TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 2A, pp. 393-402, April 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ISOMORPHIC PATH DECOMPOSITIONS OF $\lambda K_{n,n,n}$ ($\lambda K^*_{n,n,n}$) FOR ODD n

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Abstract. In this paper, the isomorphic path decompositions of λ-fold balanced complete tripartite graphs $\lambda K_{n,n,n}$ and λ-fold balanced complete tripartite digraphs $\lambda K^*_{n,n,n}$ are investigated for odd n. We prove that the obvious necessary conditions for such decompositions in the undirected case are also sufficient; we also provide sufficient conditions for the directed case.

1. Introduction and Preliminaries

Let G and H be multigraphs. If there exist edge-disjoint subgraphs $H_1, H_2, \cdots, H_r$ of G such that every edge of G appears in some H_i, and each H_i ($i = 1, 2, \cdots, r$) is isomorphic to H, then we say that G has an H-*decomposition*. For multidigraphs G and H, H-decomposition of G is similarly defined. The H-decomposition problems of a multigraph G are widely investigated when G is a complete graph or a complete r-partite graph and H is a path or a cycle.

For a multigraph G, we use the symbol G^* to denote the multidigraph obtained from G by replacing each edge e by two opposite arcs connecting the endvertices of e. Let λ be a positive integer. For a multigraph H, we use the symbol λH to denote the multigraph obtained from H by replacing each edge e by λ edges each of which has the same endvertices as e. Similarly, for a multidigraph H, we use the symbol λH to denote the multidigraph obtained from H by replacing each arc e by λ arcs each of which has the same tail and head as e.

For a positive integer k, let P_k denote a path on k vertices and let $\overrightarrow{P_k}$ denote a directed path on k vertices.

Received October 16, 2006, accepted August 23, 2007.
Communicated by Xu-Ding Zhu.
2000 *Mathematics Subject Classification*: 05C38, 05C70.
Key words and phrases: Decomposition, Path, Complete tripartite graph.
This research was supported by the NSC of R.O.C. under grant 94-2115-M-008-014.
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Let K_n denote the complete graph on n vertices. Tarsi [6] established criteria for P_k-decompositions of λK_n. Recently Meszka and Skupień [4] solved the $\overrightarrow{P_k}$-decomposition problem of λK_n^*.

Let $K_{m_1,m_2,\cdots,m_r}$ denote the complete r-partite graph with parts of sizes $m_1, m_2, \cdots, m_r$, respectively. In [7] Truszczyński solved the $\overrightarrow{P_k}$-decomposition problem of $\lambda K_{m,n}^*$, and considered the P_k-decomposition of $\lambda K_{m,n}$. The P_k-decomposition problem of $K_{m,n}$ was completely solved by Parker [5]. The condition for P_4-decomposition of $K_{m_1,m_2,\cdots,m_r}$ was obtained by Kumar [2].

In this paper, we consider the P_k-decomposition of $\lambda K_{n,n,n}$ and the $\overrightarrow{P_k}$-decomposition of $\lambda K_{n,n,n}^*$. For a multigraph (multidigraph, respectively) G, we also use $E(G)$ to denote the edge set (arc set, respectively) of G. We will obtain the following results.

Theorem A. *Let n be an odd integer. Then $\lambda K_{n,n,n}$ has a P_k-decomposition if and only if $2 \leq k \leq 3n$ and $|E(\lambda K_{n,n,n})| \equiv 0 \pmod{k-1}$.*

Theorem B. *Let $n \geq 3$ be an odd integer. Suppose that k is an integer such that $2 \leq k \leq 3n-1$ and $|E(\lambda K_{n,n,n}^*)| \equiv 0 \pmod{k-1}$. Then $\lambda K_{n,n,n}^*$ has a $\overrightarrow{P_k}$-decomposition.*

For our discussions we need the following notations and terms. Let G be a multigraph. Suppose that W_1 is a walk $v_0v_1\cdots v_k$ and W_2 is a walk $v_kv_{k+1}\cdots v_l$ in G. Then the sum of W_1 and W_2, denoted by W_1+W_2, is a walk $v_0v_1\cdots v_kv_{k+1}\cdots v_l$. Suppose that W is a walk $v_0v_1\cdots v_k$ in G (no matter W is closed or not). The $girth$ of W, denoted by $g(W)$, is the minimum number of edges between two appearances of the same vertex along W, i.e., the minimum of $j-i$ such that $v_i = v_j$ where $0 \leqslant i < j \leqslant k$. A $trail$ is a walk without repeated edges. An $Euler\ trail$ of G is a trail in G which traverses every edge of G. For multidigraphs, the following terms are similarly defined: the sum of directed walks, the girth of a directed walk, the directed trail, and the directed Euler trail.

