TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 1, pp. 157-168, February 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

# APPROXIMATION WITH JACOBI WEIGHTS BY BASKAKOV OPERATORS

Jian-Jun Wang* and Zong-Ben Xu

**Abstract.** Using the modulus of smoothness $\omega^2_{\varphi^\lambda}(f,t)_\omega$, direct theorem with Jacobi weights of Baskakov operators is established in this paper; In addition, a weak-type inverse theorem of Baskakov operators is obtained in the weighted norm.

## 1. Introduction

The *Baskakov* operator is defined by

$$V_n(f;x) = \sum_{k=0}^{\infty} f(\frac{k}{n}) v_{n,k}(x), \qquad f \in C_B[0,\infty), \tag{0.1}$$

where $v_{n,k}(x) = C^k_{n+k-1} x^k (1+x)^{-(n+k)}$.

Since we only consider the *Baskakov* operator, let us suppose that $\varphi^2(x) = x(1+x)$. First, we give some notations,

$$\begin{aligned} C_{a,b} &= \{f | f \in C_B[0,\infty), \omega f \in L_\infty[0,\infty)\}, \\ \| f \|_\omega &= \sup_{0 \le x < \infty} |\omega(x) f(x)| + |f(0)|, \end{aligned} \tag{0.2}$$

where $C_B[0,\infty)$ represents the set of bounded continuous functions in $[0,\infty)$, $\omega(x) = x^a(1+x)^{-b} (0 < a < 1, \quad b > 0)$ is a *Jacobi* weight function.

In the norm (1.2), the r-th modulus of smoothness of Ditzian-Totik with *Jacobi* weights is given by (see [1])

Received March 12, 2007, accepted July 20, 2007.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 41A36, 41A25/CLC.
*Key words and phrases*: Weighted approximation, Baskakov operators, Weighted modulous of smoothness.
*Corresponding author.

$$(0.3)\quad \begin{aligned}\Omega^2_{\varphi^\lambda}(f;t)_\omega =&\sup_{0<h\le t} \| \omega(x)\Delta^2_{h\varphi^\lambda} f(x) \|_{[(2t)^{\frac{2}{2-\lambda}},\infty)}\\ &+ \sup_{0<h\le (2t)^{\frac{2}{2-\lambda}}} \| \omega(x)\vec{\Delta}^2_{h\varphi^\lambda} f(x) \|_{[0,12(2t)^{\frac{2}{2-\lambda}}]},\end{aligned}$$

$\Delta^2_h f(x) = f(x+h) - 2f(x) + f(x-h)$, $\vec{\Delta}^2_h f(x) = f(x+2h) - 2f(x+h) + f(x)$, and the K-functional by

$$(0.4)\qquad K^2_{\varphi^\lambda}(f;t^2)_\omega = \inf_g\{\| f-g \|_\omega + t^2 \| \varphi^{2\lambda} g'' \|_\omega, g' \in A.C.loc\}.$$

From the reference [1], we have the following relationship£º

$$(0.5)\qquad C^{-1}\Omega^2_{\varphi^\lambda}(f;t)_\omega \le K^2_{\varphi^\lambda}(f;t^2)_\omega \le C\Omega^2_{\varphi^\lambda}(f;t)_\omega.$$

In the paper, the letter $C$, appearing in various formulas, denotes a positive constant independent of $n$, $x$ and $f$. Its value may be different at different occurrences, even within the same formula.

As *Baskakov* operators has the property of preserves linear, for convenience the following discussion, we may suppose $f \in C^0_{a,b}$, the space $C^0_{a,b}$ is

$$C^0_{a,b} = \{f | f \in C_{a,b}, f(0) = 0\}.$$

For *Baskakov* operators (1.1), L.S Xie([2]) gave an interesting direct estimate,

$$(0.6)\qquad |V_n(f;x) - f(x)| = O\{\omega^2_{\varphi^\lambda}(f; n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))\},$$

where $0 \le \lambda \le 1$, $\omega^2_{\varphi^\lambda}(f;t) = \sup_{0<h\le t} \| \Delta^2_{h\varphi^\lambda} f(x) \|$, which unifies the classical estimate for $\lambda = 0$ and norm estimate for $\lambda = 1$.

