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GLOBAL EXISTENCE OF SOLUTIONS OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper global existence results for certain higher order differential equations are established. Our analysis is based on a simple and classical application of the Leray-Schauder alternative.

1. Introduction

This paper is concerned with the global existence of solutions for initial value problems for higher order differential equations of the forms

$$L_m\, x(t) = f(t,\, x(t)), \tag{1}$$

$$x(0) = x_0, \quad L_{i-1}\, x(0) = 0, \quad i = 2,\, 3,\, \cdots,\, m, \tag{2}$$

and

$$L_r\, x(t) = f(t,\, x(t)), \tag{3}$$

$$\begin{aligned} &x(0) = x_0, \quad x^{(i-1)}(0) = 0, \quad i = 2,\, 3,\, \cdots,\, r, \\ &(p(0)x^{(r)}(0))^{(i-1)} = 0, \quad i = 1,\, 2,\, \cdots,\, r, \end{aligned} \tag{4}$$

where $m \geq 1, r \geq 1$ are integers. As usual, R^n denotes Euclidean n-space and $|\,.\,|$ denotes the Euclidean norm. In (1)-(2) and (3)-(4), $f : [0, T] \times R^n \to R^n$

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is a continuous function and x_0 is a given constant, the differential operators L_m, L_r are defined respectively by

$$L_m x(t) = \frac{1}{p_m(t)} \frac{d}{dt} \frac{1}{p_{m-1}(t)} \cdots \frac{1}{p_1(t)} \frac{d}{dt} x(t),$$
$$L_r x(t) = (p(t) x^{(r)}(t))^{(r)},$$

for $x \in R^n$, in which $p_i(t),\ i = 1, 2, \cdots, m;\ p_i(t)$ are positive continuous and sufficiently smooth functions defined on $[0,\ T]$. In [4] Kusano and Trench have considered the question of global existence of solutions of special version of equation (1) with prescribed asymptotic behavior, by using Schauder-Tychonoff fixed point theorem (see, also [8]). The problems of existence and growth rates of positive monotonic bounded solutions of the slight variant of equation (3) have been studied by Edelson and Schuur [3] by using Schauder's fixed point theorem.

The main purpose of this paper is to study the global existence of solutions of equations (1)-(2) and (3)-(4) by using a simple and classical application of the topological transversality theorem of Granas [2, p. 61], known as Leary-Schauder alternative. An interesting feature of this method, is that this yields simultaneously the existence of a solution and the maximal interval of existence. In fact, our results in this paper are motivated by the earlier work of Wintner [10] and its extensions recently given by Bobisud and O'Regan [1], Lee and O'Regan [5, 6], Ntouyas, Sficas and Tsamatos [7] and others by using topological arguments based on the Leray-Schauder alternative.

2. Statement of Results

Our existence theorems are based on the following theorem, which is a version of the topological transversality theorem given by A. Granas in [2, p. 61].

Theorem G. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) = \{x \in B : x = \lambda F x \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $U(F)$ is unbounded or F has a fixed point.

Now we present our main result which deals with the global existence of solutions of the equations (1)-(2).

Theorem 1. *Let $f : [0, T] \times R^n \to R^n$ be a continuous function. Assume that:*

(A) *There exists a continuous function* $q : [0, T] \to R_+ = [0, \infty)$ *such that*

$$| f(t, x) | \le q(t)H(| x |), \quad 0 \le t \le T, \quad x \in R^n,$$

where $H : [0, \infty) \to (0, \infty)$ *is a continuous nondecreasing function.*

Then the initial value problem (1)-(2) *has a solution* x *defined on* $[0, T]$ *provided* T *satisfies*

$$\int_0^T M(t_1)dt_1 < \int_c^\infty \frac{dt_1}{H(t_1)}, \tag{5}$$

where $c = | x_0 |$ *and*

$$\begin{aligned} M(t) = \; & p_1(t) \int_0^t p_2(t_2) \int_0^{t_2} p_3(t_3) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ & \times \int_0^{t_{m-1}} p_m(t_m)q(t_m)dt_m dt_{m-1} \cdots dt_3 dt_2, \end{aligned} \tag{6}$$

for $t \in [0, T]$.

Remark 1. We note that our result given in Theorem 1 extends the well known theorem of Wintner [10] on the existence of global solutions of initial value problems for first order differential equations, to higher order differential equations of the form (1)-(2). For further extensions of Wintner's theorem for first order differential equations, see [1, 5].