In [6] Tarsi obtained the path decomposition of λK_n by cutting Euler trails into paths. We state the result of cutting method in the following remark. This remark was henceforth used in many papers, e.g. [4, 5, 7].

Remark 1.1. Suppose that a multigraph $(multidigraph,\ respectively)$ G contains an Euler trail $(a\ directed\ Euler\ trail,\ respectively)$ with girth g, and that for $i = 1, 2, \cdots, r$, k_i is an integer such that $2 \leq k_i \leq g$ and $|E(G)| = k_1+k_2+\cdots+k_r - r$. Then G can be decomposed into r paths $(directed\ paths,\ respectively)$ on $k_1, k_2, \cdots, k_r$ vertices, respectively. ■

Letting $k_1 = k_2 = \cdots = k_r = k$ in the above remark, we have the following.

Remark 1.2. Suppose that a multigraph $(multidigraph, respectively)$ G contains an Euler trail $(a\ directed\ Euler\ trail,\ respectively)$ with girth g, and that k is an integer such that $2 \leq k \leq g$ and $|E(G)| \equiv 0 \pmod{k-1}$. Then G has a P_k-decomposition $(\overrightarrow{P_k} - decomposition,\ respectively)$. ■

2. Path Decompositions of $\lambda K_{n,n,n}$ for ODD n

In this section, we investigate the P_k-decomposition of $\lambda K_{n,n,n}$ for odd n. For a multigraph G, and nonempty subsets A, B of $V(G)$ with $A \cap B = \emptyset$, we use $G(A, B)$ to denote the set of all edges in G which have one end in A and the other end in B. We begin with some lemmas.

Lemma 2.1. *Let $n \geq 3$ be an odd integer. Then*

(1) $K_{n,n,n}$ *has an Euler trail with girth* $3n-6$,

(2) $\lambda K_{n,n,n}$ *has an Euler trail with girth* $3n-3$ *if* $\lambda \geq 2$.

Proof. For $\lambda = 1, 2, 3, \cdots$, let (A, B, C) be the tripartition of $\lambda K_{n,n,n}$ where $A = \{a_0, a_1, \cdots, a_{n-1}\}$, $B = \{b_0, b_1, \cdots, b_{n-1}\}$ and $C = \{c_0, c_1, \cdots, c_{n-1}\}$.

An edge joining a_i and b_{i+k} $(i = 0, 1, \cdots, n-1; k = 0, 1, \cdots, n-1)$ where the indices are taken modulo n is said to be *an edge between A and B with label k.* Similarly an edge joining b_i and c_{i+k} is said to be *an edge between B and C with label k,* and an edge joining c_i and a_{i+k} is said to be *an edge between C and A with label k.*

(1) Let $\lambda = 1$. For each $i = 0, 1, 2, \cdots, n-1$, let D_i be the following walk in $K_{n,n,n}$: $a_0 b_i c_{2i} a_1 b_{i+1} c_{2i+1} a_2 b_{i+2} c_{2i+2} \cdots a_{n-1} b_{i+n-1} c_{2i+n-1} a_0$ where the indices are taken modulo n. Note that each D_i consists of all edges between A and B with label i, all edges between B and C with label i, and all edges between C and A with label $(1-2i) \pmod n$. Thus $K_{n,n,n}(A, B)$ is a disjoint union of $D_0(A, B)$, $D_1(A, B)$,$\cdots$, $D_{n-1}(A, B)$, and $K_{n,n,n}(B, C)$ is a disjoint union of $D_0(B, C)$, $D_1(B, C)$, $\cdots$, $D_{n-1}(B, C)$. Also since n is odd, we have $\{(1-2i) \pmod n : i = 0, 1, 2, \cdots, n-1\} = \{0, 1, 2, \cdots, n-1\}$; thus $K_{n,n,n}(C, A)$ is a disjoint union of $D_0(C, A)$, $D_1(C, A)$, $\cdots$, $D_{n-1}(C, A)$. Hence $E(K_{n,n,n})$ is a disjoint union of $E(D_0)$, $E(D_1)$,$\cdots$, $E(D_{n-1})$. Let T be the walk $D_0 + D_1 + \cdots + D_{n-1}$. We thus see that T is an Euler trail in $K_{n,n,n}$.