As the inverse result, S.S Guo, H.Z Tong etc ([8]) obtained the Stechkin-Marchaud-Type inequalities for the *Baskakov* operators as follows [1]

$$(0.7)\qquad \omega^2_\varphi(f; n^{-\frac{1}{2}}) \le C\frac{1}{n}\sum_{k=1}^{n}\{\|V_k f - f\| + \| f \|\},$$

where $\| f \| = \sup_{x\in[0,\infty)} |f(x)|$.

Naturally, we will consider the following problems: " are there the similar results ( (1.6) and (1.7)) in the approximation with *Jacobi* weights by *Baskakov* operators ?" As we known, approximation with weights is not a simple generalization of normal approximation means; In the norm $\| \omega f \|_\infty$, both *Baskakov* operators and *Bernstein*

Here we take the special situation of the results of the reference [8], which is corresponding to our natation.

operators are not bounded( see [3, 6]); Introducing the norm (1.2) in [3], P.C.Xuan and D.X.Zhou obtained the bounded of *Baskakov* operators in the approximation with *Jacobi* weights. The purpose of this paper are to prove the following results which is similar to (1.6) and (1.7) by *Baskakov* operators in the norm (1.2). That is

**Theorem.** *For* $f \in C_{a,b}$*,* $0 \leq \lambda \leq 1$ *and* $0 < q < 1$*, we have*

$$\| V_n f - f \|_\omega \leq C\Omega^2_{\varphi^\lambda}(f; n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)))_\omega; \tag{0.8}$$

$$\Omega^2_{\varphi^\lambda}(f; n^{-\frac{1}{2}})_\omega \leq C\frac{\delta_{n,\lambda}(x)}{n}\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_k f - f\|_\omega + \frac{1}{n} \| f \|_\omega\}, \tag{0.9}$$

*where* $\delta_{n,\lambda}(x) = \{\min(n^{-\frac{1}{2}}; \varphi(x))\}^{2(\lambda-1)}$.

**Corollary.** *For* $f \in C_{a,b}$*, we have*

$$\Omega^2_{\varphi}(f; n^{-\frac{1}{2}})_\omega \leq C\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_k f - f\|_\omega + \frac{1}{n} \| f \|_\omega\}. \tag{0.10}$$

*Obviously, if* $1 > q > \alpha > 0$*, then (1.8) and (1.10) give a characterization for the approximation order of* $n^{-\alpha}$ *with* $0 < \alpha < 1$.

## 2. Fundamental Lemma

Now we give some lemmas:

**Lemma 2.1.** *If* $c \geq 0$*,* $d \in \mathbf{R}$*,* $0 < \gamma < 1$*, then we have*

$$|\sum_{k=1}^{\infty}(\frac{k}{n})^{-c}(1+\frac{k}{n})^d v_{n,k}(x)| \leq Cx^{-c}(1+x)^d, \quad x > 0, \tag{2.1}$$

$$|\sum_{k=0}^{\infty}(\frac{n}{n+k})^\gamma v_{n+2,k}(x)| \leq C(1+x)^{-\gamma}. \tag{2.2}$$

*Proof.* In [3], the authors gave the inequality (2.1) for $c \geq 0, \ d \geq 0$; For $c \geq 0, \ d < 0$, using Cauchy-Schwarz inequality, we have

$$|\sum_{k=0}^{\infty}(\frac{n}{n+k})^m v_{n+2,k}(x)| \leq C(1+x)^{-m} \quad (m \in N),$$

and using the methods of [3], It is not difficult to show (2.1).

Next we prove the inequality (2.2), by the Cauchy-Schwarz and the Hölder inequality,

$$\begin{aligned}|\sum_{k=0}^{\infty}(\tfrac{n}{n+k})^{\gamma}v_{n+2,k}(x)| &\le |\sum_{k=0}^{\infty}(\frac{n}{n+k})^{2\gamma}v_{n+2,k}(x)|^{\frac{1}{2}}|\sum_{k=0}^{\infty}v_{n+2,k}(x)|^{\frac{1}{2}}\\ &\le |\sum_{k=0}^{\infty}(\frac{n}{n+k})^{2}v_{n+2,k}(x)|^{\frac{\gamma}{2}}|\sum_{k=0}^{\infty}v_{n+2,k}(x)|^{\frac{1-\gamma}{2}}\\ &\le C(1+x)^{-\gamma}.\end{aligned}$$

