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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $v''(x) + q(x) \sin v(x) = 0$

Tzy-Wei Hwang and Shin-Feng Hwang

Abstract. In this paper we study the asymptotic behavior of the solution $v(x)$ of initial value problem (1.1) which arises from a mathematical model describing the large deformations of a nonumiform cantilever.

1. Introduction

In this paper we are concerned with the asymptotic behavior of the solutions of the following initial value problem:

$$\begin{aligned} v''(x) + q(x)\sin v(x) &= 0, \\ v'(0) &= 0, \\ v(0) &= a, \quad a \in R. \end{aligned} \tag{1.1}$$

where $q(0) \geq 0$ and $q'(x) > 0$ for all $x \in (0, \infty)$. The qualitative behavior of the solution $v(x, a)$ of (1.1) is important to the studies of the following mathematical model (1.2) which can describe the deformation of a nonuniform cantilever [6, 8].

$$\begin{aligned} v''(x) + q(x)\sin v(x) &= 0, \\ v'(0) &= 0, \\ v(K) &= \alpha - \pi, \quad -\pi \leq \alpha \leq \pi, K > 0. \end{aligned} \tag{1.2}$$

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We shall study the two-point boundary value problem (1.2) by using shooting method. From the uniqueness of the solution of the initial value problem (1.1), it follows that

$$
\begin{aligned}
v(x, 2\pi + a) &= 2\pi + v(x, a), \\
v(x, 2\pi - a) &= 2\pi - v(x, a), \\
v(x, a) &= -v(x, -a), \\
v(x, 0) &= 0, \\
v(x, \pi) &= \pi.
\end{aligned} \tag{1.3}
$$

From (1.3), we shall consider $v(x, a)$ only for the case $0 < a < \pi$.

Lemma 1.1. *Let $0 < a < \pi$. If $q(0) \geq 0, q'(x) > 0$ for all $x \in (0, \infty)$, then we have*

(i) $-\pi/2 < v(x, a) < \pi/2$ *for* $0 < a < \pi/2, x \geq 0$.

(ii) $-\pi < v(x, a) < \pi$ *for* $\pi/2 \leq a < \pi, x \geq 0$.

(iii) $|v(x, a)| \leq a$ *for all* $x \geq 0$*, moreover,* $v(x, a)$ *is oscillatory over* $[0, \infty)$ *with the decreasing amplitudes.*

Proof. Multiplying (1.1) by $v'(x)$ and integrating the resulting equation from 0 to x, we obtain

$$
\frac{1}{2}(v'(x))^2 = q(x) \cos v(x) - q(0) \cos a - \int_0^x q'(\xi) \cos v(\xi)\, d\xi \geq 0. \tag{1.4}
$$

If $0 < a < \pi/2$, then $\cos a = \cos v(0) > 0$. We claim that $\cos v(x) > 0$ for all $x \geq 0$. If not, then there exists $x_0 > 0$ such that $\cos v(x) > 0$ for all $0 \leq x < x_0$ and $\cos v(x_0) = 0$. Then this contradicts (1.4) with $x = x_0$ and we complete the proof for (i).

If $\pi/2 \leq a < \pi$, then $\cos a = \cos v(0) \in (-1, 0]$. We claim that $\cos v(x) \neq -1$ for all $x \geq 0$. If not, then there exists $x_0 > 0$ such that $\cos v(x_0) = -1$ and $\cos v(x) > -1$ for $0 \leq x < x_0$. Again from (1.4) we obtain a contradiction. Hence $-\pi < v(x, a) < \pi$ for all $x \geq 0$ and we established (ii).

Next we introduce the following Liapunov function

$$
V(x) = (1 - \cos v(x)) + \frac{1}{2}\frac{(v'(x))^2}{q(x)} \tag{1.5}
$$

where $v(x) = v(x, a)$. It is easy to verify that

$$V'(x) = -\frac{q'(x)}{2}\frac{(v'(x))^2}{(q(x))^2} \leq 0. \tag{1.6}$$

Then we have

$$1 - \cos v(x,a) \leq V(x) \leq V(0) = 1 - \cos a. \tag{1.7}$$

We note that $V(0) = 1 - \cos a$ follows directly from L'Hospital rule. So from (1.7) follows that $|\, v(x,a)\,| \leq a$ for all $x \geq 0$. We rewrite the equation in (1.1) as

