequilibrium problems contain many optimization-related problems as particular cases, our results have obvious consequences for such cases. As an example, in section 5 we go in details to apply the theorems in section 3 only to well-posedness for equilibrium problems with equilibrium constraints. These problems are rather general kinds of bilevel optimization-related problems which have become increasingly attracted to researchers recently.

Our notations are usual. R and R_+ denote the real straight line $(-\infty, +\infty)$ and the nonnegative halfline $[0, +\infty)$, respectively. For a set A in a topological space, $\text{int}A$ and $\text{cl}A$ stand for the interior and closure, respectively, of A . A cone C in a topological vector space is called pointed if $C \cap (-C) = \{0\}$ and called solid if $\text{int}C$ is nonempty. We will use partial orders defined by different solid convex cones in vector spaces. The following notations written for cone C but may be used in the sequel for different ordering cones, since the context ensures that no confusion threatens.

$$\begin{aligned}x \geq y &\Leftrightarrow x - y \in C, \\x > y &\Leftrightarrow x - y \in \text{int}C, \\x \not\geq y &\Leftrightarrow x - y \notin C, \\x \not> y &\Leftrightarrow x - y \notin \text{int}C\end{aligned}$$

and similarly for $\leq, <, \not\leq, \not<$.

Recall that a multimap $Q : X \rightarrow Y$ between two topological spaces is said to be upper semicontinuous (usc in short) at $x \in X$ if, for each open superset $U \supseteq Q(x)$, there is a neighborhood N of x such that $Q(N) \subseteq U$. Q is called lower semicontinuous at x if, for each open subset $U \subseteq Y$ with $Q(x) \cap U \neq \emptyset$, there is a neighborhood N of x such that $Q(x') \cap U \neq \emptyset$, for all $x' \in N$. We also use the following two equivalent formulations for lower semicontinuity:

- (a) Q is lsc at x if, for all $x_\alpha \rightarrow x$ and $y \in Q(x)$, there exists $y_\alpha \in Q(x_\alpha)$ such

that $y_\alpha \rightarrow y$;

(b) Q is lsc at x if, for all $x_\alpha \rightarrow x$, one has

$$Q(x) \subseteq \liminf Q(x_\alpha),$$

where

$$\liminf Q(x_\alpha) := \{y \in Y : \exists y_\alpha \in Q(x_\alpha), y_\alpha \rightarrow y\}.$$

2. QUASI-SEMICONtinuities OF VECTOR FUNCTIONS

In this section we propose several kinds of quasi-semicontinuity of vector functions and investigate their properties. These notions and properties will be in use later for discussing well-posedness. Let, in this section, X be a Hausdorff topological space, Y be a Hausdorff topological vector space, $A \subseteq X$ be a nonempty open subset and $C \subseteq Y$ be a solid closed convex cone.

There have been several notions of semicontinuity of vector functions in the literature, but they may appear under slightly different terms. In this paper we call $h : A \rightarrow Y$ to be C -lower semicontinuous (shortly, C -lsc) at x_0 if for any neighborhood U of $h(x_0)$, there is a neighborhood N of x_0 such that $h(N) \subseteq U + C$. h is called C -upper semicontinuous (C -usc) at x_0 if for any neighborhood U of $h(x_0)$, there is a neighborhood N of x_0 such that $h(N) \subseteq U - C$.

We note the following simple relation to semicontinuity of scalar functions. Let $c \in C$, $f : A \rightarrow R$ and $h : A \rightarrow Y$ defined by $h(x) = f(x)c$, for $x \in A$. If f is lsc (usc, respectively) at x_0 , then h is C -lsc (C -usc, respectively) at x_0 . Furthermore, relations to level sets are also easily obtained as follows. For $b \in Y$, we use the following notations for level sets of h , for different orderings \leq (by the context, no confusion occurs).

$$L_{\leq b} h = \{x \in A : h(x) \leq b\},$$

$$L_{\not\leq b} h = \{x \in A : h(x) \not\leq b\}$$

and similarly for other level sets $L_{\not\leq b} h$, $L_{> b} h$, $L_{\not> b} h$, $L_{> b} h$, etc. Then it is not hard to verify the following equivalent relations.

Proposition 2.1. *Let $h : A \rightarrow Y$.*

- (i) *h is C -lsc at x_0 if and only if $L_{\not\leq b} h$ is closed at x_0 for all $b \in Y$.*
- (ii) *h is C -usc at x_0 if and only if $L_{\not> b} h$ is closed at x_0 for all $b \in Y$.*

This equivalence suggests the following natural properties weaker than C -semicontinuities, by imposing requirements only at a given level. Here we propose also similar definitions for other preorders \leq and \geq .

Definition 2.1. Let $h : A \rightarrow Y$ and $b \in Y$ be given.

- (i) h is called b -level C -lower semicontinuous ((b, C) -lsc) at x_0 if $L_{\not\leq b}h$ is closed at x_0 .
- (ii) h is called b -level C -upper semicontinuous ((b, C) -usc) at x_0 if $L_{\not\geq b}h$ is closed at x_0 .
- (iii) h is called b -level C -quasilower semicontinuous ((b, C) -qlsc) at x_0 if $L_{\leq b}h$ is closed at x_0 .
- (iv) h is called b -level C -quasiupper semicontinuous ((b, C) -qusc) at x_0 if $L_{\geq b}h$ is closed at x_0 .

If a statement of (i)-(iv) holds for all $b \in Y$, we omit b in the corresponding terminology.

To make these definitions clearer we present equivalent statements in terms of neighborhoods as follows.

Proposition 2.2. Let $h : A \rightarrow Y$ and $b \in Y$ be fixed.

- (i) h is (b, C) -lsc at x_0 if and only if there is a neighborhood N of x_0 with $h(N) \subseteq b + \text{int}C$ whenever $h(x_0) \in b + \text{int}C$.
- (ii) h is (b, C) -usc at x_0 if and only if there is a neighborhood N of x_0 with $h(N) \subseteq b - \text{int}C$ whenever $h(x_0) \in b - \text{int}C$.
- (iii) h is (b, C) -qlsc at x_0 if and only if there exists a neighborhood N of x_0 such that $h(N) \cap (b - C) = \emptyset$ whenever $h(x_0) \notin b - C$.
- (iv) h is (b, C) -qusc at x_0 if and only if there exists a neighborhood N of x_0 such that $h(N) \cap (b + C) = \emptyset$ whenever $h(x_0) \notin b + C$.

Proof. By the similarity we verify only (iii) as an example.

“Only if”. Suppose ad absurdum that h is (b, C) -qlsc at x_0 and $h(x_0) \not\leq b$, but for any neighborhood N of x_0 , there is $x \in N$ such that, $h(x) \leq b$, i.e., there is a net $x_\alpha \rightarrow x_0$ with $h(x_\alpha) \leq b$. Then, as h is (b, C) -qlsc at x_0 , we get the contradiction that $h(x_0) \leq b$.

“If”. Let $x_\alpha \rightarrow x_0$ with $h(x_\alpha) \leq b$. We show that $h(x_0) \leq b$. If $h(x_0) \not\leq b$, by the assumption there is a neighborhood N of x_0 such that, for all $x \in N$, $h(x) \not\leq b$. Since $x_\alpha \rightarrow x_0$, one can assume that $x_\alpha \in N$, i.e., $h(x_\alpha) \not\leq b$, a contradiction. ■

The following relationships between C -semicontinuity and C -quasisemicontinuity resemble those for the scalar counterparts, though not very clear from the above definitions and discussions.

Proposition 2.3. Let $h : A \rightarrow Y$.