The proof of Lemma 2.1 is completed. ■

**Lemma 2.2.** *If $f\in D$, $n\in N$, then*

$$|\omega(x)\varphi^{2\lambda}V_n''(f;x)|\le C\parallel\varphi^{2\lambda}f''\parallel_{\omega}. \tag{2.3}$$

*where* $D=\{g|g\in C^0_{a,b}, g'\in A.C.loc, \parallel\varphi^{2\lambda}g''\parallel_{\omega}<\infty\}$.

*Proof.* In view of (2.1),

$$\begin{aligned}&|\omega(x)\varphi^{2\lambda}V_n''(f;x)|\\ &=|\omega(x)x^{\lambda}(1+x)^{\lambda}n(n+1)\sum_{k=0}^{\infty}v_{n+2,k}(x)\vec{\triangle}^{2}_{\frac{1}{n}}f(\frac{k}{n})|\\ &\le C\omega(x)x^{\lambda}(1+x)^{\lambda}\{|(n+1)\sum_{k=1}^{\infty}v_{n+2,k}(x)\int_0^{\frac{2}{n}}\frac{(1+\frac{k}{n}+u)^{b-\lambda}}{(\frac{k}{n}+u)^{a+\lambda}}du|\\ &\quad+|n(n+1)v_{n+2,0}(x)\int_0^{\frac{2}{n}}u^{1-a-\lambda}(1+u)^{b-\lambda}du|\}\parallel\varphi^{2\lambda}f''\parallel_{\omega}\\ &\le C\omega(x)x^{\lambda}(1+x)^{\lambda}\{\sum_{k=1}^{\infty}v_{n+2,k}(x)(\frac{k}{n})^{-a-\lambda}(1+\frac{k}{n})^{b-\lambda}\\ &\quad+n(n+1)v_{n+2,0}(x)\frac{(\frac{2}{n})^{2-a-\lambda}}{2-a-\lambda}\}\parallel\varphi^{2\lambda}f''\parallel_{\omega}\\ &\le C\{\omega(x)x^{\lambda}(1+x)^{\lambda}\sum_{k=1}^{\infty}v_{n+2,k}(x)(\frac{k}{n+2})^{-a-\lambda}(1+\frac{k}{n+2})^{b-\lambda}\\ &\quad+x^{a+\lambda}(1+x)^{-n-2-b+\lambda}n(n+1)(\frac{2}{n})^{2-a-\lambda}\}\parallel\varphi^{2\lambda}f''\parallel_{\omega}\\ &\le C\parallel\varphi^{2\lambda}f''\parallel_{\omega}.\end{aligned}$$

In the above course of proof, we have used the following inequalities (see [1]),

$$\vec{\triangle}^{2}_{\frac{1}{n}}f(\frac{k}{n})\le Cn^{-1}\int_0^{\frac{2}{n}}|f''(\frac{k}{n}+u)|du,\quad k=1,2,\cdots$$

$$\vec{\triangle}^2_{\frac{1}{n}} f(0) \leq C \int_0^{\frac{2}{n}} u|f''(u)|du.$$

The proof of Lemma 2.2 is completed. ∎

**Lemma 2.3.** *If* $0 \leq \lambda \leq 1,\ \ x, t \in (0, \infty)$, *then*

$$\begin{aligned} &|\int_x^t |t-u|\varphi^{-2\lambda}(u)\omega^{-1}(u)du| \\ &\leq C(t-x)^2(\varphi^{-2\lambda}(x)\omega^{-1}(x) + x^{-a-\lambda}(1+t)^{b-\lambda}). \end{aligned} \tag{2.4}$$

