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EXISTENCE OF VECTOR EQUILIBRIA VIA EKELAND'S VARIATIONAL PRINCIPLE

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Abstract. In this paper, we prove Ekeland's type of variational principle for a vector equilibrium problem, and present a Caristi-Kirk type fixed point theorem and an existence result for vector equilibrium solution.

1. Introduction

We consider the following vector equilibrium problem:

(VEP) Find $x_0 \in X$ satisfying $f(x_0, y) \notin -\text{int}C$ for all $y \in X$,

where X is a given set, Y a topological vector space, int$C$ a topological interior of a set $C \subset Y$ and $f : X \times X \to Y$ a given vector-valued function. A solution x_0 of (VEP) is called a vector equilibrium point, and the constraint is equivalent to $f(x_0, X) \cap (-\text{int}C) = \emptyset$ where $f(x_0, X) = \bigcup_{y \in X}\{f(x_0, y)\}$. The study of vector equilibrium problem was initiated in [15, 4] with the form (VEP) above. Afterward various types of equilibrium problems were studied for vector-valued and set-valued bifunctions $\phi : X \times X \to 2^Y$ with several conical constraints like $\phi(x_0, y) \subset C(x_0)$, $\phi(x_0, y) \cap C(x_0) \neq \emptyset$, $\phi(x_0, y) \cap \{-(C(x_0) \setminus \{\theta_Y\})\} = \emptyset$, $\phi(x_0, y) \not\subset -(C(x_0) \setminus \{\theta_Y\})$, $\phi(x_0, y) \cap \{-\text{int}C(x_0)\} = \emptyset$, $\phi(x_0, y) \not\subset -\text{int}C(x_0)$ where $C : X \to 2^Y$ is a convex cone-valued map; see [13] and references therein. The last constraint coincides with that of (VEP) when the ordering moving cone $C(x)$ is fixed and pointed for each vector $x \in X$ and the map ϕ is vector-valued; in the case, the problem (VEP) is the most general. Equilibrium problems have been investigated extensively from different direction by many researchers; see [1, 2, 14]

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and references therein. Because such equilibrium problems include various problems as special cases, for example, complementarity problems, optimization problems, and variational inequality problems. In 2005, Bianchi, Kassay and Pini [5] obtain a variational principle for an equilibrium problem. Furthermore, they generalize it to the case of vector-valued function in [6].

In this paper, we generalize their results in [5] to the case of vector-valued function and lead to an variational principle, different from [6], related to vector equilibrium problems. As a corollary we obtain a Caristi-Kirk type fixed point theorem (see [8]) and an existence result for the vector equilibrium problem (VEP).

We give the preliminary terminology and notation used throughout this paper. Let (X, d) be a complete metric space and Y a topological vector space. For a set $A \subset Y$, core$A$, inte$A$ and cl$A$ denote the algebraic interior, the topological interior and the topological closure of A, respectively. We assume that a set $C \subset Y$ is a solid closed convex cone, that is,

(a) int$C \neq \emptyset$,

(b) cl$C = C$,

(c) $C + C \subseteq C$,

(d) $\lambda C \subseteq C$ for all $\lambda \in [0, \infty)$.

A cone C is pointed if $C \cap (-C) = \{\theta_Y\}$ where θ_Y denotes the origin of Y. If a pointed convex cone $C \subseteq Y$ is given, we can define an ordering in Y by "$x \leq_C y$ when $y - x \in C$." This ordering is compatible with the vector structure of Y, that is, for every $y_1, y_2 \in Y$,

(1) $y_1 \leq_C y_2$ implies that $y_1 + z \leq_C y_2 + z$ for all $z \in Y$,

(2) $y_1 \leq_C y_2$ implies that $\alpha y_1 \leq_C \alpha y_2$ for all $\alpha \geq 0$.

Lemma 1.1. [Lemma 7 in [12] and Theorem 2.3.1 in [11]] *Let C be a closed convex cone. We take $k^0 \in C \setminus (-C)$ and define $h_{C,k^0} : Y \to [-\infty, \infty]$ by*

$$h_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,\big|\, y \in tk^0 - C\}.$$

Then the function h_{C,k^0} has the following six properties:

(i) *h_{C,k^0} is proper;*

(ii) *h_{C,k^0} is lower semicontinuous;*

(iii) *h_{C,k^0} is sublinear;*

(iv) *h_{C,k^0} is C-monotone (that is, $y_1 \leq_C y_2$ implies $h_{C,k^0}(y_1) \leq h_{C,k^0}(y_2)$);*