$$v''(x,a) + q(x)\left(\frac{\sin v(x)}{v(x)}\right) v(x) = 0. \tag{1.8}$$

Let $0 < \delta < min_{0 \leq v \leq a}(\frac{\sin v}{v})$. Using Sturm's comparison theorem, we compare (1.8) with (1.9)

$$\phi''(x) + \delta q(x)\,\phi(x) = 0 \tag{1.9}$$

which is oscillatory over $[0,\infty)$. Thus the solution $v(x,a)$ is oscillatory over $[0,\infty)$ for $0 < a < \pi$. Moreover, from (1.6) and (1.7) the solution $v(x,a)$ is oscillatory with the decreasing amplitudes, so we established (iii). Q.E.D.

In the next section we shall given some condition on $q(x)$, so that

$$\lim_{x\to\infty} v(x,a) = 0. \tag{1.10}$$

Consequently, if we denote the zeros of $v(x)$ by $x_1 < x_2 < \cdots < x_l < \cdots$, then we have

$$\lim_{l\to\infty} |x_l - x_{l-1}| = 0. \tag{1.11}$$

2. Main results

The purpose of this section is to establish (1.10). For all $0 < a < \pi$, the initial value problem

$$\begin{aligned} v''(x) + q(x)\sin v(x) &= 0, \\ v'(0) &= 0, \\ v(0) &= a, \quad a \in (0,\pi), \end{aligned} \tag{2.1}$$

where $q(x) \in C^1([0,\infty)) \cap C^2((0,\infty))$ and satisfies the following assumptions:

(A1) : $q(0) \geq 0$ and $q'(x) > 0$ for all $x \in (0,\infty)$;

(A2) : $\lim_{x\to\infty} q(x) = \infty$;

(A3) : $\exists x_0 \geq 0$ such that $q''(x)q(x) - \frac{5}{4}(q'(x))^2 \leq 0$ for all $x \in [x_0,\infty)$;

(A4) : $\lim_{x\to\infty} \frac{q''(x)}{\sqrt{q(x)}q'(x)} = 0$.

There are so many functions which satisfy condition (A1), (A2), (A3) and (A4), for example $q(x) = e^{Ax}; A > 0$ [7], and $q(x) = x^p; p > 0$ [8].

Theorem 2.1. *Assume that* $q(x)$ *satisfies conditions* (A1), (A2), (A3), (A4)*. Then the solution of* (2.1) *satisfies*

$$\lim_{x\to\infty} v(x,a) = 0, \quad for \quad all \quad a \in (0,\infty). \tag{2.2}$$

Moreover, the zeros of v*, denoted by* $x_1 < x_2 < \cdots < x_l < \cdots$*, satisfy*

$$\lim_{l\to\infty} |x_l - x_{l-1}| = 0. \tag{2.3}$$

We let $x^* > x_0$ be the 1-st zero such that $v'(x^*,a) = 0$ and $v(x^*,a) > 0$. From lemma 1.1, we have $|v(x,a)| \leq v^* = v(x^*,a)$ for all $x \geq x^*$. Consider $y(x) = \int_{x^*}^{x} \sqrt{q(t)}\,dt$ and $x(y)$ the inverse of $y(x)$. Let $\psi(y) = [q(x(y))]^{\frac{1}{4}}$. Then we have

$$\frac{\psi'(y)}{\psi(y)} = \frac{1}{4} \quad \frac{q'(x)}{q^{\frac{3}{2}}(x)}, \tag{2.4}$$

$$\frac{\psi''(y)}{\psi'(y)} = \frac{y'''(x)}{y'(x)y''(x)} - \frac{3}{2} \quad \frac{y''(x)}{(y'(x))^2}, \tag{2.5}$$

$$\frac{\psi''(y)}{\psi(y)} = \frac{q''(x)}{4q^2(x)} - \frac{5}{16} \quad \frac{(q'(x))^2}{q^3(x)}\,. \tag{2.6}$$

Before we prove the main Theorem 2.1, we need several lemmas.