Now we evaluate $g(T)$. Note that each D_i is a Hamiltonian cycle of $K_{n,n,n}$. Let $i = 0, 1, \cdots, n-2$. Then $D_i + D_{i+1}$ is the trail $a_0 b_i c_{2i}\ a_1 b_{i+1} c_{2i+1}\ a_2 b_{i+2} c_{2i+2}\ \cdots\ a_{n-1} b_{i+n-1}\ c_{2i+n-1}\ a_0 b_{i+1} c_{2i+2}\ a_1 b_{i+2} c_{2i+3}\ \cdots\ a_{n-1} b_i\ c_{2i+1}\ a_0$. In $D_i + D_{i+1}$, there are $3n-6$ edges between two appearances of

c_j if $j = 0, 1, 2, \cdots, 2i-1, 2i+2, 2i+3, \cdots, n-1$, and more than $3n-6$ edges between two appearances of any other vertex. Thus $g(D_i + D_{i+1}) = 3n - 6$. Hence $g(T) = 3n - 6$, and T is a required Euler trail of $K_{n,n,n}$.

(2) Let $\lambda \geq 2$. For $i = 0, 1, 2, \cdots, n-1$, let D_i be, as in the proof of (1), the trail: $a_0 b_i c_{2i} a_1 b_{i+1} c_{2i+1} a_2 b_{i+2} c_{2i+2} \cdots a_{n-1} b_{i+n-1} c_{2i+n-1} a_0$, and let E_i be the trail: $a_0 b_{i+1} c_{2i+1} a_1 b_{i+2} c_{2i+2} a_2 b_{i+3} c_{2i+3} \cdots a_{n-1} b_{i+n} c_{2i+n} a_0$ where the indices are taken modulo n.

Let $G = K_{n,n,n}$ be a subgraph of $\lambda K_{n,n,n}$. As in (1), $E(G)$ is a disjoint union of $E(D_0)$, $E(D_1)$,$\cdots$, $E(D_{n-1})$. Note also that each E_i consists of all edges between A and B with label $(i+1)$(mod n), all edges between B and C with label i, and all edges between C and A with label $(-2i)$(mod n). By similar arguments as in (1), $E(G)$ is a disjoint union of $E(E_0)$, $E(E_1)$,$\cdots$, $E(E_{n-1})$. Let T be the following trail:

$$\underbrace{D_0 + D_0 + \cdots + D_0}_{\lambda-1 \text{ copies of } D_0} + E_0 + \underbrace{D_1 + D_1 + \cdots + D_1}_{\lambda-1 \text{ copies of } D_1} + E_1 + \cdots\cdots\cdots$$

$$+ \underbrace{D_{n-2} + D_{n-2} + \cdots + D_{n-2}}_{\lambda-1 \text{ copies of } D_{n-2}} + E_{n-2} + \underbrace{D_{n-1} + D_{n-1} + \cdots + D_{n-1}}_{\lambda-1 \text{ copies of } D_{n-1}} + E_{n-1}.$$

Then T is an Euler trail of $\lambda K_{n,n,n}$. To determine $g(T)$, we show in the following that (i) $g(D_i + D_i) = 3n$ for $i = 0, 1, \cdots, n-1$, (ii) $g(D_i + E_i) = 3n-3$ for $i = 0, 1, \cdots, n-1$, and (iii) $g(E_i + D_{i+1}) = 3n-3$ for $i = 0, 1, \cdots, n-2$. Note that both D_i and E_i are Hamiltonian cycles in $\lambda K_{n,n,n}$.

(i) This is trivial.

(ii) Let $i = 0, 1, \cdots, n-1$. We see that $D_i + E_i$ is the trail $a_0 b_i c_{2i}\ a_1 b_{i+1} c_{2i+1}\ a_2 b_{i+2} c_{2i+2} \cdots a_{n-1}\ b_{i+n-1} c_{2i+n-1}\ a_0 b_{i+1} c_{2i+1}\ a_1 b_{i+2} c_{2i+2}\ a_2 b_{i+3} c_{2i+3} \cdots a_{n-1} b_i c_{2i} a_0$. In $D_i + E_i$, there are $3n-3$ edges between two appearances of b_j if $j = 0, 1, 2, \cdots, i-1, i+1, i+2, \cdots, n-1$, and of c_j if $j = 0, 1, 2, \cdots, 2i-1, 2i+1, 2i+2, \cdots, n-1$, and there are more than $3n-3$ edges between two appearances of any other vertex. Thus $g(D_i + E_i) = 3n-3$.