(v) *$\{y \in Y | h_{C,k^0}(y) \leq t\} = tk^0 - C$;*

(vi) *$h_{C,k^0}(y + \lambda k^0) = h_{C,k^0}(y) + \lambda$ for every $y \in Y$ and $\lambda \in \mathbb{R}$.*

Moreover, if k^0 $\in$*core*C *then* h_{C,k^0} *has the following four properties:*

(vii) h_{C,k^0} *achieves a real value;*

(viii) h_{C,k^0} *is continuous;*

(ix) $\{y \in Y | h_{C,k^0}(y) < t\} = tk^0 -$*core*$C$*;*

(x) h_{C,k^0} *is strictly core*C*-monotone* (*that is,* $y_2 - y_1$ $\in$*core*C *implies* $h_{C,k^0}(y_1) < h_{C,k^0}(y_2)$)*.*

Remark 1. Because of the solidness and convexity of C in the setting of this paper, for each $k^0 \in \text{int}C$ and $y \in Y$, the function $h_{C,k^0}(y)$ above achieves a real value and each symbol "core" in Lemma 1.1 can be replaced by "int."

As a corollary of the above lemma, Gerth and Weidner present the following nonconvex separation theorem.

Lemma 1.2. [Theorem 2.3.6 in [11]] *Assume that* Y *is a topological vector space,* C *a closed solid convex cone and* $A \subset Y$ *a nonempty set such that* $A \cap (-\text{int}C) = \emptyset$*. Then* h_{C,k^0} *is a finite-valued continuous function such that*

$$h_{C,k^0}(-y) < 0 \le h_{C,k^0}(x) \quad \text{for all} \quad x \in A \quad \text{and} \quad y \in \text{int}C,$$

moreover, $h_{C,k^0}(x) > 0$ *for all* $x \in \text{int}A$*.*

2. Main Results

By a similar approach to [2, 6] we obtain a vectorial version of Ekeland's variational principle related to equilibrium problem.

Theorem 2.1. *Let* $f : X \times X \to Y$ *be a function and* $k^0 \in \text{int}C$*. Assume that* f *satisfies the following four conditions:*

(i) *for each* $x \in X$*, there exists* $\tilde{y} \in Y$ *such that* $f(x, X) \cap \big(\tilde{y} - \text{int}C\big) = \emptyset$*;*

(ii) *for each* $x \in X$*,* $\{x' \in X | f(x, x') + d(x, x')k^0 \in -C\}$ *is closed;*

(iii) *for each* $x \in X$*,* $f(x, x) = \theta_Y$*;*

(iv) *for each* $x, y, z \in X$*, the following inequality (vectorial triangle inequality) holds:*

$$f(x, z) \le_C f(x, y) + f(y, z).$$

Then for every $x_0 \in X$ *there exists* $\bar{x} \in X$ *such that*

(1) $f(x_0, \bar{x}) + d(x_0, \bar{x})k^0 \in -C$ *and*

(2) $f(\bar{x}, x) + d(\bar{x}, x)k^0 \notin -C$ *for all* $x \in X \setminus \{\bar{x}\}$.

Proof. First we show that $h_{C,k^0}(f(x, X)) := \bigcup_{z \in f(x,X)} \left\{h_{C,k^0}(z)\right\}$ is bounded from below for all $x \in X$. Given $x \in X$, by Lemma 1.2, we have

$$0 \leq h_{C,k^0}(z - \tilde{y}) \quad \text{for any} \quad z \in f(x, X).$$

Using (iii) of Lemma 1.1, we have

$$-\infty < -h_{C,k^0}(-\tilde{y}) < h_{C,k^0}(z) \quad \text{for any} \quad z \in f(x, X),$$

which implies that $h_{C,k^0}(f(x, X))$ is bounded from below.

Next, we define the following set-valued map $F : X \to 2^X$

$$F(x) := \{y \in X | f(x, y) + d(x, y)k^0 \in -C\}.$$

By assumption (ii), $F(x)$ is a closed set for each $x \in X$ and F has the following two properties:

(a) $y \in F(y)$ (reflexivity),

(b) if $y \in F(x)$ then $F(y) \subset F(x)$ (transitivity).