Lemma 2.1. *Assume that* $q(x)$ *satisfies conditions* (A1), (A2), (A3)*. Then*

$$\lim_{y\to\infty} \frac{\psi'(y)}{\psi(y)} = \lim_{x\to\infty} \frac{1}{4} \quad \frac{q'(x)}{q^{\frac{3}{2}}(x)} = 0.$$

Proof. By conditions (A1) and (A3), we have

$$\left(\frac{q'(x)}{q^{\frac{5}{4}}(x)}\right)' = \frac{q''(x)q(x) - \frac{5}{4}(q'(x))^2}{q^{\frac{9}{4}}(x)} < 0$$

for all $x \geq x_0$, so $\frac{q'(x)}{q^{\frac{5}{4}}(x)}$ is decreasing for all $x \geq x_0$. Then we have

$$q'(x) \leq \frac{q'(x_0)}{q^{\frac{5}{4}}(x_0)}\, q^{\frac{5}{4}}(x) \quad \text{for all} \quad x \geq x_0. \tag{2.7}$$

Multiplying (2.7) by $q^{-\frac{3}{2}}(x)$ we have

$$0 < q'(x)q^{-\frac{3}{2}}(x) \leq \frac{q'(x_0)}{q^{\frac{5}{4}}(x_0)}\, q^{-\frac{1}{4}}(x)$$

for all $x \geq x_0$. By applying (A2) we have

$$\lim_{x\to\infty} \frac{q'(x)}{q^{\frac{3}{2}}(x)} = 0.$$

Q.E.D

Lemma 2.2. *Assume that $q(x)$ satisfies condition* (A4). *Then*

$$\lim_{y\to\infty} \frac{\psi''(y)}{\psi'(y)} = \lim_{x\to\infty} \left(\frac{y'''(x)}{y'(x)y''(x)} - \frac{3}{2}\frac{y''(x)}{(y'(x))^2}\right) = 0.$$

Proof. Since $y'(x) = q^{\frac{1}{2}}(x)$, $y''(x) = \frac{1}{2}\, q'(x)q^{-\frac{1}{2}}(x)$, and $y'''(x) = \frac{1}{2}q^{-\frac{3}{2}}(x)$ $[q''(x)q(x) - \frac{1}{2}(q'(x))^2]$, we have

$$\frac{y'''(x)}{y'(x)y''(x)} - \frac{3}{2}\frac{y''(x)}{(y'(x))^2} = \frac{q''(x)}{\sqrt{q(x)}q'(x)} - \frac{q'(x)}{q^{\frac{3}{2}}(x)}.$$

Now apply (A4) and Lemma 2.1, and we obtain the desired result. Q.E.D.

Lemma 2.3. *Assume that $q(x)$ satisfies condition* (A3). *Then*

$$\frac{d}{dy}\left(\frac{\psi'(y)}{\psi(y)}\right) < 0.$$

Proof. Since

$$\begin{aligned}
\frac{d}{dy}\left(\frac{\psi'(y)}{\psi(y)}\right) &= \frac{\psi''(y)}{\psi(y)} - \left(\frac{\psi'(y)}{\psi(y)}\right)^2 \\
&= \frac{-6(q'(x))^2 + 4q(x)q''(x)}{16q^3(x)} \\
&< 0 \qquad \text{(by condition (A3))}.
\end{aligned}$$

Q.E.D.

Lemma 2.4. *Assume that* $q(x)$ *satisfies condition* (A3). *Then*

$$\frac{\psi''(y)}{\psi(y)} \leq 0.$$

Proof. From (2.6) we have

$$\begin{aligned} \frac{\psi''(y)}{\psi(y)} &= \frac{1}{4q^3}[q''(x)q(x) - \frac{5}{4}(q'(x))^2] \\ &< 0 \quad \text{(by condition A3)}. \end{aligned}$$

Q.E.D.

Now, let $u(y,a) = v(x,a)$. Then (2.1) becomes

$$\begin{aligned} u_{yy} + 2\,\frac{\psi'(y)}{\psi(y)}\,u_y + \sin u(y) &= 0, \\ u_y(0) &= 0, \\ u(0) &= v(x^*). \end{aligned} \tag{2.8}$$

We note that $u_y(0) = 0$ follows directly from L'Hospital rule. Let $w(y) = q^{\frac{1}{4}}(x(y))u(y)$. Then (2.8) becomes

$$w_{yy} + \left(-\frac{\psi''(y)}{\psi(y)} + \frac{\sin u(y)}{u(y)}\right) w = 0. \tag{2.9}$$