*Proof.* Let $u = t + \tau(x-t),\ 0 \leq \tau \leq 1$, we have

$$\begin{aligned} &|\int_x^t |t-u|\varphi^{-2\lambda}(u)\omega^{-1}(u)du| \\ &= |\int_x^t |t-u|\frac{(1+u)^{b-\lambda}}{u^{a+\lambda}}du| \\ &\leq |\int_x^t \frac{|t-u|}{u^{a+\lambda}}du|((1+x)^{b-\lambda} + (1+t)^{b-\lambda}) \\ &\leq |\int_0^1 \frac{\tau(x-t)^2}{(\tau x + (1-\tau)t)^{a+\lambda}}d\tau|((1+x)^{b-\lambda} + (1+t)^{b-\lambda}) \\ &\leq (x-t)^2 \int_0^1 \frac{\tau^{1-a-\lambda}}{x^{a+\lambda}}d\tau((1+x)^{b-\lambda} + (1+t)^{b-\lambda}) \\ &\leq \frac{1}{2-a-\lambda}(t-x)^2(\varphi^{-2\lambda}(x)\omega^{-1}(x) + x^{-a-\lambda}(1+t)^{b-\lambda}), \end{aligned}$$

which verifies Lemma 2.3.

**Lemma 2.4.** *If* $f \in C^0_{a,b},\ n \in \mathbf{N}$, *then*

$$\left|\omega(x)\varphi^{r\lambda}(x)V_n^{(r)}(f;x)\right| \leq Cn^{\frac{r}{2}}\left\{\min\{n^{-\frac{1}{2}}, \varphi(x)\}\right\}^{r(\lambda-1)}\|f\|_\omega \tag{2.5}$$

*Proof.* To prove (2.5), we consider the following two conditions,

(i) If $0 \leq \varphi(x) < \frac{1}{\sqrt{n}}$, we write

$$\begin{aligned} \left|\omega(x)\varphi^{r\lambda}(x)V_n^{(r)}(f;x)\right| &\leq C\omega(x)n^{-\frac{r\lambda}{2}}n^r\|f\|_\omega \sum_{k=0}^{\infty}\omega^{-1}(\frac{k}{n})v_{n+r,k}(x) \\ &\leq Cn^{-\frac{r\lambda}{2}}n^r\|f\|_\omega \\ &\leq Cn^{\frac{r}{2}}n^{\frac{r}{2}(1-\lambda)}\|f\|_\omega \end{aligned}$$

(ii) If $\varphi(x) \geq \frac{1}{\sqrt{n}}$, using the representation of $V_n^{(r)}(f;x)$ (see([1], P127)),

$$V_n^{(r)}(f;x) = \varphi^{-2r}(x) \sum_{i=0}^{r} Q_i(n,x) n^i \sum_{k=0}^{\infty} v_{n,k}(x)(\frac{k}{n}-x)^i f(\frac{k}{n})$$

where $Q_i(n,x)$ is a polynomial in $nx(1+x)$ of degree $\frac{r-i}{2}$ with consistent boundary, and

$$\left| n^i \varphi^{-2r}(x) Q_i(n,x) \right| \leq C \left( \frac{n}{\varphi^2(x)} \right)^{\frac{r+i}{2}}.$$

So we have

$$\begin{aligned}
&\left| \omega(x)\varphi^{r\lambda}(x) V_n^{(r)}(f;x) \right| \\
&\leq C\omega(x)\varphi^{r\lambda}(x) \sum_{i=0}^{r} (\frac{n}{x(1+x)})^{\frac{r+i}{2}} \sum_{k=0}^{\infty} v_{n,k}(x) \left| \frac{k}{n} - x \right|^i \left| f(\frac{k}{n}) \right| \\
&\leq C\varphi^{r\lambda}(x) \left\| f \right\|_{\omega} \sum_{i=0}^{r} (\frac{n}{\varphi^2(x)})^{\frac{r+i}{2}} \sum_{k=0}^{\infty} v_{n,k}(x) \left| \frac{k}{n} - x \right|^i \omega(x)\omega^{-1}(\frac{k}{n}) \\
&\leq C\varphi^{r\lambda}(x) \left\| f \right\|_{\omega} \sum_{i=0}^{r} (\frac{n}{\varphi^2(x)})^{\frac{r+i}{2}} \\
&\quad \left\{ \sum_{k=0}^{\infty} v_{n,k}(x) \left| \frac{k}{n} - x \right|^{2i} \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{\infty} v_{n,k}(x)\omega^2(x)\omega^{-2}(\frac{k}{n}) \right\}^{\frac{1}{2}} \\
&\leq C\varphi^{r\lambda}(x) \left\| f \right\|_{\omega} \sum_{i=0}^{r} (\frac{n}{\varphi^2(x)})^{\frac{r+i}{2}} \left\{ \frac{\delta_n(x)}{\sqrt{n}} \right\}^i \\
&\leq C n^{\frac{r}{2}} \varphi^{r(\lambda-1)} \left\| f \right\|_{\omega}.
\end{aligned}$$

