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HILBERT C^*-MODULES : A USEFUL TOOL

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Abstract. In this article, we show how the concept of Hilbert C^*-module can be used to investigate completely positive linear maps. We show when two unital pure completely positive linear maps of a C^*-algebra into M_n are unitarily equivalent. We also develop and characterize a concept of weak containment between two completely positive linear maps of a C^*-algebra into a von Neumann algebra. In preparation, we exhibit some basic known properties of Hilbert C^*-modules. In addition, we explore the norm of the standard Hilbert column C^*-modules and show it is the Haagerup tensor norm of two operator spaces.

1. Introduction

The subject of this article first appeared in the work of the induced representations of C^*-algebras by Rieffel [16] and the doctoral dissertation of Paschke [15]. Later on, it was used to study the Morita equivalence of C^*-algebras by Rieffel *et al.* [6, 7, 17] and KK-theory of C^*-algebra by Kasparov [12, 13]. More recently, Woronowicz and others use this notion in studying C^*-algebra quantum group theory [23, 24, 25]. It is apparent that the subject has been used in the study of other seemingly unrelated subjects in operator algebras. One of the original topics investigated in Paschke's paper is completely positive linear maps on operator algebras and it is there he used the concept of Hilbert C^*-module to generalize the GNS construction of a representation generated by a state to a "Hilbert C^*-module" representation generated by a completely positive linear map. In this article, we start out with this connection to derive some properties of completely positive linear

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maps and then expand to connections with tensor products. Earlier works of studying extreme n-positive linear maps using Hilbert C^*-modules can be found in [15, 20, 21, 22]. With regard to the references for the subject we have no intension to provide a comprehensive list, and therefore some omissions are inevitable.

In Section 3 we investigate the role of the Hilbert B-module X_ϕ generated by a completely positive map ϕ of a C^*-algebra A into another B. Any state φ on B can induce a representation $\hat{\pi}_\phi$ of A on $X\phi \otimes_B \mathcal{H}_\varphi$ where $\mathcal{H}_\varphi$ is the GNS representation generated by φ. In case $\mathcal{B} = \mathcal{B}(\mathcal{H})$ and φ is a vector state, then $\hat{\pi}_\phi$ is the Stinespring representation. In general such a pair ϕ, φ can induce a representation $\hat{\pi}_{\phi,\varphi}$ of $A \otimes_{\max} B$ on $X_\phi \otimes_B \mathcal{H}_\varphi$. We show that for pure unital maps ψ, ϕ in $CP(A, M_n)$, $\hat{\pi}_\phi$, and $\hat{\pi}_\varphi$ are unitarily equivalent if and only if there is a unitary element u in A such that $\psi(x) = \phi(u^*xu)$ for all x in A. For ψ, ϕ in $CP(A, N)$ where N is a von Neumann algebra, we provide a more concise proof of the necessary and sufficient condition for $\hat{\pi}_{\phi,\varphi}$ being weakly contained in $\hat{\pi}_{\psi,\varphi}$, where φ is a semi-finite normal faithful weight of N. In Section 4 we exhibit an "operator space" like property on the standard column A-module in terms of the Haagerup tensor product. Some similar discussions can also be found in the recent work of D. Blecher [3, 4]. In Section 2, we lay out some basic properties of Hilbert C^*-modules. Readers may find more detailed expositions on Hilbert C^*-modules in [4, 14, 15].

2. Hilbert C^*-modules

Let A be a C^*-algebra (not necessarily unital), and $\mathbf{C}$ be the complex field.

Definition 2.1. A *pre-Hilbert A-module* is a right A-module X equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : X \times X \to A$ satisfying

(i) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0$ only if $x = 0$ for all x in X.

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all x, y, z in X, α, β in $\mathbf{C}$.

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all x, y in X.

(iv) $\langle x, y \cdot a \rangle = \langle x, y \rangle$a for all x, y in X, a in A.

The map $\langle \cdot, \cdot \rangle$ is called an *A-valued inner product* of X, and for $x \in X$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. □

Proposition 2.2. *Let X be a pre-Hilbert A-module.*

(i) $\| \cdot \|$ *is a norm on X;*

(ii) $\|x \cdot a\| \leq \|x\| \, \|a\|$ *for all* $x \in X, \ a \in A$;

(iii) $\langle x, y\rangle\langle y, x\rangle \leq \|y\|^2\langle x, x\rangle$ *for all* $x, y \in X$.

Proof. (i) It follows from (i) of 2.1.

(ii) $\langle xa, xa\rangle = a^*\langle x, x\rangle a \leq \|x\|^2 a^* a$ for all $x \in X, \ a \in A$. Thus $\|xa\|^2 \leq \|x\|^2\|a\|^2$ and $\|xa\| \leq \|x\| \, \|a\|$ for all $x \in X, \ a \in A$.

(iii) For $a \in A, \ x, y \in X$, we have

$$\begin{aligned} 0 &\leq \langle x + ya, x + ya\rangle = \langle x, x\rangle + \langle ya, x\rangle + \langle x, ya\rangle + \langle ya, ya\rangle \\ &= \langle x, x\rangle + a^*\langle y, x\rangle + \langle x, y\rangle a + a^*\langle y, y\rangle a \\ &\leq \langle x, x\rangle + a^*\langle y, x\rangle + \langle x, y\rangle a + \|y\|^2 a^* a. \end{aligned}$$

Set $a = \frac{-\langle y, x\rangle}{\|y\|^2}$ in the above inequality and get

$$0 \leq \langle x, x\rangle - \frac{2\langle x, y\rangle\langle y, x\rangle}{\|y\|^2} + \frac{\langle x, y\rangle\langle y, x\rangle}{\|y\|^2} = \langle x, x\rangle - \frac{\langle x, y\rangle\langle y, x\rangle}{\|y\|^2}. \qquad \square$$

Definition 2.3. The completion of a pre-Hilbert A-module with respect to the norm induced by the A-valued inner product is called a *Hilbert A-module*.

Examples of Hilbert C^*-modules are abound.

Example 2.4. Let A be a C^*-algebra. A is a Hilbert A-module if an A-valued inner product is defined by $\langle x, y\rangle = x^* y$ for all $x, y \in A$. Any closed right ideal of A is a Hilbert sub-A-module under the above inner product.