(iii) Let $i = 0, 1, \cdots, n-2$. We see that $E_i + D_{i+1}$ is the trail $a_0 b_{i+1} c_{2i+1}\ a_1 b_{i+2} c_{2i+2}\ a_2 b_{i+3} c_{2i+3} \cdots a_{n-1} b_i c_{2i}\ a_0 b_{i+1} c_{2i+2}\ a_1 b_{i+2} c_{2i+3}\ a_2 b_{i+3} c_{2i+4} \cdots a_{n-1}\ b_i\ c_{2i+1}\ a_0$. In $E_i + D_{i+1}$ there are $3n-3$ edges between two appearances of c_j if $j = 0, 1, 2, \cdots, 2i, 2i+2, 2i+3, \cdots, n-1$, and more than $3n-3$ edges between two appearances of any other vertex. Thus $g(E_i + D_{i+1}) = 3n-3$.

From (i), (ii) and (iii), we obtain $g(T) = 3n-3$. Thus T is a required Euler trail of $\lambda K_{n,n,n}$. ∎

Lemma 2.2. *Let G be a graph of order t such that G can be decomposed into Hamiltonian cycles. Suppose that λ and k are integers with $2 \le k \le t$ and $(k-1)|\lambda t$. Then λG has a P_k-decomposition.*

Proof. Suppose that G is decomposed into Hamiltonian cycles $H_1, H_2, \cdots, H_v$. Then λG is decomposed into $\lambda H_1, \lambda H_2, \cdots, \lambda H_v$. Since $k \le t$, $(k-1)|\lambda t$, and λH_i $(1 \le i \le v)$ has an Euler trail with girth t, each λH_i has a P_k-decomposition. Thus λG has a P_k-decomposition. ∎

In the proof of (1) in Lemma 2.1, we see that if $n \ge 3$ is an odd integer, then $K_{n,n,n}$ can be decomposed into Hamiltonian cycles $D_0, D_1, \cdots, D_{n-1}$. More generally, Laskar and Auerbach [3] proved that the complete m-partite graph $K_{n,n,\cdots,n}$ can be decomposed into Hamiltonian cycles if and only if $(m-1)n$ is even. Thus $K_{n,n,n}$ can be decomposed into Hamiltonian cycles for any positive integer n. We are ready to prove the main result of this section.

Theorem A. *Let n be an odd integer. Then $\lambda K_{n,n,n}$ has a P_k-decomposition if and only if $2 \le k \le 3n$ and $|E(\lambda K_{n,n,n})| \equiv 0 \pmod{k-1}$.*

Proof. The necessity is trivial. Now we prove the sufficiency.

The case $n = 1$ is trivial. We assume that $n \ge 3$. By the assumptions, k is an integer with $2 \le k \le 3n$ and $|E(\lambda K_{n,n,n})| \equiv 0 \pmod{k-1}$ (i.e., $(k-1)|3\lambda n^2$). We distinguish two cases for $\lambda = 1$ and $\lambda \ge 2$.

Case 1. $\lambda = 1$.

By Lemma 2.1(1), $K_{n,n,n}$ has an Euler trail with girth $3n-6$. Hence by Remark 1.2, $K_{n,n,n}$ has a P_k-decomposition if $k \le 3n-6$. So we only need to consider $3n-5 \le k \le 3n$. Since n is odd and $(k-1)|3n^2$, we have that k is even. So it remains to consider the following subcases: $k = 3n-5$, $3n-3$, $3n-1$.

Subcase 1.1. $k = 3n-5$**.**

From the assumption that $(3n-6)|3n^2$, we have $(n-2)|n^2$, which implies $(n-2)|4$ for $4 = n^2-(n+2)(n-2)$. This implies $n-2 = 1$ since n is odd. Thus $n = 3$ and $k = 4$. As mentioned in the paragraph preceding this theorem, $K_{3,3,3}$ can be decomposed into Hamiltonian cycles. Then by Lemma 2.2, $K_{3,3,3}$ has P_4-decomposition. This completes Subcase 1.1.

Subcase 1.2. $k = 3n-3$**.**

From the assumption that $(3n-4)|3n^2$, we have $(3n-4)|16$ for $16 = 3\cdot 3n^2 - (3n+4)(3n-4)$. This is impossible since n is odd.