- (i) If h is C -lsc at x_0 then h is C -qlsc at x_0 .
(ii) If h is C -usc at x_0 then h is C -qusc at x_0 .

Proof.

- (i) Suppose that h is C -lsc at x_0 and $x_\alpha \rightarrow x_0$ with $h(x_\alpha) \leq b$ for some $b \in Y$, but $h(x_0) \in Y \setminus (b - C)$. Since h is C -lsc at x_0 and $Y \setminus (b - C)$ is open, one can assume that $h(x_\alpha) \in Y \setminus (b - C) + C$. So there are $t_\alpha \in Y \setminus (b - C)$ and $c_\alpha \in C$, such that $h(x_\alpha) = t_\alpha + c_\alpha$. As $h(x_\alpha) \in b - C$. There is $c'_\alpha \in C$, such that $h(x_\alpha) = b - c'_\alpha$. Therefore, $t_\alpha = b - c'_\alpha - c_\alpha \in b - C$, which is impossible as $t_\alpha \in Y \setminus (b - C)$.
(ii) It may be checked similarly. ■

The following example shows that the converse of Proposition 2.3 is not true.

Example 2.1. Let $X = R, Y = R^2, A = R, C = R_+^2, x_0 = 0$ and

$$h(x) = \begin{cases} (x, 1) & \text{if } x \leq 0, \\ \left(-\frac{1}{x}, \frac{1}{x}\right) & \text{otherwise.} \end{cases}$$

Then h is both C -qlsc and C -qusc at 0. Indeed, by the similarity we check only the latter property. For any sequence $x_n \rightarrow 0$ such that $h(x_n) \geq b$ for some $b \in R$ and all n , there is a subsequence x_{n_k} such that $x_{n_k} \leq 0$, for all n_k , (if for all subsequences $x_{n_k}, x_{n_k} > 0$, since $x_{n_k} \rightarrow 0$, one has $h(x_{n_k}) \rightarrow (-\infty, +\infty) \not\geq b$). $h(x_{n_k}) \geq b$ means that $x_{n_k} \geq b_1$ and $1 \geq b_2$, and hence $0 \geq b_1$. Therefore, $h(x_0) = (0, 1) \geq b$, i.e., h is (b, C) -qusc at 0. As b is arbitrary, h is C -qusc at 0.

However, h is neither C -lsc nor C -usc at 0. Indeed, we check only the latter. Let $b = (2, 2)$ and $x_n = \frac{1}{n}$. Then $h(x_n) \not\leq b$ for large n . But $h(0) = (0, 1) < b$. Consequently, h is not (b, C) -usc at 0.

The relations asserted in Proposition 2.3 cannot be similarly stated for a particular level b (unlike the scalar case) as ensured by the example below.

Example 2.2. Let X, Y, A, C and $x_0 = 0$ be as in Example 2.1 and

$$h(x) = \begin{cases} (-1, 1) & \text{if } x = 0, \\ (-1, -1) & \text{otherwise.} \end{cases}$$

Then, for all $x \in R, h(x) \not\geq 0$ and hence h is $(0, C)$ -lsc at 0, since $L_{\not\geq 0}h$ is closed. But h is not $(0, C)$ -qlsc at 0, since $L_{\leq 0}h$ is not closed. Indeed, let $x_n = \frac{1}{n} \rightarrow 0$. Then $h(x_n) = (-1, -1) \leq 0$, but $h(0) = (-1, 1) \not\leq 0$. So $L_{\leq 0}h$ is not closed.

Now we modify h as

$$h(x) = \begin{cases} (-1, 1) & \text{if } x = 0, \\ (1, 1) & \text{otherwise.} \end{cases}$$

Then, for all $x \in R$, $h(x) \not\leq 0$, whence h is $(0, C)$ -usc at 0, since $L_{\not\leq 0}h$ is closed. But h is not $(0, C)$ -qlsc at 0, since $L_{\geq 0}h$ is not closed. Indeed, let $x_n = \frac{1}{n}$. Then $h(x_n) = (1, 1) \geq 0$, but $h(0) = (-1, 1) \not\leq 0$. Hence, $L_{\geq 0}h$ is not closed.

For our well-posedness consideration in sections 3 and 4 we need to develop sum rules for generalized semicontinuous and quasi-semicontinuous functions.

Proposition 2.4. *Let X, Z be topological spaces, $A \subseteq X$ and $B \subseteq Z$ be nonempty open sets, $C \subseteq Y$ be a solid closed convex cone and $f : A \rightarrow Y, g : B \rightarrow Y, A \subseteq X, B \subseteq Z$. If f and g are C -lsc (or C -usc) at x_0 and y_0 , respectively, then $h = f + g$ is C -lsc (or C -usc) at (x_0, y_0) .*

Proof. By the similarity and symmetry we demonstrate only for the C -upper semicontinuity. Let U be any open superset of $f(x_0) + g(y_0)$. There are open neighborhoods U_1 and U_2 of $f(x_0)$ and $g(y_0)$, respectively, such that $U_1 + U_2 \subseteq U$. Since f and g are C -lsc, there are open neighborhoods N_1 and N_2 of x_0 and y_0 , respectively, such that $f(x) \in U_1 - C$ and $g(y) \in U_2 - C$, for all $(x, y) \in N_1 \times N_2$. Hence $h(x, y) = f(x) + g(y) \in U_1 + U_2 - C \subseteq U - C$, for all $(x, y) \in N_1 \times N_2$, i.e., h is C -usc at (x_0, y_0) . ■

Unfortunately, Proposition 2.4 cannot be extended for C -quasi semicontinuous functions as shown by the following example.

Example 2.3. Let X, Y, A, C and x_0 be as in Example 2.1. Let $Z = B = X, y_0 = x_0$ and f be defined as h in Example 2.1. Let

$$g(x) = \begin{cases} (x, 1) & \text{if } x \leq 0, \\ \left(\frac{1}{x}, -\frac{1}{x}\right) & \text{otherwise.} \end{cases}$$

By similar arguments as those of Example 2.1, we see that f and g are C -qlsc at 0. Furthermore,

$$(f + g)(x) = \begin{cases} (2x, 2) & \text{if } x \leq 0, \\ (0, 0) & \text{otherwise.} \end{cases}$$

For $x_n = \frac{1}{n} \rightarrow 0$, one has $(f + g)(x_n) = (0, 0) \leq (1, 1)$, but $(f + g)(0) = (0, 2) \not\leq (1, 1)$. So $f + g$ is not C -qlsc at 0.

Let us now modify f and g by setting

$$f(x) = \begin{cases} (x, -1) & \text{if } x \leq 0, \\ \left(-\frac{1}{x}, \frac{1}{x}\right) & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} (x, -1) & \text{if } x \leq 0, \\ \left(\frac{1}{x}, -\frac{1}{x}\right) & \text{otherwise.} \end{cases}$$

Then f and g are C -qusc at 0. But

$$(f + g)(x) = \begin{cases} (2x, -2) & \text{if } x \leq 0, \\ (0, 0) & \text{otherwise} \end{cases}$$

is not C -qusc at 0 because, for $x_n = \frac{1}{n} \rightarrow 0$, we have $(f + g)(x_n) = (0, 0) \geq (-1, -1)$, but $(f + g)(0) = (0, -2) \not\geq (-1, -1)$.

Proposition 2.5. Assume that X, Y, Z, A, B, C, f and g are as in Proposition 2.4.

- (i) If f is C -qlsc at x_0 and g is C -lsc at y_0 , then $h = f + g$ is C -qlsc at (x_0, y_0) .
- (ii) If f is C -qusc at x_0 and g is C -usc at y_0 , then $h = f + g$ is C -qusc at (x_0, y_0) .