Example 2.5. Let $\{X_i\}$ be a finite family of Hilbert A-modules. Then $\bigoplus_i X_i$ is a Hilbert A-module with its inner product defined by $\langle (a_i), (b_i)\rangle = \sum_i \langle a_i, b_i\rangle$. When $\{X_i\}$ is an infinite family of Hilbert A-modules we define $\bigoplus_i X_i = \left\{(x_i) | \sum_i \langle x_i, x_i\rangle \text{ converges in norm in } A\right\}$. Thus $\bigoplus_i X_i$ is a Hilbert A-module with the inner product defined by $\langle (a_i), (b_i)\rangle = \sum_i \langle a_i, b_i\rangle$. In case A is a von Neumann algebra we may also consider $\overline{\bigoplus_i} X_i = \left\{(x_i) | \sum_i \langle x_i, x_i\rangle \text{ converges in } \sigma\text{−topology in } A\right\}$. Then, $\bigoplus_i X_i \subseteq \overline{\bigoplus_i} X_i$.

Example 2.6. Standard Hilbert A-modules.

We consider $\bigoplus_{i\in I} X_i$, where $X_i = A$. However, if $A \subseteq B(K)$ for some Hilbert space K, then elements in $\bigoplus_{i\in I} X_i$ can be considered as operators on $l^2(I) \bigoplus K$ with "matrix" expressions of column matrices $\mathbf{x}$ with entries in A. The inner product is simply $\langle \mathbf{y}, \mathbf{z}\rangle = \mathbf{y}^*\mathbf{z}$. This Hilbert A-module, denoted by $C_I(A)$, is called *a standard Hilbert column A-module*. Likewise, we may consider a standard Hilbert row A-module $R_I(A)$ which consists of finite or infinite rows with entries in A with an apparent inner product. Actually, the definition can easily extend to operator spaces $X_i = B \subseteq A$. Another way to describe $C_I(A)$ is as follows. Consider the algebraic tensor $l^2(I) \bigotimes A$ of $l^2(I)$ and A. Thus $l^2(I) \bigotimes A$ has an A-valued inner product defined in terms of the simple tensor product as $\langle \xi \bigotimes a, \eta \bigotimes b\rangle = \langle \xi, \eta\rangle a^* b$ for ξ, η, in $l^2(I)$ and a, b in A. Then the completion of $l^2(I) \bigotimes A$ is $C_I(A)$.

Suppose that X, Y are Hilbert A-modules. We define $L(X, Y)$ to be the set of all maps $T : X \to Y$ for which there is a map $T^* : Y \to X$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x in X and y in Y. It is easy to see that such T must be A-linear (i.e., T is lineal and $T(xa) = T(x)a$ for all x in X, a in A). For each x in the unit ball X_1 of X, define $f_x : Y \to A$ by $f_x(y) = \langle Tx, y\rangle$ for y in Y. Then $\|f_x(y)\| \le \|T^*y\|$ for all x in X_1. It follows from the Banach-Steinhaus theorem that the set $\{\|f_x\| : x \in X_1\}$ is bounded, and hence T is bounded. T^* is called the *adjoint* of T. We note that not every A-linear bounded map from X into Y has an adjoint. In particular $L(X, X) \equiv L(X)$ is a *-algebra and in fact a C^*-algebra, for it is closed under the operator norm and, for T in $L(X)$,

$$\begin{aligned}\|T^*T\| &\ge \sup\{\|\langle T^*Tx, x\rangle\| : x \in X_1\}\\ &= \sup\{\|\langle Tx, Tx\rangle\| : x \in X_1\} = \|T\|^2.\end{aligned}$$

We define X' (or $\mathrm{Hom}_A(X, A)$) to be the set of all bounded A-linear maps of X into A. Any element $x \in X$ induces a bounded A-linear map ϕ_x by $\phi_x(y) = \langle x, y\rangle$ for y in X.

Definition 2.7. A Hilbert A-module is said to be self-dual if X' comprises exactly bounded A-linear maps induced by elements in X.

To have X' in the same category, one asks if X' can be made an A-module. At least for a von Neumann algebra A, X' can be a Hilbert A-module. The procedure is as follows. Let M be a von Neumann algebra with its predual M_* and X, a Hilbert M-module. For each $f \in M_*^+$, the positive cone of M_*, we denote the GNS representation of M generated by f as $\{\pi_f, \mathcal{H}_f\}$. It is easy to check that $f(\langle\cdot,\cdot\rangle)$ induces a pseudo inner product on X. We denote $N_f = \{x \in X : f(\langle x, x\rangle) = 0\}$ which is a closed subspace of X. It follows that X/N_f is a pre-Hilbert space with an inner product defined by $\langle [x],[y]\rangle = f(\langle x,y\rangle)$ which is a result of Proposition 2.2-(iii). We denote the completion of X/N_f by $\mathcal{H}_f$. The following is taken from [15].

Proposition 2.8. *Let T be a bounded A-map from a Hilbert A-module X into another Y. Then $\|T\|^2 = \inf\{k : \langle Tx,Tx\rangle \le k\langle x,x\rangle \;\; \forall \;\; x \in X\}$.*

Proof. We may assume $\|T\| = 1$ and A is unital. First, we show that $\langle Tx,Tx\rangle \le \langle x,x\rangle$ for all x in X. For $x \in X$ and $n = 1,2,\cdots$ we set $b_n = (\langle x,x\rangle + n^{-1})^{-\frac{1}{2}}$ and $x_n = x\cdot b_n$. We have $\langle x_n,x_n\rangle = b_n^*\langle x,x\rangle b_n = \langle x,x\rangle(\langle x,x\rangle + n^{-1})^{-1}$ and $\|x_n\| \le 1$. It follows that $\|Tx_n\| \le 1$ and $\langle TX_n,Tx_n\rangle \le 1$ for $n = 1,2,\cdots$. Since $\langle Tx_n,Tx_n\rangle = \langle T(x\cdot b_n),\, T(x\cdot b_n)\rangle = \langle (Tx)b_n,\; (Tx)b_n\rangle = b_n^*\langle Tx,Tx\rangle b_n$, it follows that $\langle Tx,Tx\rangle = b_n^{-1}\langle Tx,Tx\rangle b_n^{-1} \le b_n^{-2} = \langle x,x\rangle + n^{-1}$ for $n = 1,2,\cdots$, and hence $\langle Tx,Tx\rangle \le \langle x,x\rangle$. Thus, $\inf\{k : \langle Tx,Tx\rangle \le k\langle x,x\rangle \;\forall\; x \in X\} \le 1$. However, suppose that there exits a positve real number $k < 1$ such that $\langle Tx,Tx\rangle \le k\langle x,x\rangle$ for all x in X. Thus it is obvious that $\|T\|^{\le}k$, which contradicts the assumption of $\|T\| = 1$. Thus, $\inf\{k : \langle Tx,Tx\rangle \le k\langle x,x\rangle \;\forall\; x \in X\} = 1$. □