The proof of Lemma 2.4 is completed. ■

**Lemma 2.5.** (see [2]). *If* $0 \leq \lambda \leq 1,\ 0 \leq \beta \leq 1,\ 0 < h < \frac{1}{2^{2+\lambda}}$ *and* $x > 2h\varphi^{\lambda}(x)$, *then*

$$\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}} \int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}} \varphi^{-2\beta}(x+u_1+u_2)du_1du_2 \leq Ch^2\varphi^{2(\lambda-\beta)}(x). \tag{2.6}$$

**Lemma 2.6.** (see [9]). *Suppose that for nonnegative sequences* $\{\mu_n\}$,$\{\phi_n\}$ *with* $\mu_1 = 0$, *the inequality* $(s > 0, Q \geq 1)$

$$\mu_n \leq Q(\frac{k}{n})^s \mu_k + M\phi_k \qquad (1 \leq k \leq n) \tag{2.7}$$

*holds for* $n \in N$*, then one has*

$$\mu_n \le M_q n^{-q} \sum_{k=1}^{n} k^{q-1} \phi_k \tag{2.8}$$

*with* $q = s$ *in case* $Q = 1$ *and with* $0 < q < s$ *else.*

## 3. The Proof of Theorem

Now we prove the main theorems.

First, we give the following two inequalities:

(1) If $x \ge \frac{1}{n}$, then

$$V_n((t-x)^2(1+t)^{b-\lambda}; x) \le C\frac{\varphi^2(x)}{n}(1+x)^{b-\lambda}. \tag{3.1}$$

(2) If $0 < x < \frac{1}{n}$, then

$$\omega(x)\sum_{k=0}^{\infty} v_{n,k}(x) \int_{\frac{k}{n}}^{x} |\frac{k}{n} - u|\omega^{-1}(u)\varphi^{-2\lambda}(u)du \le C\frac{\varphi^{2-2\lambda}(x)}{n}. \tag{3.2}$$

In fact, if $0 < b \le \lambda$, from the (9.5.10) of [1], we have

$$\begin{aligned} V_n((t-x)^4; x) &= \frac{\varphi^4(x)}{n^2}q_0(x) + \frac{\varphi^2(x)}{n^3}q_1(x) \\ &= \frac{\varphi^4(x)}{n^2}(q_0(x) + \frac{q_1(x)}{n\varphi^2(x)}) \le C\frac{\varphi^4(x)}{n^2}. \end{aligned}$$

where $q_0(x),\ q_1(x)$ is the polynomial in $x$ of degree zero and two, respectively.

Therefore, by the Cauchy-Schwarz, the Hölder inequalities and the equation (9.6.3) of [1], we have

$$\begin{aligned} V_n((t-x)^2(1+t)^{b-\lambda}; x) &\le (V_n((t-x)^4; x))^{\frac{1}{2}}(V_n((1+t)^{2(b-\lambda)}; x))^{\frac{1}{2}} \\ &\le (V_n((t-x)^4; x))^{\frac{1}{2}}(V_n((1+t)^{-2}; x))^{\frac{\lambda-b}{2}} \\ &\le C\frac{\varphi^2(x)}{n}(1+x)^{b-\lambda}. \end{aligned}$$

If $b > \lambda$, by (2.1)

$$V_n((1+t)^{b-\lambda}; x) \le C(1+x)^{b-\lambda},$$

the Cauchy-Schwarz and the Hölder inequalities, we can directly compute

$$\begin{aligned} V_n((t-x)^2(1+t)^{b-\lambda}; x) &\le (V_n((t-x)^4; x))^{\frac{1}{2}}(V_n((1+t)^{2(b-\lambda)}; x))^{\frac{1}{2}} \\ &\le C\frac{\varphi^2(x)}{n}(1+x)^{b-\lambda}. \end{aligned}$$

Next we prove (3.2).