Proof. By the similarity and symmetry we demonstrate only (i). For arbitrary $b \in Y$, assume that $(x_\alpha, y_\alpha) \rightarrow (x_0, y_0)$ and $h(x_\alpha, y_\alpha) \leq b$. For each $e \in \text{int}C$, as g is C -lsc at y_0 , there is a subnet y_β such that $g(y_\beta) > g(y_0) - e$, for all β . Hence

$$f(x_\beta) + g(y_0) - e \leq b.$$

As f is C -qlsc at x_0 ,

$$f(x_0) + g(y_0) - e \leq b,$$

Consequently, $f(x_0) + g(y_0) \leq b$. Thus, h is (b, C) -qlsc at (x_0, y_0) . By the arbitrariness of b , h is C -qlsc at (x_0, y_0) . ■

The following examples prevent us to state results similar to Propositions 2.4 and 2.5 for a given fixed level b .

Example 2.4. Let $A = R, Y = R^2, C = R_+^2, x_0 = 0$ and $f, g : R \rightarrow R^2$ be defined by $g(x) = (2, -1)$ for all x , and

$$f(x) = \begin{cases} (-1, 2) & \text{if } x = 0, \\ (-3, -3) & \text{otherwise.} \end{cases}$$

Then f is $(0, C)$ -lsc at 0, since $L_{\not\geq 0}f$ is closed. g is continuous, whence $(0, C)$ -lsc at 0. But

$$(f + g)(x) = \begin{cases} (1, 1) & \text{if } x = 0, \\ (-1, -4) & \text{otherwise} \end{cases}$$

is not $(0, C)$ -lsc at 0, since $L_{\not\geq 0}h$ is not closed. Indeed, let $x_n = \frac{1}{n}$. Then $h(x_n) = (-1, -4) \not\geq 0$, but $h(0) = (1, 1) > 0$. So $L_{\not\geq 0}h$ is not closed. Therefore, from the fact that f is (b_1, C) -lsc at x_0 and g is (b_2, C) lsc at x_0 , we cannot deduce that $f + g$ is $(b_1 + b_2, C)$ -lsc at x_0 . If $b_1 = b_2$, f is (b, C) -lsc at x_0 and g is even continuous at x_0 , we do not have that $f + g$ is (b, C) -lsc at x_0 .

To get an counterexample for (b, C) -upper semicontinuity we modify f and g as follows. Put $g(x) = (1, -2)$ and

$$f(x) = \begin{cases} (-2, 1) & \text{if } x = 0, \\ (3, 3) & \text{otherwise.} \end{cases}$$

Then f is $(0, C)$ -usc at 0 and g is continuous (and hence $(0, C)$ -usc) at 0. But

$$(f + g)(x) = \begin{cases} (-1, -1) & \text{if } x = 0, \\ (4, 1) & \text{otherwise} \end{cases}$$

is not $(0, C)$ -usc at 0, since $L_{\not\leq 0}h$ is not closed. Indeed, let $x_n = \frac{1}{n} \rightarrow 0$. Then $h(x_n) = (4, 1) \not\leq 0$, but $h(0) = (-1, -1) < 0$, whence $L_{\not\leq 0}h$ is not closed. This example shows that from f being (b, C) -usc and g being even continuous at x_0 , it does not follow that $f + g$ is (b, C) -usc at x_0 ,

Passing to discussions for C -quasi semicontinuity we have similar counterexamples as follows.

Example 2.5. Let Y, A, C and $x_0 = 0$ be as in Example 2.4. Let $f, g : R \rightarrow R^2$ be defined by $g(x) = (2, 2)$ for all x , and

$$f(x) = \begin{cases} (-1, -1) & \text{if } x = 0, \\ (-3, -3) & \text{otherwise.} \end{cases}$$

Then f is $(0, C)$ -qlsc and g is continuous (and hence $(0, C)$ -lsc) at 0. But

$$(f + g)(x) = \begin{cases} (1, 1) & \text{if } x = 0, \\ (-1, -1) & \text{otherwise} \end{cases}$$

is not $(0, C)$ -qlsc at 0 (to see this let $x_n = \frac{1}{n}$).

A similar assertion for (b, C) -quasi upper semicontinuity is seen by setting $g(x) = (-2, -2)$ and

$$f(x) = \begin{cases} (1, 1) & \text{if } x = 0, \\ (3, 3) & \text{otherwise.} \end{cases}$$

The following example indicates that, for a fixed level b , the sum of a continuous function and a (b, C) -lsc function may not be (b, C) -qlsc.

Example 2.6. Let us modify from the previous example only $f(x) = (2, 2)$ and

$$g(x) = \begin{cases} (-1, -1) & \text{if } x = 0, \\ (-3, -3) & \text{otherwise.} \end{cases}$$

Then, f is continuous and g is $(0, C)$ -lsc at 0. But

$$(f + g)(x) = \begin{cases} (1, 1) & \text{if } x = 0, \\ (-1, -1) & \text{otherwise} \end{cases}$$

is not $(0, C)$ -qlsc at 0.

Now let $f(x) = (-2, -2)$, $x_0 = 0$ and

$$g(x) = \begin{cases} (1, 1) & \text{if } x = 0, \\ (3, 3) & \text{otherwise.} \end{cases}$$

Then f is continuous and g is $(0, C)$ -usc at 0. But

$$(f + g)(x) = \begin{cases} (-1, -1) & \text{if } x = 0, \\ (1, 1) & \text{otherwise} \end{cases}$$

is not $(0, C)$ -qusc at 0.

3. WELL-POSEDNESS FOR QUASIEQUILIBRIUM PROBLEMS

We are concerned with the following quasiequilibrium problems, which are perturbed. Let the decision space X be a Hausdorff topological space and the range space Y be a Hausdorff topological vector space. Let $A \subseteq X$ be a nonempty set and the ordering cone $C \subseteq Y$ be nonempty convex, closed, pointed and solid. Let the constrained sets be depending on $x \in X$ and defined by $K_i : A \rightarrow 2^A$, $i = 1, 2$.

Let the machinery of the problems be expressed by $f : A \times A \rightarrow Y$. The following two quasiequilibrium problems are concerned

(WQEP) Find $\bar{x} \in K_1(\bar{x})$ such that, for all $y \in K_2(\bar{x})$,

$$f(\bar{x}, y) \not\leq 0;$$

(SQEP) Find $\bar{x} \in K_1(\bar{x})$ such that, for all $y \in K_2(\bar{x})$,

$$f(\bar{x}, y) \geq 0.$$

The notations (WQEP) and (SQEP) mean “Weak Quasiequilibrium Problem” and “Strong Quasiequilibrium Problem”, respectively. The solution sets of (WQEP) and (SQEP) are denoted by S^w and S^s , respectively.

Now assume that these problems suffer perturbations, which are expressed in terms of a perturbing parameter λ in a Hausdorff topological space Λ . This means that our problems are embedded into the following families, for $\lambda \in \Lambda$,

(WQEP $_\lambda$) Find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that, for all $y \in K_2(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \not\leq 0;$$

(SQEP $_\lambda$) Find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that, for all $y \in K_2(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \geq 0.$$

Assume that the original problems are corresponding to $\lambda = \bar{\lambda}$, i.e., (WQEP) = (WQEP $_{\bar{\lambda}}$) and (SQEP) = (SQEP $_{\bar{\lambda}}$). We adapt the notions of asymptotically minimizing sequences and well-posedness under perturbations introduced by Zolezzi for unconstrained optimization in [37] and developed for various problem settings in [22, 25, 38, 41] to our problems as follows.