Consider $\varphi \in X'$ and $f \in M_*^+$. It follows from Proposition 2.8 that $\varphi(x)^*\varphi(x) \le \|\varphi\|^2\langle x,x\rangle$ for all x in X, and $|f(\varphi(x)^*\varphi(x))| \le \|\varphi\|^2 f(\langle x,x\rangle)$. Thus $f\circ\varphi(N_f) = 0$, and $f\circ\varphi$ defines a bounded linear functional on $\mathcal{H}_f$. By the Riesz representation theorem, there is a_φ in $\mathcal{H}_f$ with $\|a_\varphi\| \le \|f\|^{\frac{1}{2}}\|\varphi\|$ such that $(f\circ\varphi)(a) = \langle a_\varphi, a\rangle_f$ for all a in $\mathcal{H}$. For a pair of φ,η in X' and $f \in M_*^+$, we first define a sesqui-linear functional by $\langle a_\varphi, a\eta\rangle_f$ via f. It can be verified that $F_{\varphi,\eta} : f \longmapsto \langle a_\varphi, a_\eta\rangle_f$ can be extended to a linear functional on M_*. Next we show that $F_{\varphi,\eta}$ is bounded. Indeed, for $g \in M_*$, $g = f_1 - f_2 + i\,(f_3 - f_4)$, $f_i \in M_*^+$, and $\sum_{i=1}^{4}\|f_i\| \le 2\|g\|$. Then we have

$$
\begin{aligned}
|F_{\varphi,\eta}\,(g)| &\le \sum_{i=1}^{4}\langle a_\varphi^i, a_\eta^i\rangle_{f_i} \le \sum_{i=1}^{4}\|a_\varphi^i\|_{f_i}\|a_\eta^i\|_{f_i} \\
&\le \sum_{i=1}^{4}\|\varphi\|\;\|\eta\|\;\|f_i\| \le 2\|\varphi\|\;\|\eta\|\;\|g\|.
\end{aligned}
$$

Thus $F_{\varphi,\eta} \in M$, and we define an M-inner product on X' by $\langle\varphi,\eta\rangle = F_{\varphi,\eta}$. We leave it to the readers to check that this is indeed an inner product on X'.

Next, we show the inner product constructed above is an extension of that on X viewed as a submodule of X'. For x,y in X and $f \in M_*^+$, $f(\langle\varphi_x,\varphi_y\rangle) = F_{\varphi_x,\varphi_y}(f) = \langle a_{\varphi_x}, a_{\varphi_y}\rangle_f = f(\langle a_{\varphi_x}, a_{\varphi_y}\rangle) = f(\langle x,y\rangle)$, and thus $\langle\varphi_x,\varphi_y\rangle = \langle x,y\rangle$.

Finally, we show $X^{'}$ is self-dual. Let $\phi \in (X^{'})^{'}$, and embed X in $X^{'}$. Thus, $\phi|_X \in X^{'}$. Namely, there is $\eta \in X^{'}$ such that $\phi(\varphi_x) = \eta(x)$ for all x in X. Let $\phi_0(\xi) = \phi(\xi) - \langle \eta, \xi \rangle$ for ξ in X. We show $\phi_0 = 0$. For $\xi \in X^{'}$ and $f \in M_*^+$, there is a sequence $\{[y_n]\}$ in X/N_f converging to a_ξ. From Proposition 2.8 we have $\phi_0(\sigma)^*\phi_0(\sigma) \leq \|\phi_0\|\langle \sigma, \sigma \rangle$ for all $\sigma \in X^{'}$, and thus, for $n = 1, 2, \cdots$

$$\begin{aligned} f(\phi_0(\xi)^*\phi_0(\xi)) &= f(\phi_0(\xi - \varphi_{y_n})^*\phi_0(\xi - \varphi_{y_n})) \\ &\leq \|\phi_0\| f(\langle \xi - \varphi_{y_n}, \xi - \varphi_{y_n} \rangle) \\ &= \|\phi_0\| \{\langle a_\xi, a_\xi \rangle_f - \langle [y_n], a_\xi \rangle_f - \langle a_\xi, [y_n] \rangle_f - \langle [y_n], [y] \rangle_f\} \\ &= \|\phi_0\| \; \|a_\xi - [y_n]\|_f^2. \end{aligned}$$

It follows that $f(\phi_0\,(\xi)^*\phi_0(\xi)) = 0$. Thus $\phi_0 = 0$.

We summarize the above in the following theorem.

Theorem 2.9. *Let X be a pre-Hilbert M-module, where M is a von Neumann algebra. Then the inner product on X extends to $X^{'}$ such that $X^{'}$ is self-dual.*

In general for a Hilbert A-module X (A being a C^*-algebra), the Riesz representation theorem for Hilbert spaces does not generalize to $X^{'}$ but to a subspace of $X^{'}$, which is to be discussed below. Let X, Y be two Hilbert A-modules. For x in X, y in Y, we define $T_{yx} : X \to Y$ by

$$T_{y,x}(z) = y\langle x, z \rangle \quad \text{for all } z \text{ in } X.$$

It is easy to check that $T_{x,y}$ in $L(X, Y)$ satisfies the following conditions:

$$(2.10) \begin{cases} (T_{y,x})^* = T_{x,y} \\ T_{u,v}T_{y,x} = T_{u\langle v,y \rangle, x} = T_{u, x\langle y,v \rangle} \\ ST_{y,x} = T_{S_{y,x}} \\ T_{y,x}P = T_{y,P_x^*} \qquad \text{for all } v \in Y,\ u \in Z,\ S \in L(Y,Z),\ P \in L(Z,X). \end{cases}$$

We denote by $K(X, Y)$ the closed subspace of $L(X, Y)$ spanned by $\{T_{y,x} : x \in X, y \in Y\}$ and we write $K(X)$ for $K(X, X)$. It follows from (2.10) that $K(X)$ is an ideal of $L(X)$. In case of $X = A$, we have $K(A) \cong A$, the isomorphism being given by identifying $T_{y,x}$ with the left multiplication by yx^*. If A is unital, then $K(A) = L(A)$. In general $L(X)$ can be proved to be the multiplier algebra of $K(X)$ (see [14]). To conclude this section we have the following proposition.

