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# A REMARK ON MEROMORPHIC FUNCTIONS SHARING FOUR VALUES

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**Abstract.** In this paper, we prove that if two distinct non-constant meromorphic functions $f$ and $g$ share four distinct values $a_1, a_2, a_3, a_4$ **DM** such that each $a_i$-point is either a $(p, q)$-fold or $(q, p)$-fold point of $f$ and $g$, then $(p, q)$ is either $(1, 2)$ or $(1, 3)$ and $f, g$ are in some particular forms.

## 1. Introduction

The relation between uniqueness and value sharing of meromorphic functions is one major problem in value distribution theory. In this paper, we study two meromorphic functions $f$ and $g$ that share four values **DM**. Throughout this paper, the term "meromorphic" means meromorphic in the whole complex plane.

We say that two meromorphic functions $f$ and $g$ share a value $a$ **CM** (counting multiplicities) if $f(z) - a$ and $g(z) - a$ have the same number of zeros with the same multiplicities, $f$ and $g$ share a value $a$ **IM** (ignoring multiplicities) if $f(z) - a$ and $g(z) - a$ have the same number of zeros without counting their multiplicities, and $f$ and $g$ share a value $a$ **DM** (different multiplicities) if $f(z) - a$ and $g(z) - a$ have the same number of zeros with different multiplicities.

We call $z_0$ a $p$-fold $a$-point of $f$ if $f(z) - a$ has $p$-fold zero at $z = z_0$. When $f(z_0) = g(z_0) = a$, we say that $z_0$ is a $(p, q)$-fold $a$-point of $f$ and $g$ if $z_0$ is a $p$-fold $a$-point of $f$ and a $q$-fold $a$-point of $g$. In case $f$ and $g$ share $a$ and all $a$-points are $(p, q)$-fold, we call $a$ a $(p, q)$-fold value of $f$ and $g$.

When $f$ and $g$ share four values **CM**, Nevanlinna proved the following well-known theorem.

**Theorem 1.** (Nevanlinna [4]). *If two distinct nonconstant meromorphic functions $f$ and $g$ share four values $a_1, a_2, a_3, a_4$* **CM**, *then $f$ is a Möbius transformation*

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*of $g$, two of shared values must be Picard exceptional values, and the cross ratio* $(a_1, a_2, a_3, a_4) = -1$.

In 1979, Gundersen [1] proved that the hypothesis of this theorem can be relaxed to that $f$ and $g$ share only three values **CM**, and one value **IM** (3**CM** and 1**IM**). Gundersen [2] improved his theorem by requiring that $f$ and $g$ share only two values **CM**, and two values **IM** (2**CM** and 2**IM**). Whether the hypothesis can be reduced to that $f$ and $g$ share only one value **IM** and three values **CM** is still open, though many people provided partial answers to this case.

Finally, if $f$ and $g$ share four values **IM**, Gundersen [1] gave a counterexample to show that the conclusion of Theorem 1 is in general false. The meromorphic functions $f$ and $g$ in Gundersen's example indeed share two $(1, 2)$-fold values and two $(2, 1)$-fold values. For two meromorphic functions $f$ and $g$ sharing four values $a_1, a_2, a_3, a_4$, if each $a_i$-point is either a $(1, 2)$-fold or $(2, 1)$-fold point of $f$ and $g$, then Reinders [5, 6] proved that $f$ and $g$ are the precise form of Gundersen's example, up to some Möbius transformation. Latter, Reinders [7] gave another pair of meromorphic functions $f$ and $g$ sharing four values $a_1, a_2, a_3, a_4$, and each $a_i$-point is either a $(1, 3)$-fold or $(3, 1)$-fold point of $f$ and $g$, which are also essentially unique.

It is interesting to know that whether there is any other example of two distinct non-constant meromorphic functions $f$ and $g$ sharing four values $a_1, a_2, a_3, a_4$ and each $a_i$-point is either a $(p, q)$-fold or $(q, p)$-fold point of $f$ and $g$, for a given pair of distinct positive integers $p$ and $q$. The main theorem of this paper shows that the above assertion holds if and only if $(p, q)$ is either $(1, 2)$ or $(1, 3)$, and $f$ and $g$ are essentially the forms given by Gundersen and Reinders.

In this paper, we assume that the reader is familiar with the standard notations and fundamental results in the Nevanlinna theory of meromorphic functions, as found in [3].

## 2. Key Examples and Facts

In order to show that there exist two meromorphic functions sharing four values **IM** but not **CM**, Gundersen [1] gave the following example.

$$f(z) = \frac{e^{h(z)} + b}{(e^{h(z)} - b)^2}, \quad g(z) = \frac{(e^{h(z)} + b)^2}{8b^2(e^{h(z)} - b)},$$

where $h(z)$ is a non-constant entire function and $b$ a non-zero complex number. It is easy to check that $f$ and $g$ share $0, \frac{1}{b}, \infty, -\frac{1}{8b}$ **DM**, where $0$ and $\frac{1}{b}$ are $(1, 2)$-fold values of $f$ and $g$, while $\infty$ and $-\frac{1}{8b}$ are $(2, 1)$-fold values of $f$ and $g$. A somewhat surprising fact is that Reinders [5, 6] proved Gundersen's example is essentially unique.