Definition 3.1. Let $\{\lambda_\alpha\} \subseteq \Lambda$ be a net converging to $\bar{\lambda}$. A net $\{x_\alpha\} \subseteq K_1(x_\alpha, \lambda_\alpha)$ is said to be an asymptotically solving net for (WQEP) ((SQEP), respectively) corresponding to $\{\lambda_\alpha\}$ if there exists a net $\{e_\alpha\} \subseteq \text{int}C$ converging to 0 such that

$$f(x_\alpha, y, \lambda_\alpha) + e_\alpha \not\leq 0, \text{ for all } y \in K_2(x_\alpha, \lambda_\alpha),$$

$$(f(x_\alpha, y, \lambda_\alpha) + e_\alpha \geq 0, \text{ for all } y \in K_2(x_\alpha, \lambda_\alpha), \text{ respectively}).$$

Definition 3.2. The problem (WQEP) ((SQEP), respectively) is said to be well-posed under perturbations if:

- (a) the solution set S^w of (WQEP) (S^s of (SQEP), respectively) is nonempty;

- (b) for any net $\{\lambda_\alpha\} \subseteq \Lambda$ converging to $\bar{\lambda}$, every asymptotically solving net for (WQEP) ((SQEP), respectively) corresponding to $\{\lambda_\alpha\}$ has a subnet converging to some point of S^w (S^s , respectively).

We consider also the following stronger property with the uniqueness requirement as in classical definitions.

Definition 3.3. The problem (WQEP) ((SQEP), respectively) is said to be uniquely well-posed under perturbations if:

- (a) there exists a unique solution \bar{x} to (WQEP) ((SQEP), respectively);
 (b) for any net $\{\lambda_\alpha\} \subseteq \Lambda$ converging to $\bar{\lambda}$, every asymptotically solving net for (WQEP) ((SQEP), respectively) corresponding to $\{\lambda_\alpha\}$ converges to \bar{x} .

From now on, for $\lambda \in \Lambda$ and $e \in \text{int}C$, let $E(\lambda)$ be the set of all fixed points of $K_1(\cdot, \lambda)$, i.e., the set $\{x \in A : x \in K_1(x, \lambda)\}$, and

$$\Pi^w(\lambda, e) = \{x \in K_1(x, \lambda) : f(x, y, \lambda) + e \not\leq 0, \forall y \in K_2(x, \lambda)\},$$

$$\Pi^s(\lambda, \varepsilon) = \{x \in K_1(x, \lambda) : f(x, y, \lambda) + e \geq 0, \forall y \in K_2(x, \lambda)\}$$

($\Pi^w(\lambda, e)$ and $\Pi^s(\lambda, e)$ are e -approximate solution sets of (WQEP) $_\lambda$ and ((SQEP) $_\lambda$), respectively).

Remark 3.1.

- (i) If (WQEP) (or (SQEP)) is (uniquely) well-posed under perturbations, it is Tikhonov well-posed (simply take $\lambda_\alpha = \bar{\lambda}$ for all α).
 (ii) The following facts are well-known and often used in sensitivity analysis (see e.g., [1-4]): Let $Q : X \rightarrow 2^Y$ be a multimap between two topological spaces.
 (ii_a) If $Q(\bar{x})$ is compact, then $Q(\cdot)$ is usc at \bar{x} if and only if for any nets $\{\bar{x}_\alpha\} \subseteq X$ with $x_\alpha \rightarrow \bar{x}$ and $\{y_\alpha\}$ with $y_\alpha \in Q(x_\alpha)$, there is a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \rightarrow y$ for some $y \in Q(\bar{x})$.
 (ii_b) If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton then, for the above nets, one has $y_\alpha \rightarrow \bar{y}$.

Thus, the definition of well-posedness under perturbations and unique well-posedness under perturbations encompass a form of upper semicontinuous dependence of solutions on the parameter. Such a dependence is investigated in stability (for recent results for quasiequilibrium problems see e.g., [1-4]).

- (iii) In [22] the special case, where $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv A$ is considered. The definition of parametric well-posedness in [22] is different from Definitions 3.2 and 3.3, since the statements are for all $\lambda \in \Lambda$ instead of for $\bar{\lambda}$ corresponding to the unperturbed problem. Our definitions are similar to those in [24,

27, 38, 41] and describe more reasonable perturbations of an original problem. However, our results below can be easily restated for the well-posedness defined in [22].

- (iv) Well-posedness for vector problems have been defined in various ways. In [9] for instance, for asymptotically solving nets, e_α is restricted to being $\epsilon_\alpha e$ for ϵ_α being positive numbers and $e \in \text{int}C$, i.e., only fixed direction e is allowed. [17] uses also the scalarizing function, for $y \in Y$,

$$\xi(y) = \inf\{t : y \leq te\}$$

to define well-posedness (called well-posedness of type III there).

A sufficient condition for the well-posedness under perturbations of problem (WQEP) is given by the following theorem. We see that the proof is based on techniques of stability considerations (cf. e.g., [1-4]), since well-posedness under perturbations has close links to stability.

Theorem 3.1. *Assume for (WQEP) that S^w is nonempty and*

- (i) *f is C -usc in $A \times A \times \{\bar{\lambda}\}$;*
- (ii) *either of the following two conditions holds*
- (ii₁) *K_1 is usc in $A \times \{\bar{\lambda}\}$, A is compact and K_2 is lsc in $A \times \{\bar{\lambda}\}$;*
- (ii₂) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact, and K_2 is lsc in $A \times \{\bar{\lambda}\}$.*

Then, (WQEP) is well-posed under perturbations. Furthermore, if S^w is a singleton, then this problem is uniquely well-posed under perturbations.

Proof. It suffices to consider the case (ii₂), since the other case is similar and simpler. We show first that Π^w is usc at $(\bar{\lambda}, 0)$. Suppose the existence of an open superset U of $\Pi^w(\bar{\lambda}, 0)$, of nets $\{(\lambda_\alpha, e_\alpha)\}$ in $A \times \text{int}C$ with $(\lambda_\alpha, e_\alpha) \rightarrow (\bar{\lambda}, 0)$ and of $\{x_\alpha\}$ in $\Pi^w(\lambda_\alpha, e_\alpha)$ such that $x_\alpha \notin U$, for all α . By the upper semicontinuity of E at $\bar{\lambda}$ and compactness of $E(\bar{\lambda})$ one can assume that $x_\alpha \rightarrow x_0$ for some $x_0 \in E(\bar{\lambda})$. If $x_0 \notin \Pi^w(\bar{\lambda}, 0) = S^w(\bar{\lambda}) = S^w$, then there is $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 in turn shows the existence of $y_\alpha \in K_2(x_\alpha, \lambda_\alpha)$ such that $y_\alpha \rightarrow y_0$. Since $x_\alpha \in \Pi^w(\lambda_\alpha, e_\alpha)$, one has

$$f(x_\alpha, y_\alpha, \lambda_\alpha) + e_\alpha \not\leq 0.$$

Let $I : C \rightarrow C$ be the identity map. Then Proposition 2.4 yields the C -upper semicontinuity of $(x, y, \lambda, e) \rightarrow f(x, y, \lambda) + e$ at $(x_0, y_0, \bar{\lambda}, 0)$. Hence

$$f(x_0, y_0, \bar{\lambda}) \not\leq 0.$$

which is a contradiction. Hence, $x_0 \in \Pi^w(\bar{\lambda}, 0) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$, for all α . Thus, Π^w is usc at $(\bar{\lambda}, 0)$.