Subcase 1.3. $k = 3n-1$**.**

From the assumption that $(3n-2)|3n^2$, we have $(3n-2)|4$ since $4 = 3\cdot 3n^2 - (3n+2)(3n-2)$. Thus $n = 1$ since n is odd. This is a contradiction since we assumed that $n \ge 3$.

Case 2. $\lambda \geq 2$.

By Lemma 2.1(2), $\lambda K_{n,n,n}$ has an Euler trail with girth $3n-3$. Hence $\lambda K_{n,n,n}$ has a P_k-decomposition if $k \leq 3n-3$. So we only need to consider $k = 3n-2, 3n-1, 3n$. We first show that $(k-1)|3\lambda$ for these k.

Subcase 2.1. $k = 3n-2$.

From the assumption $(3n-3)|3\lambda n^2$, we have $(n-1)|\lambda n^2$, which implies $(n-1)|\lambda$ since $\gcd(n-1,n) = 1$. Thus $3(n-1)|3\lambda$ (i.e., $(k-1)|3\lambda$).

Subcase 2.2. $k = 3n-1$.

Since n is odd, it is easy to see that $\gcd(3n-2,n) = 1$, and hence $\gcd(3n-2,n^2) = 1$. Thus the assumption $(3n-2)|3\lambda n^2$ implies $(3n-2)|3\lambda$ (i.e., $(k-1)|3\lambda$).

Subcase 2.3. $k = 3n$.

It is trivial that $\gcd(3n-1,n) = 1$, and hence $\gcd(3n-1,n^2) = 1$. Thus the assumption $(3n-1)|3\lambda n^2$ implies $(3n-1)|3\lambda$ (i.e., $(k-1)|3\lambda$).

Now we have that $K_{n,n,n}$ has order $3n$ and can be decomposed into Hamiltonian cycles, and that $k \leq 3n$, $(k-1)|\lambda \cdot 3n$. Thus by Lemma 2.2, $\lambda K_{n,n,n}$ has a P_k-decomposition. This completes Case 2. ∎

3. Directed Path Decompositions of $\lambda K^*_{n,n,n}$ for ODD n

In this section, we investigate the $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n,n}$ for odd n. Let us begin with $n = 1$. First the result for the decomposition of λK^*_n into directed Hamiltonian paths is the following [1, 4]: λK^*_n can be decomposed into directed Hamiltonian paths if and only if neither $n = 3$ and λ is odd nor $n = 5$ and $\lambda = 1$. It follows from the case $n = 3$ that λK^*_3 has a $\overrightarrow{P_3}$-decomposition if and only if λ is even. Thus we can see that $\lambda K^*_{1,1,1} = \lambda K^*_3$ has a $\overrightarrow{P_k}$-decomposition if and only if either $k = 2$ or $k = 3$ and λ is even.

Remark 3.1. If a multigraph G has a P_k-decomposition, then G^* has a $\overrightarrow{P_k}$-decomposition. ∎

For a multidigraph G and nonempty subsets A, B of $V(G)$ with $A \cap B = \emptyset$, let $G(A,B)$ denote the set of all arcs of G which have their tails in A and their heads in B.

Lemma 3.2. *Let $n \geq 3$ be an odd integer. Then $\lambda K^*_{n,n,n}$ has a directed Euler trail with girth $3n-4$.*

Proof. Let (A,B,C) be the tripartition of $\lambda K^*_{n,n,n}$ where $A = \{a_0, a_1, \cdots, a_{n-1}\}$, $B = \{b_0, b_1, \cdots, b_{n-1}\}$ and $C = \{c_0, c_1, \cdots, c_{n-1}\}$.

An arc joining a_i to b_{i+k} $(i = 0, 1, \cdots, n-1, k = 0, 1, \cdots, n-1)$ where the indices are taken modulo n is said to be *an arc from A to B with label k*. An arc from B to C with label k and an arc from C to A with label k are similarly defined.

For $i = 0, 1, 2, \cdots, n-1$, let $\overrightarrow{D}_i$ be the directed trail: $a_0 \to b_i \to c_{2i} \to a_1 \to b_{i+1} \to c_{2i+1} \to a_2 \to b_{i+2} \to c_{2i+2} \to \cdots\cdots \to a_{n-1} \to b_{i+n-1} \to c_{2i+n-1} \to a_0$, and let $\overrightarrow{F}_i$ be the directed trail: $a_0 \to c_{2i+1} \to b_{i+1} \to a_1 \to c_{2i+2} \to b_{i+2} \to a_2 \to c_{2i+3} \to b_{i+3} \to \cdots\cdots \to a_{n-1} \to c_{2i+n} \to b_{i+n} \to a_0$ where the indices are taken modulo n.