Property (a) is a consequence of assumption (iii). To prove property (b), we take $y \in F(x)$ and suppose that $z \in F(y)$. Then we have that

$$f(x, y) + d(x, y)k^0 \in -C \quad \text{and} \quad f(y, z) + d(y, z)k^0 \in -C.$$

Adding the above inequalities

$$f(x, y) + f(y, z) + d(x, y)k^0 + d(y, z)k^0 \in -C$$

and using the usual triangle inequlity for d and assumption (iv) for f, we have that

$$f(x, z) + d(x, z)k^0 \leq_C f(x, y) + f(y, z) + d(x, y)k^0 + d(y, z)k^0$$

and hence $f(x, z) + d(x, z)k^0 \in -C$, which implies $z \in F(x)$. Therefore, condition (b) holds.

We define

$$v(x) := \inf_{z \in F(x)} h_{C,k^0}(f(x, z)).$$

For every $z \in F(x)$ we have that $-d(x, z)k^0 \geq_C f(x, z)$. By using assumptions (iv) and (vi) of Lemma 1.1, we have

$$-d(x, z) \geq h_{C,k^0}(f(x, z)) \geq \inf_{z \in F(x)} h_{C,k^0}(f(x, z)) = v(x)$$

and hence $d(x,z) \le -v(x)$. In particular, if we take $x_1, x_2 \in F(x)$, we have that

$$0 \le d(x_1,x_2) \le d(x_1,x)+d(x,x_2) \le d(x,x_1)+d(x,x_2) \le -v(x)-v(x) = -2v(x),$$

which implies that the following upper bound on the diameter of $F(x)$

$$\text{Diam}(F(x)) \le -2v(x) \tag{2.1}$$

for all $x \in X$.

Next we define the following sequence beginning with x_0. Because of the definition of v, we can take $x_n \in F(x_{n-1})$ such that

$$h_{C,k^0}(f(x_{n-1},x_n)) \le v(x_{n-1}) + 2^{-n}. \tag{2.2}$$

Since $F(x_n) \subset F(x_{n-1})$ by property (b), we have

$$\inf_{y\in F(x_{n-1})} h_{C,k^0}(f(x_n,y)) \le \inf_{y\in F(x_n)} h_{C,k^0}(f(x_n,y)) = v(x_n). \tag{2.3}$$

By assumption (iv) of Theorem 2.1, we have

$$f(x_{n-1},y) \le h_C f(x_{n-1},x_n) + f(x_n,y).$$

Applying (iv) and (iii) of Lemma 1.1 to the above inequality, we have

$$\begin{aligned} h_{C,k^0}(f(x_{n-1},y)) \;&\le\; h_{C,k^0}\Big(f(x_{n-1},x_n)+f(x_n,y)\Big) \\ &\le\; h_{C,k^0}(f(x_{n-1},x_n)) + h_{C,k^0}(f(x_n,y)). \end{aligned}$$

By taking infimum in terms of $y \in F(x_{n-1})$, we have

$$\begin{aligned} v(x_{n-1}) \;&=\; \inf_{y\in F(x_{n-1})} h_{C,k^0}(f(x_{n-1},y)) \\ &\le\; h_{C,k^0}(f(x_{n-1},x_n)) + \inf_{y\in F(x_{n-1})} h_{C,k^0}(f(x_n,y)). \end{aligned} \tag{2.4}$$

Combining inequality (2.3) with inequality (2.4), we have

$$v(x_{n-1}) \le h_{C,k^0}(f(x_{n-1},x_n)) + v(x_n).$$

By inequality (2.2) we get

$$-v(x_{n-1}) \le -h_{C,k^0}(f(x_{n-1},x_n)) + 2^{-n} \le v(x_n) - v(x_{n-1}) + 2^{-n},$$

which implies that

$$-v(x_n) \le 2^{-n}. \tag{2.5}$$

Therefore, it follows from (2.1) and (2.5) that

$$\mathrm{Diam}(F(x_n)) \leq -2v(x_n) \leq 2 \cdot 2^{-n},$$

which implies that the diameter of the closed sets $F(x_n)$ converges to 0. By Cantor's theorem, there is $\bar{x}$ satisfying

$$\bigcap_{n=0}^{\infty} F(x_n) = \{\bar{x}\}.$$

Since $\bar{x}$ belongs to $F(x_0)$, conclusion (1) holds. Since $\bar{x}$ belongs to all of the $F(x_n)$, we have that $F(\bar{x}) \subset F(x_n)$ and consequently that $F(\bar{x}) = \{\bar{x}\}$, which implies that

$$f(\bar{x}, x)+d(\bar{x}, x)k^0 \notin -C \quad \text{for all} \quad x \in X\setminus\{\bar{x}\} \qquad \blacksquare$$

Remark 2. We note that taking $Y = \mathbb{R}$, $C = \mathbb{R}_+ = [0, \infty)$ and $k^0 = \varepsilon \in \mathbb{R}_+ \setminus \{0\}$ in Theorem 2.1, we obtain Theorem 2.1 of [5]. We also note that taking $f(x, y) = g(y) - g(x)$ in Theorem 2.1, we obtain a vectorial Ekeland's variational principle. We also note that the pointedness of C and Hausdorff condition of Y are not needed to prove Theorem 2.1.