Next we check that S^w is compact by showing simply its closedness (as $E(\bar{\lambda})$ is compact). Let $x_\alpha \in S^w$ with $x_\alpha \rightarrow x_0$. If $x_0 \notin S^w$, there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that

$$f(x_0, y_0, \bar{\lambda}) < 0.$$

The lower semicontinuity of K_2 gives the existence of $y_\alpha \in K_2(x_\alpha, \bar{\lambda})$ such that $y_\alpha \rightarrow y_0$. Since $x_\alpha \in S^w$ for each α , one has

$$f(x_\alpha, y_\alpha, \bar{\lambda}) \not< 0.$$

The C -upper semicontinuity of $f(\cdot, \cdot, \bar{\lambda})$ now implies that

$$f(x_0, y_0, \bar{\lambda}) \not< 0,$$

which is impossible. Consequently, S^w is compact.

Finally, taking Remark 3.1 (ii_a) into account, we see that the upper semicontinuity of Π^w and the compactness of S^w together imply the well-posedness under perturbations of (WQEP). In the particular case, where S^w is a singleton, the unique well-posedness under perturbations is followed by Remark 3.1 (ii_b). ■

Remark 3.2. In the special case, where $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv A$, assumption (i) of Theorem 3.1 can clearly be reduced to the C -upper semicontinuity of $f(\cdot, y, \cdot)$, for each $y \in A$. Therefore, this theorem contains the well-posedness part of Theorem 3.3 and Corollary 3.1 of [22] (with abuse of the fact that the latter paper considers well-posedness for all $\lambda \in \Lambda$, since such results can be obtained by modifying Theorem 3.1). The other part of that theorem is about the existence of solutions for all $\lambda \in \Lambda$. But in our approach both in the definition of well-posedness and in Theorem 3.1, we need the existence of solutions only for $\bar{\lambda}$. Moreover, in the literature there have been many papers on the solution existence for quasiequilibrium problems (unparametrized, or what is the same, depending on one fixed parameter $\bar{\lambda}$). Therefore, we focus on well-posedness, leaving our needed existence as an assumption.

The following example shows that, even in the mentioned special case, the C -upper semicontinuity in assumption (i) should be imposed with respect to (shortly, wrt) both x and λ .

Example 3.1. Let $X = Y = R$, $A = [0, 1]$, $\bar{\lambda} = 0$, $A = [0, 1]$, $C = R_+$, $K_1(x, \lambda) = K_2(x, \lambda) = [0, 1]$ and

$$f(x, y, \lambda) = \begin{cases} x - y & \text{if } \lambda = 0, \\ y - x & \text{otherwise.} \end{cases}$$

Then assumption (ii₁) is satisfied. As for (i), f is continuous wrt x but not R_+ -usc wrt λ . Indeed, taking $x_n = 0, y_n = 1, \lambda_n = \frac{1}{n}$ and $e_n = 0$, we have $(x_n, y_n, \lambda_n, e_n) \rightarrow (0, 1, 0, 0)$ and $f(x_n, y_n, \lambda_n) + e_n = f(0, 1, \frac{1}{n}) = 1 > 0$, but $f(0, 1, 0) = -1 < 0$. We see directly that $S(0) = \{1\}$ and, for $\lambda_n = \frac{1}{n}, e_n = \frac{1}{n}$, the sequence $\{x_n\}$ in $\Pi(\lambda_n, e_n)$, with $x_n = 0$ for all n , is an asymptotically solving sequence for (QEP) corresponding to λ_n and e_n . But $x_n \rightarrow 0 \notin S(0)$ and hence (QEP) is not well-posed under perturbations.

For the strong problem (SQEP) we have the following result.

Theorem 3.2. *Impose the assumptions of Theorem 3.1 except (i), which is replaced by*

(i') f is C -qusc in $A \times A \times \{\bar{\lambda}\}$.

Then, (SQEP) is well-posed under perturbations. Furthermore, if S^s is a singleton, then (SQEP) is uniquely well-posed under perturbations.

Proof. The proof is technically similar to that of Theorem 3.1. We present it shortly for the sake of completeness. To prove first the upper semicontinuity of Π^s by contraposition, suppose the existence of an open superset U of $\Pi^s(\bar{\lambda}, 0)$ and nets $\{(\lambda_\alpha, e_\alpha)\}$ and $\{x_\alpha\}$ in $\Pi^s(\lambda_\alpha, e_\alpha)$ such that $(\lambda_\alpha, e_\alpha) \rightarrow (\lambda_0, 0)$, and $x_\alpha \notin U$, for all α . By assumption (ii) one can assume that $x_\alpha \rightarrow x_0$ for some $x_0 \in \Pi^s(\bar{\lambda}, 0)$, a contradiction. (Instead of Proposition 2.4, here we use the sum rule in Proposition 2.5 (ii) for C -qusc function f .)

Next suppose for $x_\alpha \in S^s$, one has $x_\alpha \rightarrow x_0$ but $x_0 \notin S^s$. Then, there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that

$$f(x_0, y_0, \bar{\lambda}) \not\leq 0.$$

The lower semicontinuity of K_2 and the C -quasi upper semicontinuity of $f(\cdot, \cdot, \bar{\lambda})$ together imply the contradiction that

$$f(x_0, y_0, \bar{\lambda}) \geq 0.$$

Therefore, S^s is closed and hence compact.

Finally, by Remark 3.1 (ii), the upper semicontinuity of Π^s and compactness or being a singleton of S^s together complete the proof. ■

4. QUASICONVEXITY OF VECTOR FUNCTIONS

Now we discuss several generalized quasiconvexity notions for vector functions and develop their properties in order to use them for our well-posedness study in the next section. In this section let X be a vector space and Y be a topological vector space partially ordered by a solid pointed closed convex cone C , $A \subseteq X$ be a convex set and $h : A \rightarrow Y$. The following definition is known. (The relations $\leq, <$, etc, are defined by C .)

Definition 4.1. (i) h is called convex if, for every $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$h(tx_1 + (1-t)x_2) \leq th(x_1) + (1-t)h(x_2).$$

(ii) h is said to be strictly convex if, for every $x_1, x_2 \in A$ and $t \in (0, 1)$,

$$h(tx_1 + (1-t)x_2) < th(x_1) + (1-t)h(x_2).$$

(iii) h is said to be quasiconvex if, for every $x_1, x_2 \in A$ and $t \in [0, 1]$, either

$$h(tx_1 + (1-t)x_2) \leq h(x_1) \quad \text{or} \quad h(tx_1 + (1-t)x_2) \leq h(x_2).$$

(iv) h is said to be strictly quasiconvex if, for every $x_1, x_2 \in A$ and $t \in (0, 1)$, either

$$h(tx_1 + (1-t)x_2) < h(x_1) \quad \text{or} \quad h(tx_1 + (1-t)x_2) < h(x_2).$$

For these definitions and for all later convexity-related definitions, by inverting the inequalities we get the corresponding concavity notions. In other words, h is concave (strictly concave, quasiconcave, etc) if and only if $-h$ is convex (strictly convex, quasiconvex, etc, respectively).

Remark 4.1. If $Y = C \cup -C$, like for scalar functions, convexity (strict convexity, respectively) implies quasiconvexity (strict quasiconvexity). However, this is not true in general as shown by the following example.