Proposition 2.11. *Suppose that X is a Hilbert A-module for a C^*-algebra A. Then $K(X, A)$ is the set of φ_x for $x \in X$.*

Proof. It is easy to see that for $x \in X$, $\varphi_x^*(a) = xa$ for all $a \in A$, and hence $\varphi_x \in L(X, A)$. For $x \in X$ of the form $x = ya^*$ for some y in X and $a \in A$, we have, for all z in X,

$$\begin{aligned}\varphi_x(z) &= \langle x, z\rangle = \langle ya^*, z\rangle = a\langle y, z\rangle \\ &= T_{a,y}(z).\end{aligned}$$

Thus, $\varphi_x = T_{a,y}$. It is clear that $x \longmapsto \varphi_x$ is an isometry. Next, we show that $\|x \cdot e_\alpha - x\| \to 0$ for x in X and an approximate unit $\{e_\alpha\}$ of A. Then it will follow that $\varphi_x = \lim_\alpha \varphi_{xe_\alpha}$ is in $K(X, A)$. Indeed, for $x \in X$, $\langle x - xe_\alpha, x - xe_\alpha\rangle = \langle x, x\rangle - e_\alpha\langle x, x\rangle - \langle x, x\rangle e_\alpha + e_\alpha\langle x, x\rangle e_\alpha$ converges to 0.

Conversely, for $a \in A$, $y \in X$, $T_{a,y} = \varphi_{ya^*}$. It is easy to see that for x, $y \in X, \alpha \in \mathbf{C}, \alpha\varphi_x + \varphi_y = \varphi_{\alpha x + y}$. Through the linear span of $T_{a,y}$, isometry of $x \mapsto \varphi_x$, and the density of XA in X, we have that every element of $K(X, A)$ is of the form φ_x for some $x \in X$. □

3. Hilbert C^*-modules Induced by Completely Positive Maps

Let Φ be a completely positive linear map from a C^*-algebra A into another B, and $A \otimes B$ be the algebraic tensor product of A and B. $A \otimes B$ becomes an A-B-module by left multiplication of elements in A and right multiplication of elements in B, i.e., $a\left(\sum_{i=1}^n a_i \otimes b_i\right) b = \sum aa_i \otimes b_i b$, for a_i, $a \in A$ and b_i, $b \in B, i = 1, \cdots, n$. We define a B-valued sesqui-linear map, $\langle , \rangle_B$, on $A \otimes B$ by $\left\langle \sum_{i=1}^n a_i \otimes b_i, \sum_{j=1}^m a_j' \otimes b_j' \right\rangle_B = \sum_{i,j} b_i^* \Phi(a_i^* a_j') b_j'$, where a_i, a_j' are in A and b_i, b_j', are in B for $i = 1, \cdots, n, j = 1, \cdots, m$. For $x \in A \otimes B$, $x = \sum_{i=1}^n a_i \otimes b_i, \langle x, x\rangle_B = \sum_{i,j=1}^n b_i^* \Phi(a_i^* a_j) b_j = \hat{b}^*[I_n \otimes \Phi(\hat{a}^*\hat{a})]\hat{b}$ is positive in B, where $\hat{a} = [a_i, \cdots, a_n]^*$ and $\hat{b} = [b_1, \cdots, b_n]$. For $x, y, z \in A \otimes B$ one can easily check $\langle x, y\rangle = \langle y, x\rangle^*, \langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $\alpha, \beta \in C$, and $\langle x, y \cdot b\rangle = \langle x, y\rangle b$ for $b \in B$. Let $N_\Phi = \{x \in A \otimes B : \Phi(x^*x) = 0\}$. It is easy to check that N_Φ is an A-B-submodule, and then $A \otimes B/N_\Phi$ is a pre-Hilbert B-module, and its completion is a Hilbert B-module denoted by X_Φ (or X if without confusion).

Next, we may construct a (natural) *-homomorphism π_Φ, from A into $L(X)$ by $\pi_\Phi(a')([a \otimes b]) = [a'a \otimes b]$ for a, a' in A and b in B. It is well-defined for N_Φ is an A-B-submodule, and $\pi_\Phi(a')^* = \pi_\Phi(a'^*)$ for $a' \in A$. This is a generalization of the GNS representation.

Theorem 3.1. *Let Φ be a completely positive linear map from a unital C^*-algebra A into another B. Then there is a Hilbert B-module X, an element* *x in X and a *-representation π_ϕ of A into $L(X)$ such that $\Phi(a) = \langle \pi_\Phi(a)x, x\rangle$ for all a in A.*

Proof. It only remains to show $[1 \otimes 1] = x$ in X gives rise to Φ. Indeed $\langle \pi_\phi(a)[1\otimes 1], [1\otimes 1]\rangle = \langle [a\otimes 1], [1\otimes 1]\rangle = \Phi(a)$ for all a in A. □

Let φ be a state on B and $\{H_\varphi, \pi_\varphi\}$ be the GNS representation pair induced by φ. Then $H_{\phi,\varphi} = X_\phi \otimes_B H_\varphi$ is the *induced* representation $\hat{\pi}_\phi$ of π_φ via X_ϕ first introduced by Rieffel [16].

Proposition 3.2. *Let $B = B(\mathcal{H})$, the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, and Φ be a completely positive linear map of a C^*-algebra A into B. Then the Stinespring representation is the induced representation $\hat{\pi}_\phi$ of π_φ via X_ϕ by a vector state φ on $B(\mathcal{H})$.*

(The proof is straightforward and left to the readers.)

