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A NOTE ON EXTENSIONS OF PRINCIPALLY QUASI-BAER RINGS

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Abstract. Let R be a ring with unity. It is shown that the formal power series ring $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and every countable subset of right semicentral idempotents has a generalized countable join.

1. Introduction

Throughout this note, R denotes a ring with unity. Recall that R is called a (*quasi-*)*Baer* ring if the right annihilator of every (right ideal) nonempty subset of R is generated, as a right ideal, by an idempotent of R. Baer rings are introduced by Kaplansky [18] to abstract various properties of AW^*-algebras and von Neumann algebras. Quasi-Baer rings, introduced by Clark [11], are used to characterize when a finite dimensional algebra over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The definition of a (quasi-) Baer ring is left-right symmetric [11, 18].

In [9], Birkenmeier, Kim and Park initiated the study of right principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated, as a right ideal, by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. If R is both right and left p.q.-Baer, then it is called *p.q.-Baer*. The class of p.q.-Baer rings include all biregular rings, all quasi-Baer rings and all abelian PP rings. See [9] for more details.

Ore extensions or polynomial extensions of (quasi-)Baer rings and their generalizations are extensively studied recently ([4-10] and [14-17]). It is proved in [8,

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Theorem 1.8] that a ring R is quasi-Baer if and only if $R[[X]]$ is quasi-Baer, where X is an arbitrary nonempty set of not necessarily commuting indeterminates. In [7, Theorem 2.1], it is shown that R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer. But it is not equivalent to that $R[[x]]$ is right p.q.-Baer. In fact, there exists a commutative von Neumann regular ring R (hence p.q.-Baer) such that the ring $R[[x]]$ is not p.q.-Baer [7, Example 2.6]. In [20, Theorem 3], a necessary and sufficient condition for semiprime ring under which the ring $R[[x]]$ is right p.q.-Baer are given. It is shown that $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join when all left semicentral idempotents are central. Indeed, for a right p.q.-Baer ring, asking the set of left semicentral idempotents $\mathcal{S}_\ell(R)$ equals to the set of central idempotents $B(R)$ is equivalent to assume R is semiprime [9, Proposition 1.17]. In this note, the condition requiring all left semicentral idempotents being central is shown to be redundant. We show that: *The ring $R[[x]]$ is right p.q.-Baer if and only if R is p.q.-Baer and every countable subset of right semicentral idempotents has a generalized countable join*. This theorem properly generalizes Fraser and Nicholson's result in the class of reduced PP rings [12, Theorem 3] and Liu's result in the class of semiprime p.q.-Baer rings [20, Theorem 3]. For simplicity of notations, denote $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers.

2. Annihilators and Left Semicentral Idempotents

Lemma 1. *Let $f(x) = \sum_{i=0}^{\infty} f_i x^i$, $g(x) = \sum_{j=0}^{\infty} g_j x^j \in R[[x]]$. Then the following are equivalent.*

(1) $f(x)R[[x]]g(x) = 0$;

(2) $f(x)Rg(x) = 0$;

(3) $\sum_{i+j=k} f_i a g_j = 0$ *for all* $k \in \mathbb{N}$, $a \in R$.

Proof. Let $h(x) = \sum_{k=0}^{\infty} h_k x^k \in R[[x]]$ and assume $f(x)R[[x]]g(x) = 0$. Then $0 = f(x)h(x)g(x) = \sum_{k=0}^{\infty}(f(x)h_k g(x))x^k$ and thus $f(x)Rg(x) = 0$ if and only if $f(x)R[[x]]g(x) = 0$. Now, let $a \in R$ be arbitrary. Observe that

$$f(x)ag(x) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} f_i a g_j \right) x^k.$$

Thus (2) is equivalent to (3). ■

Recall that an idempotent $e \in R$ is called *left* (resp. *right*) *semicentral* [3] if $re = ere$ (resp. $er = ere$) for all $r \in R$. Equivalently, $e = e^2 \in R$ is left (resp.

right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a right ideal is generated by a left semicentral idempotents in a right p.q.-Baer ring. The set of left (resp. right) semicentral idempotents of R is denoted $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$). The following result is used frequently later in this note.