**Theorem 2.** (Reinders [5, 6]). *Let $f$ and $g$ be meromorphic functions that share four distinct values $a_1, a_2, a_3, a_4$, such that each $a_i$-point is either a $(1,2)$-fold or $(2,1)$-fold point of $f$ and $g$. Define*

$$\hat{f}(z) = \frac{e^{h(z)}+1}{(e^{h(z)}-1)^2},\ \hat{g}(z) = \frac{(e^{h(z)}+1)^2}{8(e^{h(z)}-1)},$$

*where $h(z)$ is a non-constant entire function. Then $f$ and $g$ are of the forms:*

$$f = L \circ \hat{f},\ g = L \circ \hat{g},$$

*where $L$ is a Möbius transformation.*

Reinders latter defined the following pair of meromorphic functions $F$ and $G$, and proved the following theorem.

**Theorem 3.** (Reinders [7]). *Let*

$$F = \frac{U'}{8\sqrt{3}} \cdot \frac{U}{U+1},\ G = \frac{U'}{8\sqrt{3}} \cdot \frac{U+4}{(U+1)^2},$$

*where $U$ is a non-constant solution of the differential equation*

$$(U')^2 = 12U(U+1)(U+4).$$

*Then $F$ and $G$ share the values $0, 1, \infty$ and $-1$. Every $0, 1, \infty$ and $-1$ point is either a $(1,3)$-fold or $(3,1)$-fold point of $F$ and $G$.*

Also, from a result of Reinders [7], we can get the following theorem.

**Theorem 4.** *Let $f$ and $g$ be meromorphic functions that share four distinct values $a_1, a_2, a_3, a_4$* **DM**, *such that each $a_i$-point is either a $(1,3)$-fold or $(3,1)$-fold point of $f$ and $g$. Let $F$ and $G$ be the meromorphic functions defined in Theorem 3, then $f$ and $g$ are of the form*

$$f = L \circ F \circ h, \quad g = L \circ G \circ h,$$

*where $h$ is a non-constant entire function and $L$ is a Möbius transformation.*

## 3. Main Theorem

In order to prove the main theorem, we need the following fact of meromorphic functions sharing four distinct values.

**Theorem 5.** (Gundersen [7]). *Let $f$ and $g$ be two distinct non-constant meromorphic functions and share four distinct values $a_1, a_2, a_3, a_4$* **IM**. *Then*

(1) $T(r, f) = T(r, g) + S(r, f), T(r, g) = T(r, f) + S(r, g)$;

(2) $\sum_{i=1}^{4} \overline{N}(r, \frac{1}{f-a_i}) = 2T(r, f) + S(r, f)$.

Now, we are ready to state and prove the main theorem.

**Main Theorem.** *Let $f$ and $g$ be two distinct, non-constant meromorphic functions that share four values $a_1, a_2, a_3, a_4$* **DM**. *Let $(p, q)$ be a pair of positive integers with $p < q$. If each $a_i$-point is either a $(p, q)$-fold or $(q, p)$-fold points of $f$ and $g$, then $(p, q)$ is either $(1, 2)$ or $(1, 3)$. Moreover, $f$ and $g$ are the forms defined in Theorem 2 and 4.*

*Proof.* Suppose $f$ and $g$ share $a_1, a_2, a_3, a_4$. Write $k = p + q$, it is easy to see that $k$ must be greater than two since $p, q$ are distinct positive integers. For $k = 3$, the only possible pair of $(p, q)$ is $(1, 2)$. Then $f$ and $g$ are given in the form of Theorem 2. When $k = 4$, the only possible pair of $(p, q)$ is $(1, 3)$. Then $f$ and $g$ are given in the form of Theorem 3.

Now, consider $k \geq 5$. For each shared value $a_i$ of $f$ and $g$, any $a_i$-point $z_0$ is either a $(p, q)$-fold or $(q, p)$-fold of $f$ and $g$. Therefore, it is clear that

$$(p + q)\overline{N}(r, \frac{1}{f - a_i}) \leq N(r, \frac{1}{f - a_i}) + N(r, \frac{1}{g - a_i}).$$

Since $N(r, \frac{1}{f-a_i}) \leq T(r, f) + O(1)$ and $N(r, \frac{1}{g-a_i}) \leq T(r, g) + O(1)$, $1 \leq i \leq 4$, and $T(r, g) = T(r, f) + S(r, g)$ by Theorem 5, we have, for $1 \leq i \leq 4$,

$$(p + q)\overline{N}(r, \frac{1}{f - a_i}) \leq T(r, f) + T(r, f) + S(r, f) = 2T(r, f) + S(r, f).$$

Therefore, we get

$$(p + q) \sum_{i=1}^{4} \overline{N}(r, \frac{1}{f - a_i}) \leq 8T(r, f) + S(r, f).$$

Apply Theorem 5 again, which says $\sum_{i=1}^{4} \overline{N}(r, \frac{1}{f-a_i}) = 2T(r, f) + S(r, f)$, we obtain

$$2(p + q)T(r, f) \leq 8T(r, f) + S(r, f),$$

which implies that $k = p + q \leq 4$. This contradicts to our assumption that $k \geq 5$. Therefore, $k$ cannot be greater than 4 and the theorem is proved. ■

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