In case B is a von Neumann algebra N, we consider the standard representation (see [19]) $L^2_\varphi(N)$ generated by a semi-finite faithful normal weight φ on N. We denote the induced representation of A by φ via X_ϕ by $\{\pi, \mathcal{H}\}$ (without the subscripts ϕ, φ for simplicity). On this Hilbert space $\mathcal{H}$, we can construct a *-representation (normal) π^0 of N^0, the opposite algebra of N, by the right multiplication of elements in N^0. We observe that π and π^0 are commuting. The triple $\{\mathcal{H}, \pi, \pi^0\}$ is called a *correspondence* from A to N by Connes. He and V. Jones used this concept in [8] to investigate property T for von Neumann algebras.

We denote the set of all completely positive linear maps from a C^*-algebra A into another B by $CP(A,B)$. An element ϕ in $CP(A,B)$ is called *pure* if for all $\psi \in CP(A,B)$ with $\phi - \psi \in CP(A,B)$ we have $\psi = \lambda\phi$ for some scalar λ. It is shown in [2] by Arveson that ϕ is pure in $CP(A, B(\mathcal{H}))$ if and only if the Stinespring representation induced by ϕ is irreducible. In this section we investigate the question that for ϕ, ψ in $CP(A,B)$ when $\hat{\pi}_{\phi,\varphi}$, $\hat{\pi}_{\psi\varphi}$ are unitarily equivalent for some state φ of B. By *unitary equivalence* of two representations π_1, π_2 of a C^*-algebra A we mean that there is an isometry U of $\mathcal{H}_1$ onto $\mathcal{H}_2$, where $\mathcal{H}_i$ is the representation space for $\pi_i, i = 1, 2$, such that $\pi_1(x) = U^*\pi_2(x)U$ for all x in A. We focus on the case where ϕ, φ are pure elements in $CP(A,B)$.

Theorem 3.3. *Let $B = M_n$, the $n \times n$ matrix algebra, A be a unital C^*-algebra, and ϕ, ψ are pure in $CP(A, M_n)$ with $\phi(I) = \psi(I) = I$. The Stinespring representations π_ϕ, π_ψ induced by ϕ and ψ are unitarily equivalent* *if and only if there exists a unitary operator u in A such that $\psi(x) = \phi(u^*xu)$ for all x in A.*

Proof. Suppose that $\psi(x) = \phi(u^*xu)$ for all x in A and a unitary u in A. Let φ be a vector state of $B(\mathcal{H})$ whose GNS representation induces the Stinespring representations π_ϕ, π_ψ generated by ϕ and ψ, respectively. The representation spaces $\mathcal{H}_\phi, \mathcal{H}_\psi$ of π_ϕ, π_ψ are given by

$$\mathcal{H}_\phi = \text{the completion of } A \otimes \mathcal{H}/N_\phi \text{ where } N_\phi = \{x \in A \otimes \mathcal{H} | \langle x, x\rangle_\phi = 0\},$$

$$\mathcal{H}_\psi = \text{the completion of } A \otimes \mathcal{H}/N_\psi \text{ where } N_\psi = \{x \in A \otimes \mathcal{H} | \langle x, x\rangle_\psi = 0\},$$

We define a map U of $A \otimes \mathcal{H}/N_\psi$, onto $A \otimes \mathcal{H}/N_\phi$ by $U([\sum a_i \otimes \xi_i]_\psi) = [\sum a_i u \otimes \xi_i]_\phi$ and check that

$$\begin{aligned}\left\|\Big[\sum a_i u \otimes \xi_i\Big]_\phi\right\|^2 &= \sum_{i,j} \langle \phi(u^* a_i^* a_j u)\xi_i, \xi_j\rangle \\ &= \sum_{i,j} \langle \psi(a_i^* a_j)\xi_i, \xi_j\rangle \\ &= \left\|[\textstyle\sum a_i \otimes \xi_i]_\psi\right\|^2.\end{aligned}$$

Thus, U has a unitary extension of $\mathcal{H}_\psi$ onto $\mathcal{H}_\phi$. One can easily check that $\pi_\psi(x) = U^*\pi_\phi(x)U$ for all $x \in A$.

Suppose $\pi_\psi(x) = U^*\pi_\phi(x)U$ for all x in A and a unitary operator U of $\mathcal{H}$ onto $\mathcal{H}_\phi$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for C^n. We map $\mathbf{C}^n$ in $\mathcal{H}_\phi$ and in $\mathcal{H}_\psi$ respectively by $V_\phi(\xi) = [1 \otimes \xi]_\phi$ and $V_\psi(\xi) = [1 \otimes \xi]_\psi$. Since ψ, ϕ are unital, it follows that V_ϕ, V_ψ are isometries. Then, there exists a unitary operator U_0 on $\mathcal{H}_\phi$ such that $U_0 U V_\psi(e_i) = V_\phi(e_i)$, $1 \le i \le n$. Since π_ϕ is irreducible on $\mathcal{H}_\phi$, it follows from a theorem in [11] that there is a unitary operator U_1 in $\pi_\phi(A)$ such that $U_1 U V_\psi(e_i) = V_\phi(e_i)$, $1 \le i \le n$, and the spectrum of U_1 is not $\{z \in \mathbf{C} | \ |z| = 1\}$. Then there exists a unitary element u in A such that $\pi_\phi(u) = U_1$ by a lemma in [11]. Thus,

$$\begin{aligned}\phi(x) &= V_\phi^* \pi_\phi(x) V_\phi = V_\psi^* U^* \pi_\phi(u^*)\pi_\phi(x)\pi_\phi(u) U V_\psi \\ &= V_\psi^* U^* \pi_\phi(u^*xu) U V_\psi \\ &= V_\psi^* \pi_\psi(u^*xu) V_\psi \\ &= \psi(u^*xu)\end{aligned}$$

for all x in A. □

In case of $n = 1$, Theorem 3.3 reduces to Corollary 8 in [11].