Example 1. Let $X = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and d the Euclidean metric of $\mathbb{R}^2$. Suppose that $g : X \to Y$ defined by

$$g(x_1, x_2) := \begin{cases} (x_1 - x_2, 0) & x_1 \leq x_2; \\ (0, x_2 - x_1) & x_1 > x_2 \end{cases}$$

and that $f : X \times X \to Y$ defined by

$$f(x, y) := g(y) - g(x).$$

Since the function of the form $a(y) - a(x)$ for some function a satisfies assumptions (iii) and (iv) of Theorem 2.1, so does f. Also, f satisfies other assumptions of Theorem 2.1 for any direction $k^0 \in \mathrm{int}C$. Besides, for $x_0 \in X$, the solution which satisfies conclusions (1) and (2) of Theorem 2.1 is itself only. This example shows that condition (i) of Theorem 2.1 is clearly weaker than condition (iv) of Theorem 1 in [6] and condition (iii) of Theorem 3.1 in [2].

Example 2. Let $X = \mathbb{N}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k^0 = \left(\frac{1}{10}, \frac{1}{10}\right)$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that $g : X \to Y$ defined by

$$g(x) := \begin{cases} \dfrac{x-1}{x}\,(10+x, 10)\,, & \text{if } x \text{ is an odd number;} \\ \dfrac{x-1}{x}\,(10, 10+x)\,, & \text{if } x \text{ is an even number} \end{cases}$$

and that $f : X \times X \to Y$ defined by

$$f(x,y) := g(x) - g(y) + \frac{1}{2}\,|g(x) - g(y)|_{\mathbb{R}^2_+}\,,$$

where for each $(a,b) \in \mathbb{R}^2$, $|(a,b)|_{\mathbb{R}^2_+}$ denotes the absolute value or modulus of (a,b) with respect to the ordering $\mathbb{R}^2_+$, that is, $|(a,b)|_{\mathbb{R}^2_+} = (a,b) \vee (-a,-b) = (|a|,|b|)$. Remark that $|(0,0)|_{\mathbb{R}^2_+} = (0,0)$ and $|z_1 + z_2|_{\mathbb{R}^2_+} \le |z_1|_{\mathbb{R}^2_+} + |z_2|_{\mathbb{R}^2_+}$ as a property of commutative lattice group for $(\mathbb{R}^2, \mathbb{R}^2_+)$. Then $f(x,y)$ satisfies all assumptions of Theorem 2.1. Hence, for each $x_0 \in X$, we can find sufficiently large number $\bar{x} \in X = \mathbb{N}$ satisfying conclusions (1) and (2) of Theorem 2.1 in the case of $x_0 < 50$. Indeed, if $x_0 = 1, 3, 5, \ldots, 49$ then $\bar{x} = 10001$, and if $x_0 = 2, 4, 6, \ldots, 48$ then $\bar{x} = 10000$, otherwise $\bar{x} = x_0$.

3. Applications

3.1. Caristi-Kirk type fixed point theorem

Theorem 3.1. *Let $f : X \times X \to Y$, $T : X \to 2^X$ and $k^0 \in \mathrm{int}C$. Assume that the following conditions:*

(i) *for each $x \in X$ there exists $\tilde{y} \in Y$ such that $f(x, X) \cap \big(\tilde{y} - \mathrm{int}C\big) = \emptyset$;*

(ii) *for each $x \in X$ $\{x' \in X | f(x,x') + d(x,x')k^0 \in -C\}$ is closed;*

(iii) *for each $x \in X$ $f(x,x) = \theta_Y$;*

(iv) *for each $x, y, z \in X$ the following inequality holds,*

$$f(x,z) \le_C f(x,y) + f(y,z);$$

(v) *for each $x \in X$ there exists $y \in X$ such that $y \in Tx$ and*

$$f(x,y) + d(x,y)k^0 \in -C.$$

Then T has at least one fixed point, that is, there exists $x \in X$ such that $x \in Tx$.