Example 4.1. Let $X = A = R, Y = R^2, C = R_+^2$ and $h(x) = (-1 - 2x, -3 + 3x)$. Then h is convex as it is linear. But h is not quasiconvex. Indeed, for $x_1 = 0, x_2 = 1, t = \frac{1}{2}$, one has $h(x_1) = (-1, -3), h(x_2) = (-3, 0)$, but $h(\frac{1}{2}x_1 + \frac{1}{2}x_2) = (-2, -\frac{3}{2}) \not\leq h(x_1)$ and $\not\leq h(x_2)$. In [14, pp. 48-49] there is the assertion that convexity implies quasiconvexity for general set-vector-valued functions. This example shows that this assertion is not adequate even for the single-valued case.

The following example shows that, for vector functions, even strict convexity does not imply quasiconvexity.

Example 4.2. Let $X = A = R, Y = R^2, C = R_+^2$ and $h(x) = (4x^2 + 2x - 1, 4x^2 - 10x + 5) := (h_1(x), h_2(x))$. Observe that both h_1 and h_2 are strictly convex scalar functions. Hence, for all $x_1, x_2 \in R$, all $t \in (0, 1)$ and for $x_t = tx_1 + (1-t)x_2$, one has

$$h(x_t) < th(x_1) + (1-t)h(x_2),$$

i.e., h is strictly convex. But h is not quasiconvex. Indeed, for $x_1 = 0, x_2 = 1, t = \frac{1}{2}$, one has $h(x_1) = (-1, 5), h(x_2) = (5, -1)$ and $h(\frac{1}{2}x_1 + \frac{1}{2}x_2) = (1, 1) \not\leq h(x_1)$ and $\not\leq h(x_2)$.

Definition 4.2. Let $b \in Y$ be fixed. If, for all $x_1, x_2 \in A$, all $t \in (0, 1)$ and for $x_t = (1 - t)x_1 + tx_2$,

- (i) $h(x_1) \not\prec b, h(x_2) \not\prec b$ implies $h(x_t) \not\prec b$, then h is called $\text{lev}_{\not\prec b}$ -convex;
- (ii) $h(x_1) \geq b, h(x_2) > b$ implies $h(x_t) > b$, then h is said to be $\text{lev}_{>b}$ -convex.

Note that, if $Y = C \cup -C$, these two properties coincide. Furthermore, if h is $\text{lev}_{\not\prec b}$ -convex ($\text{lev}_{>b}$ -convex, respectively), then $L_{\not\prec b}h$ ($L_{>b}h$, respectively) is convex, but not vice versa as shown by the following example.

Example 4.3. Let $A = [-2, 2], Y = R, C = R_+$ and $h(x) = \max\{0, x\}$ for $x \in A$. Then $L_{\not\prec 0}h = (0, 2]$ is convex, but h is not $\text{lev}_{\not\prec 0}$ -convex. Indeed, let $x_1 = -1, x_2 = 1$ and $t = \frac{1}{2}$. Then $h(x_1) = 0 \not\prec 0, h(x_2) = 1 \not\prec 0$, but $h(x_t) = h(\frac{1}{2}x_1 + \frac{1}{2}x_2) = h(0) = 0 \leq 0$.

Remark 4.2. If h is strictly quasiconcave, then h is $\text{lev}_{\not\prec b}$ -convex, for all $b \in Y$. Indeed, let $x_1, x_2 \in A$ be such that $h(x_1) \not\prec b, h(x_2) \not\prec b$. Suppose the existence of $c_1 \in C$ such that $h(x_t) - b = -c_1$. Since h is strictly quasiconcave, $h(x_t) > h(x_1)$ or $h(x_t) > h(x_2)$, for all $t \in (0, 1)$. If $h(x_t) > h(x_1)$, there is $c_2 \in \text{int}C$ with $h(x_t) = h(x_1) + c_2$. Hence, $h(x_1) - b = -c_1 - c_2 \in -\text{int}C$, which is impossible as $h(x_1) \not\prec b$. If $h(x_t) > h(x_2)$ we have a similar contradiction.

Example 4.2 shows that $\text{lev}_{\not\prec b}$ -convexity does not follow neither from strict concavity nor from strict convexity. Indeed, we know that $h(x) = (4x^2 + 2x - 1, 4x^2 - 10x + 5)$ is strictly convex (and $-h$ is strictly concave). For $b = 0, x_1 = 0, x_2 = 1$ and $t = \frac{1}{2}$, we have $h(x_1) = (-1, 5) \not\prec 0, h(x_2) = (5, -1) \not\prec 0$, but $h(x_{\frac{1}{2}}) = (1, 1) \leq 0$. Similarly, $-h$ is not $\text{lev}_{\not\prec b}$ -convex either. The next example indicates that neither quasiconcavity nor quasiconvexity implies $\text{lev}_{\not\prec b}$ -convexity.

Example 4.4. Let $X = Y = R, A = R, C = R_+$ and

$$h(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ x - 1 & \text{otherwise.} \end{cases}$$

Then, h is both quasiconvex and quasiconcave. But h is not $\text{lev}_{\not\prec 0}$ -convex, since for $x_1 = 0, x_2 = 2$, one has $h(x_1) = 0 \not\prec 1, h(x_2) = 1 \not\prec 1$, but $h(x_{\frac{1}{2}}) = 0 \not\prec 1$.

Remark 4.3. If h is strictly quasiconcave or concave, then h is $\text{lev}_{>b}$ -convex for all $b \in Y$. Indeed, assume that h is strictly quasiconcave and $x_1, x_2 \in A$ such that $h(x_1) \geq b, h(x_2) > b$. For any $t \in (0, 1)$, if $h(x_t) > h(x_1)$, there is $c_1 \in \text{int}C$ with $h(x_t) - h(x_1) = c_1$. Since $h(x_1) \geq b$, there is $c_2 \in C$ with $h(x_1) = b + c_2$. Consequently, $h(x_t) - b = c_1 + c_2 \in \text{int}C$, i.e., $h(x_t) > b$. Similarly, $h(x_t) > b$ if $h(x_t) > h(x_2)$. Thus, h is $\text{lev}_{>b}$ -convex. Now assume that h is concave and $x_1, x_2 \in A$ such that $h(x_1) \geq b, h(x_2) > b$. Then, there

is $c_1 \in C$ and $c_2 \in \text{int}C$ such that $h(x_1) = b + c_1, h(x_2) = b + c_2$. Since h is concave, there exists $c_3 \in C$ such that $h(x_t) - th(x_1) - (1-t)h(x_2) = c_3$. Hence, $h(x_t) - b = tc_1 + (1-t)c_2 + c_3 \in \text{int}C$. Thus, $h(x_t) > b$.

Example 4.4 shows also that neither quasiconcavity nor quasiconvexity yields $\text{lev}_{>b}$ -convexity, since in this case $\text{lev}_{\leq b}$ -convexity coincides with $\text{lev}_{>b}$ -convexity.

5. WELL-POSEDNESS FOR EQUILIBRIUM PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

Let X, Y and Z be Hausdorff topological vector spaces. $K \subseteq X$ be a nonempty convex subset, $C \subseteq Y$ and $D \subseteq Z$ are solid pointed convex cones, $f : K \times K \rightarrow Y$ and $g : K \times K \rightarrow Z$. Consider the following two equilibrium problems with equilibrium constraints

(WEPEC) Find $\bar{x} \in S_1^w$ such that, for all $y \in S_1^w$,

$$g(\bar{x}, y) \not\leq 0,$$

where $S_1^w = \{x \in K : f(x, y) \not\leq 0, \forall y \in K\}$;

(SEPEC) Find $\bar{x} \in S_1^s$ such that, for all $y \in S_1^s$,

$$g(\bar{x}, y) \geq 0,$$

where $S_1^s = \{x \in K : f(x, y, \lambda) \geq 0, \forall y \in K\}$.