Let $\{\mathcal{H}, \pi, \pi^0\}$ be a correspondence from a unital C^*-algebra A to a von Neumann algebra N. Then there is a representation $\hat{\pi}$ of $A \otimes_{\rm nor} N$ into $B(\mathcal{H})$ such that $\hat{\pi}(a \otimes 1) = \pi(a)$ and $\hat{\pi}(1 \otimes n) = \pi^0(n)$ (for definition of $A \otimes_{\rm nor} N$, see [10]). In case the correspondence $\mathcal{H} = X_\phi \otimes_N L^2(N)$, where X_ϕ is the Hilbert N-module X_ϕ generated by a completely positive linear map ϕ of A into N in Theorem 3.1, we denote the corresponding representation $\hat{\pi}$ of $A \otimes_{\rm nor} N$ by $\hat{\pi}_\phi$. In this case the correspondence H from A to N can be constructed as below. Consider the algebraic tensor product $A \otimes L^2(N)$. For $u = \sum m_i \otimes \eta_i$, $v = \sum m_j^{'} \otimes \eta_j^{'}$, $m_i, m_j^{'} \in A$, and $\eta_i, \eta_j^{'} \in L^2(N)$, $i = 1, \cdots, k, j = 1, \cdots, 1$, we define

$$\langle u, v\rangle = \sum_{i,j} \langle \phi(m_i^* m_j^{'})\eta_i, \eta_j^{'}\rangle.$$

Let $\mathcal{I} = \{u \in A \otimes L^2(N) \mid \langle u, u\rangle = 0\}$. It is clear that $\mathcal{I}$ is a closed A-submodule of $A \otimes L^2(N)$. Thus $\mathcal{H}$ is the completion of $A \otimes L^2(N)/\mathcal{I}$.

Definition 3.4. Let ϕ, ψ be in $CP(A, N)$. ϕ is said to be *weakly contained in* ψ, if $\hat{\pi}_\phi$ is weakly contained in $\hat{\pi}_\psi$, and it is denoted by $\phi \subset \psi$ and $\hat{\pi}_\phi \subset \hat{\pi}_\psi$. For the definition of the weak containment of representations of C^*-algebra, please see [9].

Definition 3.5. Let ϕ be a completely positive linear map of a C^*-algebra A into another B, and $\overline{a} = \{a_i\}_{i=1}^n \subseteq A, \overline{b} = \{b_i\}_{i=1}^n \subseteq B$. The (completely positive) linear map defined below is called a *completely positive linear map associated with* ϕ and it is denoted by ${}_{\overline{a}}\phi_{\overline{b}}$. For x in A, ${}_{\overline{a}}\phi_{\overline{b}}(x) = \sum_{i,j} b_i^* \phi(a_i^* x a_j) b_j$.

First we have to justify the complete positivity of $\overline{a}\phi\overline{b}$. As a matter of fact, $\overline{a}\phi\overline{b}$ is the composition of the following maps each of which is easily recognized as completely positive:

$$A \to A \otimes M_n \to A \otimes M_n \longrightarrow B \otimes M_n \longrightarrow B \otimes M \longrightarrow B$$

$$x \mapsto x \otimes I_n \mapsto \hat{a}^*(x \otimes I_n)\hat{a} \mapsto [\phi(a_i^* x a_j)]_{ij} \mapsto \hat{b}^*[\quad]_{ij}\hat{b} \mapsto e_{11}[\quad]_{ij}e_{11},$$

where $\hat{a} = \begin{bmatrix} a_1 \cdots a_n \\ 0 \end{bmatrix}, \hat{b} = \begin{bmatrix} b_1 & \\ \vdots & 0 \\ b_n & \end{bmatrix}$ with $a_i \in A, b_i \in B, i = 1, \cdots, n$.

In case $B = \mathbf{C}$ in the above definition, one gets a weak* dense subset of positive linear functional associated with π_ϕ, (the GNS representation generated by ϕ).

The following theorem and its proof can be found in [8].

Theorem 3.6. *Let π_1, π_2 be two representations of a C^*-algebra A. Then the following conditions are equivalent:*

(1) $\pi_1 \subset \pi_2$.

(2) *ker* $\pi_1 \supseteq \ker \pi_2$, *where ker* $\pi = \{x \in A \,|\, \pi(x) = 0\}$.

(3) *Every positive linear functional associated with* π_1 *is in the weak***-limit of the convex set generated by the positive linear functionals associated with* π_2.

Proposition 3.7. *Let* ϕ *be a completely positive map of* A *into* N. *Then* ${}_{\overline{a}}\phi_{\overline{b}} \subset \phi$.

Proof. We may assume A is unital. Then $\hat{\pi}_\psi$ and $\hat{\pi}_{\overline{a}\psi\bar{b}}$ have a cyclic vector ξ, ξ', respectively. By Theorem 3.6, it suffices to show that the linear functional $u \mapsto \langle \hat{\pi}_{\overline{a}\phi\bar{b}}(u)(m \otimes \eta), (m \otimes \eta)\rangle$, for u in $A \otimes_{\mathrm{Nor}} N$, $m \otimes \eta \in A \otimes L^2(N)$, is in the weak* convex hull of all positive linear functionals associated with $\hat{\pi}_\phi$. For $u = \sum_{i=1}^{l} x_i \otimes y_i, x_i \in A, y_i \in N^0, i = 1, \cdots, l$, we have for $m \in A,\ \eta \in L^2(N)$,

$$\begin{aligned}
&\langle \hat{\pi}_{\overline{a}\phi\bar{b}}(\textstyle\sum x_i \otimes y_i)(m \otimes \eta),\ (m \otimes \eta)\rangle_{\overline{a}\phi\bar{b}} \\
&\quad = \sum_i \langle x_i m \otimes \eta y_i,\ m \otimes \eta\rangle_{\overline{a}\phi\bar{b}} \\
&\quad = \sum_i \langle {}_{\overline{a}}\phi_{\overline{b}}(m^* x_i m)\eta y_i,\ \eta\rangle_{L^2(N)} \\
&\quad = \sum_i \sum_{k,j} \langle b_j^* \phi(a_j^* m^* x_i m a_k) b_k \eta y_i,\ \eta\rangle \\
&\quad = \sum_i \left\langle \hat{\pi}(x_i \otimes y_i) \sum_j m a_j \otimes b_j \eta, \sum_k m a_k \otimes b_k \eta \right\rangle_\phi \\
&\quad = \left\langle \hat{\pi}_\phi \left(\sum_i x_i \otimes y_i\right) \sum_j m a_j \otimes b_j \eta, \sum_k m a_k \otimes b_k \eta \right\rangle_\phi .
\end{aligned}$$

Thus $u \mapsto \langle \hat{\pi}_{\overline{a}\phi\bar{b}}(u)(m \otimes \eta), (m \otimes \eta)\rangle$ is associated with $\hat{\pi}_\phi$. □

Proposition 3.8. *Let* $\{\phi_i\}$ *be a net of completely positive linear maps of a* C^**-algebra* A *into a von Neumann algebra* N, *and* ξ_i *be the canonical cyclic vector for* $\hat{\pi}_{\phi_i}(A), \hat{\pi}(A)$ *for short.* $\{\phi_i\}$ *converges to* ϕ *in the point-*σ *topology if and only if* $\{w_{\xi_i} \circ \hat{\pi}_i\}$ *converge to* $w_\xi \circ \hat{\pi}$ *in the weak** *topology of* $(A \otimes_{\mathrm{nor}} N)^*$, *where* ξ *is the canonical cyclic vector for* $\hat{\pi}_\phi(A)$, *and* $w_{\epsilon_i},\ w_\epsilon$ *are the vector states defined by* ϵ_i *and* ϵ.