(i) If $k=0$ and $0<x<\frac{1}{n}$, then

$$\begin{aligned}&\omega(x)v_{n,0}(x)\int_0^x u\omega^{-1}(u)\varphi^{-2\lambda}(u)du\\&=\omega(x)(1+x)^{-n}\int_0^x u^{1-a-\lambda}(1+u)^{b-\lambda}du.\end{aligned}$$

If $0<b\leq\lambda$, then

$$\begin{aligned}&\omega(x)(1+x)^{-n}\int_0^x u^{1-a-\lambda}(1+u)^{b-\lambda}du\\&\leq\omega(x)(1+x)^{-n}\int_0^x u^{1-a-\lambda}du\\&\leq C\omega(x)(1+x)^{-n}x^{2-a-\lambda}\\&\leq C\varphi^{2-2\lambda}(x)(1+x)^{-(n+b-\lambda)}\\&\leq C\frac{\varphi^{2-2\lambda}(x)}{n}.\end{aligned}$$

If $b>\lambda$, then

$$\begin{aligned}&\omega(x)(1+x)^{-n}\int_0^x u^{1-a-\lambda}(1+u)^{b-\lambda}du\\&\leq\omega(x)(1+x)^{-n+b-\lambda}\int_0^x u^{1-a-\lambda}du\\&\leq C\varphi^{2-2\lambda}(x)(1+x)^{-n}\\&\leq C\frac{\varphi^{2-2\lambda}(x)}{n}.\end{aligned}$$

(ii) If $k\geq 1$ and $0<x<\frac{1}{n}$, then

$$\begin{aligned}&\omega(x)\sum_{k=1}^{\infty}v_{n,k}(x)\int_{\frac{k}{n}}^x|\frac{k}{n}-u|\omega^{-1}(u)\varphi^{-2\lambda}(u)du\\&\leq\omega(x)\sum_{k=1}^{\infty}v_{n,k}(x)(\frac{k}{n}-x)\varphi^{-2\lambda}(x)(1+\frac{k}{n})^b\int_{\frac{k}{n}}^x u^{-a}du\\&\leq\omega(x)\sum_{k=1}^{\infty}v_{n,k}(x)(\frac{k}{n}-x)\varphi^{-2\lambda}(x)(1+\frac{k}{n})^b\frac{(\frac{k}{n})^{1-a}-x^{1-a}}{1-a}\\&\leq\frac{\omega(x)}{1-a}\varphi^{-2\lambda}(x)\sum_{k=1}^{\infty}v_{n,k}(x)(\frac{k}{n}-x)^{2-a}(1+\frac{k}{n})^b\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\omega(x)}{1-a}\varphi^{-2\lambda}(x)\{(\sum_{k=1}^{\infty} v_{n,k}(x)(\frac{k}{n}-x)^2)^{1-\frac{a}{2}}(\sum_{k=1}^{\infty} v_{n,k}(x)(1+\frac{k}{n})^{\frac{2b}{a}})^{\frac{a}{2}}\} \\
&\leq \frac{\omega(x)}{1-a}\varphi^{-2\lambda}(x)(\frac{\varphi^2(x)}{n})^{1-\frac{a}{2}}(1+x)^b \\
&\leq Cx^{\frac{a}{2}}\varphi^{2-2\lambda}(x)n^{-1+\frac{a}{2}}(1+x)^{-\frac{a}{2}} \\
&\leq C\frac{1}{n}\varphi^{2-2\lambda}(x)(nx)^{\frac{a}{2}} \\
&\leq C\frac{1}{n}\varphi^{2-2\lambda}(x).
\end{aligned}
$$

From (i),(ii), we can get the inequality (3.2).