We next establish the following theorem on the global existence of solutions of the equations (3)-(4).

Theorem 2. *Let* $f : [0, T] \times R^n \to R^n$ *be a continuous function which satisfies the hypothesis (A) in Theorem* 1. *Then the initial value problem* (3)-(4) *has a solution* x *defined on* $[0, T]$ *provided* T *satisfies*

$$\int_0^T N(t_{2r-1})dt_{2r-1} < \int_c^\infty \frac{dt_{2r-1}}{H(t_{2r-1})}, \tag{7}$$

where $c = | x_0 |$ *and*

$$\begin{aligned} N(t) = & \int_0^t \int_0^{t_{2r-2}} \cdots \int_0^{t_{r+1}} \frac{1}{p(t_r)} \int_0^{t_r} \cdots \int_0^{t_1} q(s)ds \\ & \times dt_1 \cdots dt_r dt_{r+1} \cdots dt_{2r-2}, \end{aligned} \tag{8}$$

for $t \in [0, T]$.

Remark 2. We note that, our result given in Theorem 2 is a further extension of the Wintner's theorem given in [10], to higher order differential equations of the form (3)-(4) which in turn yields the global existence of the solution of slight variant of the equations studied by Edelson and Schuur in [3]. For further properties of the solutions of the equations of the form (3)-(4), see [9].

3. Proofs of Theorems 1 and 2

To prove the existence of a solution of initial value problem (1)-(2) we apply Theorem G. First we establish the priori bounds for the initial value problem $(1)_\lambda$–(2), $\lambda \in (0,1)$, where

$$L_m x(t) = \lambda f(t, x(t)). \tag{$(1)_\lambda$}$$

Let $x(t)$ be a solution of $(1)_\lambda$-(2). Then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) = {} & x_0 + \lambda \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ & \times \int_0^{t_{m-1}} p_m(t_m) f(t_m,\, x(t_m)) dt_m dt_{m-1} \cdots dt_2 dt_1. \end{aligned} \tag{9}$$

From (9) and using the hypothesis (A) we have

$$\begin{aligned} |x(t)| \le |x_0| & + \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ & \times \int_0^{t_{m-1}} p_m(t_m) q(t_m) H(|x(t_m)|) dt_m dt_{m-1} \cdots dt_2 dt_1. \end{aligned} \tag{10}$$

Define a function $z(t)$ by the right side of (10), then $|x(t)| \le z(t)$ and

$$\begin{aligned} z(t) \le |x_0| & + \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ & \times \int_0^{t_{m-1}} p_m(t_m) q(t_m) H(z(t_m)) dt_m dt_{m-1} \cdots dt_2 dt_1. \end{aligned} \tag{11}$$

Since $z(t)$ is nondecreasing in t, from (11) we observe that

$$\begin{aligned} z(t) \le |x_0| & + \int_0^t p_1(t_1) H(z(t_1)) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ & \times \int_0^{t_{m-1}} p_m(t_m) q(t_m) dt_m dt_{m-1} \cdots dt_2 dt_1. \end{aligned} \tag{12}$$

Define a function $u(t)$ by the right hand side of (12), then we have

$$z(t) \le u(t), \quad t \in [0,T], \quad u(0) = c,$$

and

$$u'(t) \le M(t)H(u(t)),$$

i.e.,

$$\frac{u'(t)}{H(u(t))} \le M(t). \tag{13}$$

Integrating (13) from 0 to t and using (5) we have

$$\int_c^{u(t)} \frac{dt_1}{H(t_1)} \le \int_0^t M(t_1)dt_1 \le \int_0^T M(t_1)dt_1 < \int_c^\infty \frac{dt_1}{H(t_1)}. \tag{14}$$

From (14) we conclude that there is a constant Q independent of $\lambda \in (0,1)$ such that $u(t) \le Q$ and hence $z(t) \le Q$ for $t \in [0,T]$. Thus we have $|x(t)| \le Q$ for $t \in [0,T]$, and consequently

$$\| x \| = \sup\{ | x(t) | : 0 \le t \le T\} \le Q.$$

We define $B = C([0,T],\ R^n)$ to be the Banach space of all continuous functions from $[0,T]$ into R^n endowed with the sup-norm

$$\| x \| = \sup\{ | x(t) | : 0 \le t \le T\}.$$

In the second step we rewrite the initial value problem (1)-(2) as follows. If $y \in B$ and $x(t) = y(t) + x_0,\ t \in [0,T]$, it is easy to see that y satisfies

$$\begin{aligned} y(0) &= y_0 = 0, \\ y(t) &= \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ &\quad \times \int_0^{t_{m-1}} p_m(t_m) f(t_m,\ y(t_m) + x_0) dt_m dt_{m-1} \cdots dt_2 dt_1, \end{aligned}$$

if and only if x satisfies (1)-(2).