Proof. Suppose φ is a normal semi-finite faithful weight on N that gives rise to the standard representation $L^2(N)$. By Proposition 2.13 in [19] we know $\{\varphi(\cdot a) : a \in \mathcal{T}_\varphi^2\}$ is norm-dense of N_*, the predual of N, where $\mathcal{T}_y$ is the Tomita algebra with respect to φ. $\mathcal{T}_\varphi$ is σ-dense in N, represented on $L^2(N)$. It follows from this that $\phi_i \to \phi$ in point-σ topology is equivalent to $w_{\xi_i} \circ \hat{\pi}_{\phi_i} \to w_\xi \circ \hat{\pi}_\phi$ in the weak* topology. □

We have given a short proof for a theorem below first proved by Anatharaman-Delaroche and Havet in 1990 (cf. [1]).

Theorem 3.9. *Let ψ, ϕ be in $CP(A, N)$. Then $\psi \subset \phi$ if and only if ψ is in the point-σ closure of the convex set generated by all completely positive linear maps associated with ϕ.*

Proof. Theorem 3.9 follows readily from Theorem 3.6, Propositions 3.7 and 3.8. □

4. Haagerup Tensor Products

An operator space X is a subspace of a C^*-algebra A. We denote the space of all $n \times m$ matrices with entries in X by $M_{n,m}(X)$, $M_{n,n}(X)$ by $M_n(X)$, and $M_{n,m}(\mathbf{C})$ by $M_{n,m}$. It is easy to see that $M_{n,m}(X)$ is a left $M_{p,n}$-module and right $M_{m,k}$ module. An operator space X inherits the following properties from the C^*-algebra A containing X. Let $X \subset A$ and let A be faithfully represented in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus $M_{n,m}(X) \subset B\left(\bigotimes_{i=1}^{m} \mathcal{H}_i, \bigotimes_{j=1}^{n} \mathcal{H}_j\right)$, where $\mathcal{H}_i = \mathcal{H}_j = \mathcal{H}$, and $M_{n,m}(X)$ inherits a natural norm $\| \quad \|_{n,m}$, satisfying the following.

Proposition 4.1.

(1) *For x in $M_{n,p}$, y in $M_{p,q}(X)$, z in $M_{q,m}$, we have $\|xyz\|_{n,m} \le \|x\| \; \|y\|_{p,q}$ $\|z\|$.*

(2) *For x in $M_n(X)$, 0 in $M_m(X)$, $\|x \otimes 0\|_{n+m} = \|x\|_n$.*

(3) *For x in $M_n(X)$, y in $M_m(X)$, $\|x \otimes y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$.*

The proof is left to the readers.

One can use the above properties to define matrix normed space as follows.

Definition 4.2. A normed linear space X is called a *matrix normed space* if, for each n, m positive integers, $M_{n,m}(X)$ is endowed with a norm $\| \cdot \|_{n,m}$ such that for x in $M_{n,p}$, y in $M_{p,q}(X)$, z in $M_{q,m}$, we have $\|xyz\|_{n,m} \le \|x\| \; \|y\|_{p,q} \|z\|$.

In 1988, Ruan [18] proved that any matrix normed space satisfing condition (3) in Proposition 4.1 can be (completely) isometrically represented as a subspace of a C^*-algebra A. Let X, Y be two operator spaces. There are many ways to define a cross norm on the algebraic tensor product $X \otimes Y$ so that its completion remains an operator space. Amongst them, there is one with the complete injective property (for definition see [5]) which is the Haagerup tensor product given below.

Let $\hat{x} = [x_1, \cdots, x_n] \in M_{1,n}(X)$ and $\hat{y} = [y_1, \cdots, y_n]^t \in M_{n,1}(Y)$. We denote $\hat{x} \odot \hat{y} = \sum\limits_{i=1}^{n} x_i \otimes y_i$.

Definition 4.3. For $u \in x \otimes Y$, the Haagerup tensor norm on $X \otimes Y$ is defined as

$$\|u\|_h = \inf\{\|\hat{x}\| \ \|\hat{y}\| : u = \hat{x} \odot \hat{y}\},$$

where infimum is taken over all such expressions for u with $\hat{x}$ in $M_{1,n}(X)$ and $\hat{y}$ in $M_{n,1}(X)$.

It is crucial to be familiar with the calculation of the norm of $\hat{x}$ in $M_{1,n}(X)$ and the norm of $\hat{y}$ in $M_{n,1}(X)$ in order to understand the Haagerup tensor norm.