Thus, for all $g' \in A.C.loc$, since $V_n(f;x)$ have the properties of preserving constants and linear, by Taylor formula, (2.4) and (3.1), for $x \geq \frac{1}{n}$, we have

$$
\begin{aligned}
&|\omega(x)(V_n(g;x)-g(x))| \\
&= |\omega(x)V_n(\int_x^t (t-u)g''(u)du;x)| \\
&\leq |\omega(x)V_n(\int_x^t |t-u|\omega^{-1}(u)\varphi^{-2\lambda}(u)du;x)| \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega \\
&\leq C \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega \{\varphi^{-2\lambda}(x)V_n((t-x)^2;x) \\
&\quad +x^{-a-\lambda}\omega(x)V_n((t-x)^2(1+t)^{b-\lambda};x)\} \\
&\leq C\frac{1}{n}\varphi^{2-2\lambda}(x) \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega .
\end{aligned}
$$

For $0 < x < \frac{1}{n}$, by (3.2),

$$
\begin{aligned}
&|\omega(x)(V_n(g;x)-g(x))| \\
&= |\omega(x)V_n(\int_x^t (t-u)g''(u)du;x)| \\
&\leq |\omega(x)V_n(\int_x^t |t-u|\omega^{-1}(u)\varphi^{-2\lambda}(u)du;x)| \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega \\
&\leq C \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega |\omega(x)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{\frac{k}{n}}^x |\frac{k}{n}-u|\omega^{-1}(u)\varphi^{-2\lambda}(u)du| \\
&\leq C\frac{1}{n}\varphi^{2-2\lambda}(x) \parallel \varphi^{2\lambda}(x)g'' \parallel_\omega .
\end{aligned}
$$

Hence, for $f \in C_B[0,\infty)$ and for all $g' \in A.C.loc$, we have

$$\begin{aligned}|\omega(x)(V_n(f;x)-f(x))| &\le |\omega(x)V_n(f-g;x)| + |\omega(x)(f(x)-g(x))| \\ &\quad +|\omega(x)(V_n(g;x)-g(x))| \\ &\le C\{\| f-g \|_\omega + (n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^2 \| \varphi^{2\lambda}(x)g'' \|_\omega\}.\end{aligned}$$

Using the inequality (1.4) and (1.5), we can easily obtain the inequality (1.8).

The following we prove the inequality (1.9).

Let

$$\begin{aligned}\mu_n &= \frac{1}{n}\|\varphi^{2\lambda}(V_n''-V_1'')f\|_\omega, \\ \phi_n &= \delta_{n,\lambda}(x)\|V_nf-f\|_\omega + \frac{\delta_{1,\lambda}(x)}{n}\|f\|_\omega,\end{aligned}$$

where $\delta_{n,\lambda}(x) = \{\min(n^{-\frac{1}{2}};\varphi(x))\}^{2(\lambda-1)}$.

By (2.3) and (2.5), we have

$$\begin{aligned}&\mu_n \\ &\le \frac{1}{n}\|\varphi^{2\lambda}V_n''f\|_\omega + \frac{1}{n}\|\varphi^{2\lambda}V_1''f\|_\omega \\ &\le \frac{1}{n}\|\varphi^{2\lambda}V_n''(V_kf-f)\|_\omega + \frac{1}{n}\|\varphi^{2\lambda}V_n''V_kf\|_\omega + \frac{C\delta_{1,\lambda}(x)}{n} \| f \|_\omega \\ &\le \frac{C}{n}\|\varphi^{2\lambda}V_k''f\|_\omega + C\delta_{n,\lambda}(x)\|V_kf-f\|_\omega + \frac{C\delta_{1,\lambda}(x)}{n} \| f \|_\omega \\ &\le \frac{C}{n}\|\varphi^{2\lambda}(V_k''-V_1'')f\|_\omega + \frac{C}{n}\|\varphi^{2\lambda}V_1''f\|_\omega + C\delta_{n,\lambda}(x)\|V_kf-f\|_\omega + \frac{C\delta_{1,\lambda}(x)}{n} \| f \|_\omega \\ &\le C\frac{k}{n}\mu_k + C\phi_k.\end{aligned}$$

Therefore Lemma 2.6 implies

$$\|\varphi^{2\lambda}(V_n''-V_1'')f\|_\omega \le C\delta_{n,\lambda}(x)\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_kf-f\|_\omega + \frac{1}{n} \| f \|_\omega\}$$

Hence,

$$\begin{aligned}\|\varphi^{2\lambda}V_n''f\|_\omega &\le C\delta_{n,\lambda}(x)\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_kf-f\|_\omega + \frac{1}{n} \| f \|_\omega\} + \|\varphi^{2}V_1''f\|_\omega \\ &\le C\delta_{n,\lambda}(x)\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_kf-f\|_\omega + \frac{1}{n} \| f \|_\omega\}.\end{aligned}$$