Define $F : B_0 \to B_0,\ B_0 = \{y \in B : y_0 = 0\}$ by

$$\begin{aligned} Fy(t) = &\int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ &\times \int_0^{t_{m-1}} p_m(t_m) f(t_m,\ y(t_m) + x_0) dt_m dt_{m-1} \cdots dt_2 dt_1 \end{aligned} \tag{15}$$

for $t \in [0,T]$. Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0, i.e.,

$$\| w_k \| \leq b \quad \text{for} \quad \text{all } k,$$

where b is a positive constant. From (15) and using the hypothesis (A) and letting $M^* = \sup\{M(t) : t \in [0,T]\}$, we have

$$\begin{aligned} |Fw_k(t)| &\leq \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ &\quad \times \int_0^{t_{m-1}} p_m(t_m) q(t_m) H(|w_k(t_m)| + |\, x_0 \,|) dt_m dt_{m-1} \cdots dt_2 dt_1 \\ &\leq T\, M^*\, H(b + |\, x_0 \,|). \end{aligned}$$

Hence we obtain

$$\| Fw_k \| \leq T\, M^*\, H(b + |\, x_0 \,|).$$

This means that $\{Fw_k\}$ is uniformly bounded.

Now we shall show that the sequence $\{Fw_k\}$ is equicontinuous. Let $0 \leq s_1 \leq s_2 \leq T$. Then

$$\begin{aligned} &|\, Fw_k(s_2) - Fw_k(s_1) \,| \\ &\quad \leq \int_{s_1}^{s_2} p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ &\quad\quad \times \int_0^{t_{m-1}} p_m(t_m) |f(t_m,\ w_k(t_m) + x_0)| dt_m dt_{m-1} \cdots dt_2 dt_1 \\ &\quad \leq \int_{s_1}^{s_2} p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{m-2}} p_{m-1}(t_{m-1}) \\ &\quad\quad \times \int_0^{t_{m-1}} p_m(t_m) q(t_m) H(|w_k(t_m)| + |x_0|) dt_m dt_{m-1} \cdots dt_2 dt_1 \\ &\quad \leq \int_{s_1}^{s_2} M^*\, H(b + |x_0|) dt_1. \end{aligned} \tag{16}$$

From (16) we conclude that $\{Fw_k\}$ is equicontinuous and hence by the Arzela-Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F) = \{y \in B_0 : y = \lambda F y,\ \lambda \in (0,1)\}$ is bounded, since for every y in $U(F)$ the function $x = y + x_0$ is a solution of $(1)_\lambda$-(2), for which we have proved $\| x \| \leq Q$ and hence $\| y \| \leq Q + |\, x_0 \,|$. Now an application of Theorem G shows that the operator F has a fixed point in B_0. This means that the initial value problem (1)-(2) has a solution. This completes the proof of Theorem 1.

The details of the proof of Theorem 2 follows by closely looking at the proof of Theorem 1 given above with suitable modifications. Here we omit the details.

Remark 3. We note that the results obtained in Theorems 1 and 2 can be extended very easily to the following higher order integrodifferential equations of the forms:

$$L_m\, x(t) = \int_0^t K(t,s) f(s, x(s)) ds, \tag{17}$$

with the initial conditions given in (2), and

$$L_r\, x(t) = \int_0^t K(t,s) f(s, x(s)) ds; \tag{18}$$

with the initial conditions given in (4), under some suitable hypotheses on the functions involved in (17)-(2) and (18)-(4). We also note that one can easily extend the ideas of this paper to the equations of the forms (1), (3), (17) and (18) when the function f depends on the delay arguments, under appropriate initial conditions. For similar results for first order differential delay equations, see [6, 7].

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