These problems can be expressed otherwise as rewritten below (we write for (WEPEC) only)

(WEPEC) Find the value $x = \bar{x}$ such that

$$g(x, y) \not\leq 0,$$

$$f(x, z) \not\leq 0, f(y, z) \not\leq 0,$$

$$z \in K.$$

Assume that the perturbations upon these problems can be expressed in terms of a perturbing parameter λ which varies in a Hausdorff topological space Λ in the way that we can embed our two problems into the corresponding parametric families as follows. The constraint set K depends on λ as a multimap $K : A \times \Lambda \rightarrow 2^A$, where $A \subseteq X$ is nonempty. The maps f and g now depend on λ . The parametric families have the form:

(WEPEC $_{\lambda}$) Find $\bar{x} \in S_1^w(\lambda)$ such that, for all $y \in S_1^w(\lambda)$,

$$g(\bar{x}, y, \lambda) \not\leq 0,$$

where $S_1^w(\lambda) = \{x \in K(\lambda) : f(x, y, \lambda) \not\leq 0, \forall y \in K(\lambda)\}$.

(SEPEC $_{\lambda}$) Find $\bar{x} \in S_1^s(\lambda)$ such that, for all $y \in S_1^s(\lambda)$,

$$g(\bar{x}, y, \lambda) \geq 0,$$

where $S_1^s(\lambda) = \{x \in K(\lambda) : f(x, y, \lambda) \geq 0, \forall y \in K(\lambda)\}$.

Assume further that the original problems are corresponding to $\bar{\lambda}$. We denote the solutions sets of (WEPEC $_{\lambda}$) and (SEPEC $_{\lambda}$) by $S_2^w(\lambda)$ and $S_2^s(\lambda)$, respectively. As auxiliary problems we use also the following "strict" versions of the constraints S_1^w and S_1^s , respectively, for $\lambda \in \Lambda$,

$$S_0^w(\lambda) = \{x \in K(\lambda) : f(x, y, \lambda) \not\leq 0, \forall y \in K(\lambda)\},$$

$$S_0^s(\lambda) = \{x \in K(\lambda) : f(x, y, \lambda) > 0, \forall y \in K(\lambda)\}.$$

We first establish a sufficient condition for the well-posedness of (WEPEC).

Theorem 5.1. *Assume for (WEPEC) that $S_2^w(\bar{\lambda})$, $S_0^w(\bar{\lambda})$ are nonempty and*

- (i) *K is continuous at $\bar{\lambda}$ and $K(\bar{\lambda})$ is convex and compact;*
- (ii) *g is C -usc in $A \times A \times \{\bar{\lambda}\}$;*
- (iii) *in $A \times A \times \{\bar{\lambda}\}$, f is 0-level C -qlsc and 0-level C -usc;*
- (iv) *for all $y \in A$, $L_{\not\leq 0}f(\cdot, y, \bar{\lambda})$ is convex and $f(\cdot, y, \bar{\lambda})$ is $\text{lev}_{\not\leq 0}$ -convex.*

Then, (WEPEC) is well-posed under perturbations. Furthermore, if the solution set $S_2^w(\bar{\lambda})$ is a singleton, then (WEPEC) is uniquely well-posed under perturbations.

Proof. To apply Theorem 3.1 we need to verify only assumption (ii₂) with $K_1(\cdot) = K_2(\cdot) = S_1^w(\cdot) = E(\cdot)$. Let us show first the compactness of S_1^w . As $K(\bar{\lambda})$ is compact, we have to show only the closedness. Let $x_{\alpha} \in S_1^w$ be such that $x_{\alpha} \rightarrow x$. For each $y \in K(\bar{\lambda})$, we have

$$f(x_{\alpha}, y, \bar{\lambda}) \not\leq 0.$$

By the closedness of $L_{\not\leq 0}f(\cdot, y, \bar{\lambda})$, $f(x_0, y, \bar{\lambda}) \not\leq 0$, i.e., $x \in S_1^w$, and hence S_1^w is closed.

Next we prove that $S_1^w(\cdot)$ is usc at $\bar{\lambda}$. Suppose the existence of nets $\{\lambda_{\alpha}\}$ with $\lambda_{\alpha} \rightarrow \bar{\lambda}$ and $\{x_{\alpha}\}$ with $x_{\alpha} \in S^w(\lambda_{\alpha})$, of a superset U of $S^w(\bar{\lambda})$ such that $x_{\alpha} \notin U$,

for all α . By the upper semicontinuity of $K(\cdot)$ and the compactness of $K(\bar{\lambda})$ one can assume that $x_\alpha \rightarrow x$, for some $x \in K(\bar{\lambda})$. If $x \notin S_1^w(\bar{\lambda})$, there is $y \in K(\bar{\lambda})$ such that $f(x, y, \bar{\lambda}) < 0$. Since K is lsc at $\bar{\lambda}$, there exists $y_\alpha \in K(\lambda_\alpha)$ such that $y_\alpha \rightarrow y$. Since $x_\alpha \in S_1^w(\lambda_\alpha)$,

$$f(x_\alpha, y_\alpha, \lambda_\alpha) \not\leq 0.$$

As f is 0-level C -usc, $f(x, y, \bar{\lambda}) \not\leq 0$, which is impossible. Thus, $x \in S_1^w(\bar{\lambda}) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$, for all α .

To verify now the lower semicontinuity of $S_1^w(\cdot)$ we prove first that $S_0^w(\cdot)$ is lsc at $\bar{\lambda}$. Suppose to the contrary that there exist $x_0 \in S_0^w(\bar{\lambda})$ and $\lambda_\alpha \rightarrow \bar{\lambda}$ such that, for all $x_\alpha \in S_0^w(\lambda_\alpha)$, $x_\alpha \not\rightarrow x_0$. Since K is lsc at $\bar{\lambda}$, there exists $x_\alpha \in K(\lambda_\alpha)$ such that $x_\alpha \rightarrow x_0$. By the above contradiction assumption, we can assume that $x_\alpha \notin S_0^w(\lambda_\alpha)$ for all α , i.e., for some $y_\alpha \in K(\lambda_\alpha)$,

$$f(x_\alpha, y_\alpha, \lambda_\alpha) \leq 0. \quad (1)$$

By assumption (i) we can assume that $y_\alpha \rightarrow y_0$ for some $y_0 \in K(\bar{\lambda})$. The closedness of $L_{\leq 0}f$ and (1) together yield that $f(x_0, y_0, \bar{\lambda}) \leq 0$, which is impossible since $x_0 \in S_0^w(\bar{\lambda})$.

Now let us check that

$$S_1^w(\bar{\lambda}) \subseteq \text{cl}S_0^w(\bar{\lambda}). \quad (2)$$

Let $x_1 \in S_1^w(\bar{\lambda})$, $x_0 \in S_0^w(\bar{\lambda})$, $t \in (0, 1)$ and $x_t = (1-t)x_1 + tx_0$. Since $K(\bar{\lambda})$ is convex, $x_t \in K(\bar{\lambda})$. We show that $x_t \in S_0^w(\bar{\lambda})$. For each $y \in K(\bar{\lambda})$, we have $f(x_0, y, \bar{\lambda}) \leq 0$ and $f(x_1, y, \bar{\lambda}) \not\leq 0$. Since $f(\cdot, y, \bar{\lambda})$ is $\text{lev}_{\leq 0}$ -convex,

$$f(x_t, y, \bar{\lambda}) \leq 0,$$

i.e., $x_t \in S_0^w(\bar{\lambda})$. Therefore (2) holds. By the lower semicontinuity of S_0^w at $\bar{\lambda}$, if $\lambda_\alpha \rightarrow \bar{\lambda}$ we have

$$S_1^w(\bar{\lambda}) \subseteq \text{cl}S_0^w(\bar{\lambda}) \subseteq \liminf S_0^w(\lambda_\alpha) \subseteq \liminf S_1^w(\lambda_\alpha),$$

i.e., S_1^w is lsc at $\bar{\lambda}$.