Since $|u(y)| = |v(x(y))| \leq v^* = v(x^*,a) < \pi$ for all $y \geq 0$, we have

$$\frac{\sin u(y)}{u(y)} \geq \frac{\sin v^*}{v^*} = \delta = \delta(a) > 0. \tag{2.10}$$

From Lemma 2.4 and (2.10) we compare (2.9) with

$$\varpi_{yy}(y) + \delta\varpi(y) = 0.$$

Let $z_1 < z_2 < \cdots < z_l < \cdots$ be the zeros of $\varpi(y)$. Then from Sturm's comparison theorem it follows that

$$|\, z_l - z_{l-1} \,| \leq \frac{\pi}{\sqrt{\delta}}\,. \tag{2.11}$$

Let $0 = \gamma_0 < \gamma_2 < \gamma_4 < \cdots < \gamma_{2k} < \cdots <$ and $\gamma_1 < \gamma_3 < \cdots < \gamma_{2k+1} < \cdots$, be the local maxima and local minima of $u(y,a)$, respectively. Thus from (2.11) we have the following lemma.

Lemma 2.5. *Assume that* $q(x)$ *satisfies conditions* (A1) *and* (A3). *Then there exists* $D = D(a) > 0$ *such that* $|\gamma_k - \gamma_{k-1}| \leq D$ *for all* $k \geq 0$.

Since $v(x,a)$ is oscillatory over $[0,\infty)$ with decreasing amplitudes, from $u(y,a) = v(x,a)$ so is $u(y,a)$. Assume

$$\lim_{k\to\infty} u(\gamma_{2k}, a) = \xi \geq 0 \tag{2.12}$$

and

$$\lim_{k\to\infty} u(\gamma_{2k-1}, a) = \eta \leq 0. \tag{2.13}$$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. From (2.12), (2.13), Lemma 2.5 and the Cauchy Schwarz inequality it follows that for each $k \geq 1$

$$\begin{aligned} \xi - \eta &\leq |u(\gamma_{2k}) - u(\gamma_{2k-1})| = \left| \int_{\gamma_{2k-1}}^{\gamma_{2k}} u_y(y)\, dy \right| \leq \int_{\gamma_{2k-1}}^{\gamma_{2k}} |u_y(y)|\, dy \\ &\leq (\gamma_{2k} - \gamma_{2k-1})^{\frac{1}{2}} \left[\int_{\gamma_{2k-1}}^{\gamma_{2k}} (u_y(y))^2\, dy \right]^{\frac{1}{2}} \\ &\leq D^{\frac{1}{2}} \left[\int_{\gamma_{2k-1}}^{\gamma_{2k}} (u_y(y))^2\, dy \right]^{\frac{1}{2}} \end{aligned}$$

or

$$\frac{(\xi - \eta)^2}{D} \leq \int_{\gamma_{2k-1}}^{\gamma_{2k}} (u_y(y))^2\, dy \tag{2.14}$$

Multiplying u_y on both side of (2.8) and integrating the resulting identity from c to d yields

$$\frac{1}{2}(u'(d))^2 - \frac{1}{2}(u'(c))^2 + \int_c^d 2\frac{\psi'(y)}{\psi(y)}(u'(y))^2\, dy + \cos\, u(c) - \cos\, u(d) = 0. \tag{2.15}$$

Let $c = 0, d = \gamma_k$ in (2.15) and let $k \to \infty$, we have that

$$\int_0^\infty 2\, \frac{\psi'(y)}{\psi(y)}(u'(y))^2\, dy < \infty. \tag{2.16}$$

From Lemma 2.3 and (2.14), we have the following inequality

$$\begin{aligned} 2\int_{\gamma_{2k-1}}^{\gamma_{2k}} \frac{\psi'(y)}{\psi(y)}(u'(y))^2\, dy &\geq 2\frac{\psi'(\gamma_{2k})}{\psi(\gamma_{2k})} \int_{\gamma_{2k-1}}^{\gamma_{2k}} (u'(y))^2 dy \\ &\geq 2\frac{\psi'(\gamma_{2k})}{\psi(\gamma_{2k})} \frac{(\xi - \eta)^2}{D}. \end{aligned} \tag{2.17}$$