Let $G = K^*_{n,n,n}$ be a subgraph of $\lambda K^*_{n,n,n}$. Note that each $\overrightarrow{D_i}$ consists of the following arcs in G: all arcs from A to B with label i, all arcs from B to C with label i, and all arcs from C to A with label $(1-2i)$(mod n). Thus $G(A, B)$ is a disjoint union of $\overrightarrow{D_0}(A, B)$, $\overrightarrow{D_1}(A, B)$, $\cdots$, $\overrightarrow{D}_{n-1}(A, B)$, and $G(B, C)$ is a disjoint union of $\overrightarrow{D_0}(B, C)$, $\overrightarrow{D_1}(B, C)$, $\cdots$, $\overrightarrow{D}_{n-1}(B, C)$. And since n is odd, we have $\{(1-2i)(\text{mod } n) : i = 0, 1, 2, \cdots, n-1\} = \{0, 1, 2, \cdots, n-1\}$; thus $G(C, A)$ is a disjoint union of $\overrightarrow{D_0}(C, A)$, $\overrightarrow{D_1}(C, A)$, $\cdots$, $\overrightarrow{D}_{n-1}(C, A)$. Hence $G(A, B) \cup G(B, C) \cup G(C, A) = E(\overrightarrow{D_0}) \cup E(\overrightarrow{D_1}) \cup \cdots \cup E(\overrightarrow{D}_{n-1})$. By similar arguments, we have $G(A, C) \cup G(C, B) \cup G(B, A) = E(\overrightarrow{F_0}) \cup E(\overrightarrow{F_1}) \cup \cdots \cup E(\overrightarrow{F}_{n-1})$. Therefore $E(G)$ is a disjoint union of $E(\overrightarrow{D_0}), E(\overrightarrow{D_1}), \cdots, E(\overrightarrow{D}_{n-1})$, $E(\overrightarrow{F_0}), E(\overrightarrow{F_1}), \cdots, E(\overrightarrow{F}_{n-1})$.

Let $\overrightarrow{T}$ be the following directed trail:

$$\underbrace{\overrightarrow{D_0}+\overrightarrow{D_0}+\cdots+\overrightarrow{D_0}}_{\lambda \text{ copies of } \overrightarrow{D_0}}+\underbrace{\overrightarrow{F_0}+\overrightarrow{F_0}+\cdots+\overrightarrow{F_0}}_{\lambda \text{ copies of } \overrightarrow{F_0}}+\underbrace{\overrightarrow{D_1}+\overrightarrow{D_1}+\cdots+\overrightarrow{D_1}}_{\lambda \text{ copies of } \overrightarrow{D_1}}+\underbrace{\overrightarrow{F_1}+\overrightarrow{F_1}+\cdots+\overrightarrow{F_1}}_{\lambda \text{ copies of } \overrightarrow{F_1}}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots+\underbrace{\overrightarrow{D}_{n-1}+\overrightarrow{D}_{n-1}+\cdots+\overrightarrow{D}_{n-1}}_{\lambda \text{ copies of } \overrightarrow{D}_{n-1}}+\underbrace{\overrightarrow{F}_{n-1}+\overrightarrow{F}_{n-1}+\cdots+\overrightarrow{F}_{n-1}}_{\lambda \text{ copies of } \overrightarrow{F}_{n-1}}.$$

We see that $\overrightarrow{T}$ is a directed Euler trail of $\lambda K^*_{n,n,n}$.

To evaluate $g(\overrightarrow{T})$, we show in the following that for $i = 0, 1, \cdots, n-1$ we have (i) $g(\overrightarrow{D}_i + \overrightarrow{D}_i) = 3n$, $g(\overrightarrow{F}_i + \overrightarrow{F}_i) = 3n$ (ii) $g(\overrightarrow{D}_i + \overrightarrow{F}_i) = 3n-4$ and (iii) $g(\overrightarrow{F}_i + \overrightarrow{D}_{i+1}) = 3n-2$. Note that each $\overrightarrow{D_i}$ is a directed Hamiltonian cycle of $\lambda K^*_{n,n,n}$, and so is each $\overrightarrow{F_i}$.