Finally, applying Theorem 3.1 completes the proof. \blacksquare

Since the assumptions of Theorem 5.1 look complicated we provide now examples showing their essentialness.

Example 5.1. ((i) is essential). Let $X = Y = R$, $A = [-1, 1]$, $\Lambda = [0, 1]$, $C = R_+$, $\bar{\lambda} = 0$,

$$K(\lambda) = \begin{cases} [-1, 1] & \text{if } \lambda = 0, \\ [0, 1] & \text{otherwise,} \end{cases}$$

and $f(x, y, \lambda) = x + y + 2$, $g(x, y, \lambda) = x + y + \lambda$. It is clear that $K(\lambda)$ is usc at 0, $K(0)$ is compact, assumptions (ii)-(iv) hold since f and g are continuous and $f(\cdot, y, \lambda)$ is convex. We have $S_1^w(0) = [-1, 1]$, $S_2^w(0) = \{1\}$ and, for all $\lambda \in (0, 1]$, $S_1^w(\lambda) = [0, 1] = S_2^w(\lambda)$ ($S_2^w(\lambda)$ is the solution set of (WEPEC $_\lambda$)). Taking $\lambda_n = e_n = \frac{1}{n}$ we see that $x_n = 0$ is an asymptotically solving sequence for (WEPEP) corresponding to λ_n and e_n . But $x_n \rightarrow 0 \notin S_2^w(0)$ and hence (WEPEC) is not well-posed under perturbations. The reason is that (i) is not satisfied, since K is not lsc at 0.

Example 5.2. ((ii) cannot be dropped) Let $X = Y = R$, $A = [-1, 1]$, $\bar{A} = [0, 1]$, $C = R_+$, $\bar{\lambda} = 0$, $K(\lambda) = [-1, 1]$, $f(x, y, \lambda) = x$ and

$$g(x, y, \lambda) = \begin{cases} x - y & \text{if } \lambda = 0, \\ y - x & \text{otherwise.} \end{cases}$$

Assumptions (i), (iii) and (iv) evidently hold. We see that $S_2^w(0) = \{1\}$ and $S_2^w(\lambda) = \{0\}$ for all $\lambda \in (0, 1]$. $x_n = 0$ is clearly an asymptotically solving sequence for (WEPEP) corresponding to $\lambda_n = e_n = \frac{1}{n}$. Then, (WEPEC) is not well-posed under perturbations as $x_n \rightarrow 0 \notin S_2^w(0)$. The reason is that (ii) is violated. Indeed, let $x_n = 0$, $y_n = 1$ and $\lambda_n = \frac{1}{n}$. Then, $(x_n, y_n, \lambda_n) \rightarrow (0, 1, 0)$ and $g(x_n, y_n, \lambda_n) = 1 \geq 0$, but $g(0, 1, 0) = -1 < 0$.

Example 5.3. ((iii) cannot be dispensed) Let $X = Y = R$, $A = [-1, 1]$, $\bar{A} = [0, 1]$, $C = R_+$, $\bar{\lambda} = 0$, $K(\lambda) = [-1, 1]$, $g(x, y, \lambda) = 1$ and

$$f(x, y, \lambda) = \begin{cases} -1 & \text{if } x \leq 0, \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that assumptions (i), (ii) and (iv) hold. We see that $S_2^w(\lambda) = (0, 1]$ for all $\lambda \in [0, 1]$, and $x_n = \frac{1}{n}$ is an asymptotically solving sequence for (WEPEP) corresponding to $\lambda_n = e_n = \frac{1}{n}$. But $x_n \rightarrow 0 \notin S_2^w(0)$ and hence (WEPEC) is not well-posed under perturbations. This is not surprising since (iii) is not fulfilled. Indeed, let $x_n = \frac{1}{n}$, $y_n = 1$ and $\lambda_n = \frac{1}{n}$. Then, $(x_n, y_n, \lambda_n) \rightarrow (0, 1, 0)$ and $f(x_n, y_n, \lambda_n) = 1 \geq 0$, but $f(0, 1, 0) = -1 < 0$, and hence f is not $(0, C)$ -usc at $(0, 1, 0)$.

Example 5.4. ((iv) is essential) Let $X = Y = R$, $A = [-1, 3]$, $\bar{A} = [0, 1]$, $C = R_+$, $\bar{\lambda} = 0$, $K(\lambda) = [-1, 3]$, $f(x, y, \lambda) = -x(x^2 - 1 - \lambda)$ and $g(x, y, \lambda) = x + y + \lambda$. Assumptions (i)-(iv) are apparently fulfilled. We easily compute that $S_1^w(0) = \{-1\} \cup [0, 1]$, $S_2^w(0) = \{1\}$ and, for all $\lambda \in (0, 1]$, $S_1^w(\lambda) = [0, \sqrt{1 + \lambda}]$ and $S_2^w(\lambda) = [0, \sqrt{1 + \lambda}]$. Taking $\lambda_n = e_n = \frac{1}{n}$ one sees that $x_n = 0$ is an asymptotically solving sequence for (WEPEP) corresponding to $\lambda_n = e_n = \frac{1}{n}$. But

$x_n \rightarrow 0 \notin S_2^w(0)$, whence (WEPEC) is not well-posed under perturbations. The reason is that (iv) is violated. Indeed, taking $x_1 = -1, x_2 = \frac{1}{2}, t = \frac{2}{3}$ and $y = 0$, we see that $f(x_1, y, 0) = 0, f(x_2, y, 0) = \frac{3}{8}$, but $f(x_t, y, 0) = f(-\frac{1}{2}, y, 0) = -\frac{3}{8} < 0$.

Theorem 5.2. *Assume for (SEPEC) that $S_2^s(\bar{\lambda}), S_0^s(\bar{\lambda})$ are nonempty and assume further that*

- (i) K is continuous at $\bar{\lambda}$ and $K(\bar{\lambda})$ is convex and compact;
- (ii) g is C -quasi in $A \times A \times \{\bar{\lambda}\}$;
- (iii) in $A \times A \times \{\bar{\lambda}\}$, f is 0-level C -lsc and 0-level C -quasi;
- (iv) for all $y \in A$, $f(\cdot, y, \bar{\lambda})$ is $\text{lev}_{>0}$ -convex.

Then, (SEPEC) is well-posed under perturbations. Furthermore, if the solution set $S_2^s(\bar{\lambda})$ is a singleton, then (SEPEC) is uniquely well-posed under perturbations.

Proof. We have to verify only assumption (ii₂) of Theorem 3.2 in order to apply it. First, the compactness of S_1^s is checked similarly as that of S_1^w , but using the 0-level C -quasi upper semicontinuity of f . Next, the upper semicontinuity of S_1^s is verified analogously as the counterpart of S_1^w . For the lower semicontinuity of S_1^s , we show first this property for the map S_0^s . Then, via proving the inclusion

$$S_1^s(\lambda) \subseteq \text{cl}S_0^s(\lambda)$$

by using the assumed $\text{lev}_{>0}$ -convexity of $f(\cdot, y, \bar{\lambda})$ we get the required semicontinuity. ■

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