By Mean Value Theorem and Lemma 2.3, we have

$$\begin{aligned}
0 &\le \frac{\psi'(\gamma_{2k-1})}{\psi(\gamma_{2k-1})} - \frac{\psi'(\gamma_{2k})}{\psi(\gamma_{2k})} \\
&= (\gamma_{2k-1} - \gamma_{2k}) \left(\frac{\psi'}{\psi}\right)'(q_k), \quad \text{where} \quad q_k \in (\gamma_{2k-1}, \gamma_{2k}), \\
&= (\gamma_{2k} - \gamma_{2k-1}) \left(\left(\frac{\psi'(q_k)}{\psi(q_k)}\right)^2 - \frac{\psi''(q_k)}{\psi(q_k)} \right) \\
&= (\gamma_{2k} - \gamma_{2k-1}) \frac{\psi'(q_k)}{\psi(q_k)} \left[\frac{\psi'(q_k)}{\psi(q_k)} - \frac{\psi''(q_k)}{\psi'(q_k)} \right] \\
&\le D \left[\frac{\psi'(q_k)}{\psi(q_k)} - \frac{\psi''(q_k)}{\psi'(q_k)} \right] \frac{\psi'(\gamma_{2k-1})}{\psi(\gamma_{2k-1})}.
\end{aligned}$$

By Lemma 2.1 and Lemma 2.2, there exists $k_0 > 0$ such that

$$\frac{\psi'(\gamma_{2k})}{\psi(\gamma_{2k})} \ge \frac{1}{2} \frac{\psi'(\gamma_{2k-1})}{\psi(\gamma_{2k-1})}. \tag{2.18}$$

for all $k \ge k_0$. From (2.14), (2.18), and Lemma 2.3, we have that for $k \ge k_0$

$$\begin{aligned}
\int_{\gamma_{2k-1}}^{\gamma_{2k}} 2\frac{\psi'(y)}{\psi(y)}(u'(y))^2\,dy &\ge 2\frac{\psi'(\gamma_{2k})}{\psi(\gamma_{2k})} \frac{(\xi-\eta)^2}{D} \\
&\ge \frac{\psi'(\gamma_{2k-1})}{\psi(\gamma_{2k-1})} \frac{(\xi-\eta)^2}{D} \\
&\ge \left(\tfrac{\xi-\eta}{D}\right)^2 \int_{\gamma_{2k-1}}^{\gamma_{2k}} \frac{\psi'(y)}{\psi(y)}\,dy, \\
&= \left(\frac{\xi-\eta}{D}\right)^2 \left[\ln \frac{\psi(\gamma_{2k})}{\psi(\gamma_{2k-1})}\right].
\end{aligned} \tag{2.19}$$

Summing up (2.19) over $k \ge k_0$ yields

$$\int_{\gamma_{k_0-1}}^{\infty} 2\frac{\psi'(y)}{\psi(y)}(u'(y))^2\,dy \ge \left(\frac{\xi-\eta}{D}\right)^2 \sum_{k=k_0}^{\infty} [ln\,\psi(\gamma_{2k}) - ln\,\psi(\gamma_{2k-1})]. \tag{2.20}$$

Therefor $\xi - \eta = 0$, since otherwise (2.16) and $\lim_{y\to\infty} \ln\,\psi(y) = \lim_{x\to\infty} \frac{1}{4} \ln\, q(x) = \infty$ would lead to a contradiction. Since $\xi \ge 0$ and $\eta \le 0$, we have that $\xi = \eta = 0$, that is $\lim_{x\to\infty} v(x,a) = 0$.

Since $w(y)$ and $u(y)$ have exactly the same zeros in $(0,\infty)$, by lemma 2.5 and $u(y,a)=v(x,a)$, we have

$$\begin{aligned} 2D \geq |y(x_k)-y(x_{k-1})| &= \int_{x_{k-1}}^{x_k} \sqrt{q(t)}\,dt \\ &= q^{\frac{1}{2}}(c_k)|x_k - x_{k-1}|, \end{aligned} \tag{2.21}$$

where $c_k \in (x_{k-1}, x_k)$. So (2.3) follows from (2.21) and $\lim_{x\to\infty} q(x)=\infty$. Thus we complete the proof. Q.E.D.

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Tzy-Wei Hwang
Department of Mathematics, Kaohsiung Normal University
Kaohsiung, Taiwan

Shin-Feng Hwang
Department of Applied Mathematics, Feng-Chia University
Taichung, Taiwan