For $n \geq 2$, there exists $l \in N$, such that $\frac{n}{2} \leq l \leq n$, and

$$\|V_l f - f\|_\omega \leq \|V_k f - f\|_\omega, \qquad \frac{n}{2} \leq k \leq n.$$

Using the definition of $K^2_{\varphi^\lambda}(f;t^2)_\omega$(see (1.4)), we have

$$\begin{aligned}
&K^2_{\varphi^\lambda}(f;\frac{1}{n})_\omega \\
&\leq \|V_l f - f\|_\omega + \frac{1}{n}\|\varphi^{2\lambda}V_l'' f\|_\omega \\
&\leq \frac{2}{n}\sum_{k=\frac{n}{2}}^{n}\|V_k f - f\|_\omega + \frac{C}{n}\delta_{n,\lambda}(x)\sum_{k=1}^{l}(\frac{k}{n})^{q-1}\{\|V_k f - f\|_\omega + \frac{1}{n}\parallel f \parallel_\omega\} \\
&\leq \frac{2}{n}\{\sum_{k=1}^{n}\|V_k f - f\|_\omega + \frac{1}{n}\parallel f \parallel_\omega\} \\
&\quad + \frac{C}{n}\delta_{n,\lambda}(x)\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_k f - f\|_\omega + \frac{1}{n}\parallel f \parallel_\omega\} \\
&\leq C\frac{\delta_{n,\lambda}(x)}{n}\sum_{k=1}^{n}(\frac{k}{n})^{q-1}\{\|V_k f - f\|_\omega + \frac{1}{n}\parallel f \parallel_\omega\}
\end{aligned}$$

Thus by (1.5), we can obtain the inverse result.

## ACKNOWLEDGMENT

This work was supported by Natural Science Foundation of China (No. 10726040, 10701062), the Key Project of Ministry of Education of China (No.108176), China Postdoctoral Science Foundation (No. 20080431237), Southwest University Development Foundation (No. SWUF2007014) and Southwest University Doctoral Foundation (No. SWUB2007006). The authors would like to thank the referee for a number of helpful comments.

## REFERENCES

1. V. Totik, Moduli of Smoothness, Springer-Verlag Press, Berlin-New York, 1987.
2. L. S. Xie, Direct and Inverse Approximation Theorems and Derivatives of Baskakov Operators, *Chinese Annals of Mathematics*, **21A(3)** (2000), 253-260.
3. P. C. Xuan and D. X. Zhou, Rate of Convergence for Baskakov Operators with *Jacobi*-Weights, *Math. Appl. Sinica*, **18** (1995), 129-139.

4. Z. Ditzian and K. G. Ivanov, Bernstein-type operators and theirs derivatives, *J. Appr. Theory*, **56** (1989), 72-90.

5. D. X. Zhou, On smoothness characterizied by Bernstein-type operators, *J. Appr. Theory*, **81** (1995), 303-305.

6. D. X. Zhou, Rate of Convergence for Bernstein operators with *Jacobi*-Weights, *ACTA Math. Sinica*, **35(3)** (1992), 331-338

7. H. Berens and G. G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math.*, **21** (1972), 693-708.

8. S. S. Guo, H. Z. Tong and G. S. Zhang, Stechkin-Marchaud-Type Inequalities for Baskakov Polynomials, *J. Appr. Theory*, **114** (2002), 33-47.

9. E. Van. Wickeren, Weak-type Inequalities for Kantorovic Polynomials and Related Operators, *Indag. Math.*, **49** (1987), 111-120.



Jian-Jun Wang
Institute for Information and System Science,
Xi'an Jiaotong University,
Xi'an 710049,
and
School of Mathematics and Statistics,
Southwest University,
Chongqing 400715,
P. R. China
E-mail: wjj@swu.edu.cn
wangjianjun@mail.xjtu.edu.cn

Zongben Xu
Institute for Information and System Science,
Xi'an Jiaotong University,
Xi'an 710049,
P. R. China
E-mail: zbxu@mail.xjtu.edu.cn