Proof. By assumptions (i)-(iv) and Theorem 2.1, there exists $\bar{x} \in X$ such that

$$f(\bar{x}, y) + d(\bar{x}, y)k^0 \notin -C \text{ for all } y \in X \setminus \{\bar{x}\}. \tag{3.1}$$

On the other hand by assumption (v), there exists $\bar{y} \in X$ such that $\bar{y} \in T\bar{x}$ and

$$f(\bar{x}, \bar{y}) + d(\bar{x}, \bar{y})k^0 \in -C.$$

By (3.1), we see that $\bar{x} = \bar{y}$. Hence $\bar{x} \in T\bar{x}$, that is, T has at least one fixed point. ■

Remark 3. We note that taking $f(x, y) = g(y) - g(x)$ in Theorem 3.1, we obtain Theorem 4.1 of [3] (vectorial Caristi-Kirk fixed point theorem).

Example 3. In Example 2, assume that $T' : X \to 2^X$ defined by

$$T'(x) := \{y \in X | f(x, y) + d(x, y)k^0 \in -C\}$$

and that $T : X \to 2^X$ defined by

$$T(x) := \begin{cases} T'(x) \setminus \{x\}, & \text{if } T'(x) \neq \{x\}; \\ T'(x), & \text{otherwise.} \end{cases}$$

Then T satifies condition (v) of Theorem 3.1, and hence T has fixed points. Indeed $\{x \in X | x \geq 50\}$ is the set of fixed points of T.

3.2. Vector equilibria on noncompact sets

Theorem 3.2. *Let $f : X \times X \to Y$ and $k^0 \in \mathrm{int}C$. Assume that the following conditions:*

(i) *for each $x \in X$ there exists $\tilde{y} \in Y$ such that $f(x, X) \cap \big(\tilde{y} - \mathrm{int}C\big) = \emptyset$;*

(ii) *for each $x \in X$ $\{x' \in X | f(x, x') + d(x, x')k^0 \in -C\}$ is closed;*

(iii) *for each $x \in X$ $f(x, x) = \theta_Y$;*

(iv) *for each $x, y, z \in X$ the following inequality holds,*

$$f(x, z) \leq_C f(x, y) + f(y, z);$$

(v) *for each $x \in X$ with $f(x, X) \cap (-\mathrm{int}C) \neq \emptyset$ there exists $y \in X$ such that $y \neq x$ and*

$$f(x, y) + d(x, y)k^0 \in -C.$$

Then f has at least one vector equilibrium point $\bar{x} \in X$ of (VEP), that is, $f(\bar{x}, X) \cap (-\mathrm{int}C) = \emptyset$ for all $y \in X$.

Proof. By assumptions (i)–(iv) and Theorem 2.1, there exists $\bar{x} \in X$ such that

$$f(\bar{x}, y) + d(\bar{x}, y)k^0 \notin -C \text{ for all } y \in X \setminus \{\bar{x}\}. \tag{3.2}$$

Suppose that $\bar{x}$ is not a vector equilibrium point of (VEP), that is,

$$f(\bar{x}, X) \cap (-\mathrm{int}C) \neq \emptyset,$$

then by assumption (v), there exists $\bar{y} \in X$ such that $\bar{y} \neq \bar{x}$ and

$$f(\bar{x}, \bar{y}) + d(\bar{x}, \bar{y})k^0 \in -C.$$

This contradicts to (3.2). Hence $f(\bar{x}, X) \cap (-\mathrm{int}C) = \emptyset$, that is, $\bar{x} \in X$ is an equilibrium point of f. Therefore f has at least one equilibrium point. ■

Remark 4. We note that condition (v) of Theorem 3.2 is weaker than condition (ii) of Theorem 5.1 in [2]. We also note that taking $f(x, y) = g(y) - g(x)$ in Theorem 3.2, we obtain Theorem 5.1 in [3] (vectorial Takahashi's nonconvex minimization theorem).

Example 4. In Example 2, f satifies condition (v) of Theorem 3.2. Hence f has at least one equilibrium point. Indeed, $\bar{x} = 10000$ is an equilibrium point of f.

4. Conclusion

Using a nonlinear scalarizing function which is introduced by Gerth and Wiedner in [10], we generalize Bianchi-Kassay-Pinis' results [5] to the case of vector-valued function and lead to an variational principle for the vector equilibrium problem. As a corollary we obtain Caristi-Kirk type fixed point theorem and an existence result for the vector equilibrium problem, which are generalizations of vectorial Caristi-Kirk fixed point theorem and vectorial Takahashi's nonconvex minimization theorem, respectively.

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