(i) This is trivial.

(ii) We see that $\overrightarrow{D}_i + \overrightarrow{F}_i$ is the directed trail $a_0 \to b_i \to c_{2i} \to a_1 \to b_{i+1} \to c_{2i+1} \to \cdots \to a_{n-1} \to b_{i+n-1} \to c_{2i+n-1} \to a_0 \to c_{2i+1} \to b_{i+1} \to a_1 \to c_{2i+2} \to b_{i+2} \to \cdots \to a_{n-1} \to c_{2i} \to b_i \to a_0$. In $\overrightarrow{D}_i+\overrightarrow{F}_i$, there are $3n-4$ arcs between two appearances of c_j if $j = 0, 1, \cdots, 2i-1, 2i+1, \cdots, n-1$, and more than $3n-4$ arcs between two appearances of any other vertex. Thus

$g(\overrightarrow{D}_i + \overrightarrow{F}_i) = 3n - 4$.

(iii) $\overrightarrow{F}_i + \overrightarrow{D}_{i+1}$ is the directed trail $a_0 \to c_{2i+1} \to b_{i+1} \to a_1 \to c_{2i+2} \to b_{i+2} \to \cdots \to a_{n-1} \to c_{2i} \to b_i \to a_0 \to b_{i+1} \to c_{2i+2} \to a_1 \to b_{i+2} \to c_{2i+3} \to \cdots \to a_{n-1} \to b_i \to c_{2i+1} \to a_0$. In $\overrightarrow{F}_i + \overrightarrow{D}_{i+1}$, there are $3n - 2$ arcs between two appearances of c_j if $j = 0, 1, \cdots, 2i, 2i+2, \cdots, n-1$, and more than $3n - 2$ arcs between two appearances of any other vertex. Thus $g(\overrightarrow{F}_i + \overrightarrow{D}_{i+1}) = 3n - 2$.

From (i), (ii) and (iii), we obtain $g(\overrightarrow{T}) = 3n - 4$. ■

Now we prove the main result of this section.

Theorem B. *Let $n \geq 3$ be an odd integer. Suppose that k is a positive integer such that $2 \leq k \leq 3n - 1$ and $|E(\lambda K^*_{n,n,n})| \equiv 0 \pmod{k-1}$. Then $\lambda K^*_{n,n,n}$ has a $\overrightarrow{P_k}$-decomposition.*

Proof. Since $|E(\lambda K^*_{n,n,n})| \equiv 0 \pmod{k-1}$ (i.e., $(k-1)|6\lambda n^2$), by Lemma 3.2 and Remark 1.2 $\lambda K^*_{n,n,n}$ has a $\overrightarrow{P_k}$-decomposition if $2 \leq k \leq 3n-4$. So we only need to consider $3n-3 \leq k \leq 3n-1$. We distinguish two cases: Case 1. $k = 3n-3$ or $k = 3n - 1$, Case 2. $k = 3n - 2$.

Case 1. $k = 3n - 3$ or $k = 3n - 1$.

Then $k - 1$ is odd. Thus $(k-1)|6\lambda n^2$ implies $(k-1)|3\lambda n^2$. By Theorem A and Remark 3.1, $\lambda K^*_{n,n,n}$ has a $\overrightarrow{P_k}$-decomposition.

Case 2. $k = 3n - 2$.

From the assumption $(3n-3)|6\lambda n^2$, we have $(n-1)|2\lambda n^2$, which implies $(n-1)|2\lambda$ since $\gcd(n, n-1) = 1$. Hence we have $(k-1)|6\lambda$.

For $i = 0, 1, \cdots, n-1$, let $\overrightarrow{D}_i$, $\overrightarrow{F}_i$ be the directed trails defined in Lemma 3.2, and let $\overrightarrow{W}_i$ be the following directed trail:

$\underbrace{\overrightarrow{F_i} + \cdots + \overrightarrow{F_i}}_{\lambda \text{ copies of } \overrightarrow{F_i}} + \underbrace{\overrightarrow{D}_{i+1} + \cdots + \overrightarrow{D}_{i+1}}_{\lambda \text{ copies of } \overrightarrow{D}_{i+1}}$, where the indices are taken modulo n.