It was shown that under this norm $X \otimes Y$ is an operator space denoted by $X \otimes_h Y$ [5]. The standard column Hilbert A-module, $C_I(A)$, which is discussed in Section 2, can be identified with $l^2(I, A)$. This is explained below. In general, for a Hilbert space $\mathcal{H}$ and a C^*-algebra A, we consider the algebra tensor product $\mathcal{H} \otimes A$ equipped with a cross norm defined below. For simplicity we assume that $\mathcal{H}$ is separable and $\{\epsilon_i\}$ is an orthonormal basis for $\mathcal{H}$. For $u \in \mathcal{H} \otimes A$, $u = \sum\limits_{i=1}^{n} \xi_i \otimes a_i$, $\xi_i \in \mathcal{H}$, $a_i \in A$, we have $\xi_i = \sum\limits_{k} \lambda_{ik} \epsilon_k$ and $u = \sum\limits_{k} \epsilon_k \otimes \left(\sum\limits_{i} \lambda_{ik} a_i \right)$. u can be identified with an A-valued function, f_u, on $\mathbf{N}$, such that $f_u(k) = \sum\limits_{i=1}^{n} \lambda_{ik} a_i$. The A-valued inner product on $\mathcal{H} \otimes A$ can be defined as below. For $u = \sum\limits_{k} \epsilon_k \otimes (\sum \lambda_{ik} a_i)$, $v = \sum\limits_{j=1}^{m} \eta_j \otimes b_j = \sum_k \epsilon_k \otimes \left(\sum\limits_{j=1}^{m} \gamma_{jk} b_j \right)$, with $\eta_j = \sum\limits_{k} \gamma_{jk} \epsilon_k$, with $a_i b_j$ in A, λ_{ik}, γ_{jk} in $\mathbf{C}$, we define

$$\begin{aligned} \langle u, v \rangle &= \sum_{k} \left(\sum_{i=1}^{n} \lambda_{ik} a_i \right)^* \left(\sum_{j=1}^{m} \gamma_{jk} b_j \right) \\ &= \sum_{ij} \left(\sum_{k} \overline{\lambda}_{ik} \gamma_{jk} \right) a_i^* b_j = \sum_{ij} \langle \xi_i, \eta_j \rangle a_i^* b_j, \end{aligned}$$

where $|\sum_k \overline{\lambda}_{ik}\gamma_{jk}| \le \left(\sum_k |\lambda_{ik}|^2\right)^{\frac{1}{2}} \left(\sum_k |\gamma_{jk}|^2\right)^{\frac{1}{2}} < \infty$. This shows that $\mathcal{H} \otimes A$ can be identified as a dense subset of square summable A-valued functions on $\mathbf{N}$, where $\mathbf{N}$ is the set of positive integers. Next, we may identify l^2 as a subspace C_{N} of $\mathrm{B}(l^2)$ and thus l^2 is an operator space. For $u = (\alpha_i) \in l^2$ we consider an infinite matrix (t_{ij}) with its only nonzero entries in the first column and $t_{i1} = \alpha_i$, $i = 1, 2, \cdots$. In this notation, we have the following theorem.

Theorem 4.4. *For a standard column Hilbert A-module $C_{\mathrm{N}}(A)$, we have $C_{\mathbf{N}}(A) = C_{\mathbf{N}} \otimes_h A$.*

Proof. It suffices to show that the two norms agree on a dense subspace $C_{\mathbf{N}} \otimes A$. Let u be an element in $C_{\mathbf{N}} \otimes A$ and $u = \sum_{i=1}^{n} h_i \otimes y_i$, where $h_i = (h_i^k)_{k=1}^{\infty}$ and $y_i \in A$, $i = 1, \cdots, n$, or $u = \hat{x} \odot \hat{y}$, where $\hat{x} = [h_1, \cdots, h_n]$ and $\hat{y} = [y_1, \cdots, y_n]^t$. Let $\epsilon_k = (0, \cdots, 0, 1, 0, \cdots)$, where the only nonzero entry in ϵ_k is at the κ-th component, for $\kappa = 1, 2, \cdots u$ can be considered as an A-valued function f_u on $\mathbf{N}$, $f_u(k) = \sum_{i=1}^{n} h_i^k y_i$. Then

$$\begin{aligned} \|u\|^2 &= \left\| \sum_{k-1}^{\infty} \left(\sum_{i=1}^{n} \overline{h}_i^k y_i^* \right) \left(\sum_{j=1}^{n} h_j^k y_j \right) \right\| \\ &= \left\| \sum_{i,j=1}^{n} \left(\sum_{k=1}^{\infty} \overline{h}_i^k h_j^k \right) y_i^* y_j \right\| \\ &= \left\| \sum_{i,j}^{n} \langle h_i, h_j \rangle y_i^* y_j \right\|. \end{aligned}$$

The above calculation is independent of the choice of an orthonormal basis for l^2.

Actually, $\sum_{ij}^{n} \langle h_i, h_j \rangle y_i^* y_j = R_1 \cdot Y^* \cdot T \cdot Y \cdot C_1$, where C_1 is the $n \times n$ matrix with first column consisting of 1's and zero elsewhere, $R_1 = C_1^t$, Y is an $n \times n$ diagonal matrix with diagonal entries $\{y_1, \cdots, y_n\}$ and T is an $n \times n$ positive matrix with (i, j)-th entry $\langle h_i, h_j \rangle$. Clearly $R_1 Y^* T Y C_1 \le \|T\| R_1 Y^* Y C_1$. T can be viewed as an operator on $\mathbf{C}^n$, and T is unitarily equivalent to $\hat{x}^* \hat{x}$. Thus,

$$\begin{aligned} \|u\|^2 &\le \|T\| \; \|R_1 Y^* Y C_1\| = \|\hat{x}\|^2 \left\| \sum_{i=1}^{n} y_i^* y_i \right\| \\ &= \|\hat{x}\|^2 \; \|\hat{y}\|^2. \end{aligned}$$

On the other hand, let $\{h_i^{'}, \cdots, h_m^{'}\}$ be an orthonormal basis for the subspace of l^2 spanned by $\{h_1, \cdots, h_n\}$. Thus $h_i = \sum_{k=1}^{m} \lambda_{ik} h_k^{'}$ and $u = \sum_{i=1}^{n} \left(\sum_{k=1}^{m} \lambda_{ik} h_k^{'} \right) \otimes y_i = \sum_{k=1}^{m} h_k^{'} \otimes \sum_{i=1}^{n} \lambda_{ik} y_i = \hat{x}^{'} \odot \hat{y}^{'}$, where $\hat{x}^{'} = [h_1^{'}, \cdots, h_m]$ and $\hat{y}^{'} = [b_1, \cdots, b_m]^t$ and $b_j = \sum_{i=1}^{n} \lambda_{ij} y_i, j = 1, \cdots, m$. Thus, repeating the above calculation we get $\|u\|^2 = \left\| \sum_{j=1}^{m} b_j^* b_j \right\| = \|\hat{y}^{'}\|^2 = \|\hat{y}^{'}\|^2 \, \|\hat{x}\|^2$, for $\|\hat{x}^{'}\| = 1$. Hence the two norms agree. □

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