In the proof of Lemma 3.2, we see that a subgraph $K^*_{n,n,n}$ of $\lambda K^*_{n,n,n}$ can be decomposed into $\overrightarrow{D_0}, \overrightarrow{D_1}, \cdots, \overrightarrow{D}_{n-1}, \overrightarrow{F_0}, \overrightarrow{F_1}, \cdots, \overrightarrow{F}_{n-1}$. Thus $\lambda K^*_{n,n,n}$ can be decomposed into $\overrightarrow{W}_0, \overrightarrow{W}_1, \cdots, \overrightarrow{W}_{n-1}$.

Also from (iii) in the proof of Lemma 3.2, we have $g(\overrightarrow{F}_i + \overrightarrow{D}_{i+1}) = 3n - 2$ for $i = 0, 1, \cdots, n-1$. Thus $g(\overrightarrow{W}_i) = 3n - 2$ for $i = 0, 1, \cdots, n-1$. Now we see that $k \leq g(\overrightarrow{W}_i)$, $(k-1)|6\lambda$ and the length of $\overrightarrow{W}_i$ is $6n\lambda$. Thus we can cut each $\overrightarrow{W}_i$ from the starting vertex into $6n\lambda/(k-1)$ directed paths of length $k-1$. Hence $\lambda K^*_{n,n,n}$ is decomposed into directed paths of order k.

For the $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n,n}$, the trivial necessities are $2 \leq k \leq 3n$ and $|E(\lambda K^*_{n,n,n})| \equiv 0 \pmod{k-1}$. Comparing with Theorem B, we see that for odd n the undetermined case for the sufficiency is $k = 3n$.

Using Remark 1.1, we can decompose a multigraph (multidigraph, respectively) into paths (directed paths, respectively) which need not to have equal orders. Thus Lemmas 2.1 and 3.2 imply the following:

1. Let $n \geq 3$ be an odd integer. Suppose that for $i = 1, 2, \cdots, r$, k_i is an integer such that $2 \leq k_i \leq 3n-6$ and $|E(K_{n,n,n})| = k_1 + k_2 + \cdots + k_r - r$. Then $K_{n,n,n}$ can be decomposed into r paths on $k_1, k_2, \cdots, k_r$ vertices, respectively.
2. Let $n \geq 3$ be an odd integer and $\lambda \geq 2$ be an integer. Suppose that for $i = 1, 2, \cdots, r$, k_i is an integer such that $2 \leq k_i \leq 3n-3$ and $|E(\lambda K_{n,n,n})| = k_1 + k_2 + \cdots + k_r - r$. Then $\lambda K_{n,n,n}$ can be decomposed into r paths on $k_1, k_2, \cdots, k_r$ vertices, respectively.
3. Let $n \geq 3$ be an odd integer. Suppose that for $i = 1, 2, \cdots, r$, k_i is an integer such that $2 \leq k_i \leq 3n-4$ and $|E(\lambda K^*_{n,n,n})| = k_1 + k_2 + \cdots + k_r - r$. Then $\lambda K^*_{n,n,n}$ can be decomposed into r paths on $k_1, k_2, \cdots, k_r$ vertices, respectively.

The decompositions into paths with even less restrictive orders are much more challenging.

In this paper the P_k-decomposition of $\lambda K_{n,n,n}$ and the $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n,n}$ have been studied for odd n. We use the property $\gcd(2, n) = 1$ in Lemmas 2.1 and 3.2. We do not have this advantage for even n. Up to now we can only deal with this case for even λ.

Acknowledgment

The Authors are grateful to the referee for the helpful comments which improve the readability of this paper.

References

1. J. Bosák, *Decompositions of Graphs*, Kluwer, Dordrecht, Netherlands, 1990.
2. C. S. Kumar, On P_4-decomposition of graphs, *Taiwanese J. Math.* **7** (2003), 657-664.
3. R. Laskar and B. Auerbach, On decomposition of r-partite graphs into edge disjoint Hamilton circuits, *Discrete Math.* **14** (1976), 265-268.

4. M. Meszka and Z. Skupień, Decompositions of a complete multidigraph into non-hamiltonian paths, *J. Graph Theory* **51** (2006), 82-91.

5. C. A. Parker, *Complete bipartite graph path decompositions*, Thesis, Auburn University, Auburn, Alabama 1998.

6. M. Tarsi, Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs, *J. Combin. Theory, Ser. A* **34** (1983), 60-70.

7. M. Truszczyński, Note on the decomposition of $\lambda K_{m,n}(\lambda K^*_{m,n})$ into paths, *Discrete Math.* **55** (1985), 89-96.

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