Lemma 2. [9, Lemma 1.1] *Let e be an idempotent in a ring R with unity. Then the following conditions are equivalent.*

(1) $e \in \mathcal{S}_\ell(R)$;
(2) $1 - e \in \mathcal{S}_r(R)$;
(3) $(1-e)Re = 0$;
(4) *eR is an ideal of R;*
(5) *$R(1-e)$ is an ideal of R.*

To prove the main result, we first characterize the left semicentral idempotents in $R[[x]]$.

Proposition 3. *Let $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in R[[x]]$. Then $\varepsilon(x) \in \mathcal{S}_\ell(R[[x]])$ if and only if*

(1) $\varepsilon_0 \in \mathcal{S}_\ell(R)$;
(2) $\varepsilon_0 r \varepsilon_i = r\varepsilon_i$ *and* $\varepsilon_i r \varepsilon_0 = 0$ *for all* $r \in R$, $i = 1, 2, \cdots$;
(3) $\sum_{\substack{i+j=k \\ i,j \geqslant 1}} \varepsilon_i r \varepsilon_j = 0$ *for all* $r \in R$ *and* $k \geqslant 2$.

Proof. Assume $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in \mathcal{S}_\ell(R[[x]])$ and $r \in R$. Then $\varepsilon(x) r \varepsilon(x) = r\varepsilon(x)$, or

$$\sum_{k=0}^{\infty} \left(\sum_{i+j=k} \varepsilon_i r \varepsilon_j \right) x^k = \sum_{k=0}^{\infty} r \varepsilon_k x^k.$$

By comparing the coefficient of each terms x^k in the above expansion, we have a system of equations

$$E(k){:} \quad \sum_{i+j=k} \varepsilon_i r \varepsilon_j = r\varepsilon_k, \quad \text{for all } k \geqslant 0.$$

From $E(0)$, we have

$$\varepsilon_0 r \varepsilon_0 = r \varepsilon_0$$

and thus $\varepsilon_0 \in \mathcal{S}_\ell(R)$ since R has unity. Consider $E(1)$: $\varepsilon_0 r \varepsilon_1 + \varepsilon_1 r \varepsilon_0 = r\varepsilon_1$, and multiply $E(1)$ by ε_0 from right yields

$$\varepsilon_0 r \varepsilon_1 \varepsilon_0 + \varepsilon_1 r \varepsilon_0^2 = r \varepsilon_1 \varepsilon_0.$$

Since $\varepsilon_0 \in \mathcal{S}_\ell(R)$, $\varepsilon_0 r\varepsilon_1\varepsilon_0 = r\varepsilon_1\varepsilon_0$ and consequently $\varepsilon_1 r\varepsilon_0 = \varepsilon_1 r\varepsilon_0^2 = 0$. Thus $\varepsilon_0 r\varepsilon_1 = r\varepsilon_1$ from $E(1)$. Multiply $E(2)$: $\varepsilon_0 r\varepsilon_2 + \varepsilon_1 r\varepsilon_1 + \varepsilon_2 r\varepsilon_0 = r\varepsilon_2$ by ε_0 from right yields

$$\varepsilon_0 r\varepsilon_2\varepsilon_0 + \varepsilon_1 r\varepsilon_1\varepsilon_0 + \varepsilon_2 r\varepsilon_0 = r\varepsilon_2\varepsilon_0.$$

Since $\varepsilon_0 \in \mathcal{S}_\ell(R)$, we have $\varepsilon_0 r\varepsilon_2\varepsilon_0 = r\varepsilon_2\varepsilon_0$, and also that $\varepsilon_1 r\varepsilon_1\varepsilon_0 = \varepsilon_1 r(\varepsilon_0\varepsilon_1\varepsilon_0) = (\varepsilon_1 r\varepsilon_0)\varepsilon_1\varepsilon_0 = 0$. It follows that $\varepsilon_2 r\varepsilon_0 = 0$. Assume that $\varepsilon_i r\varepsilon_0 = 0$ for $i = 1, 2, \cdots, k-1$. Inductively, multiply $E(k)$ by ε_0 from right yields

$$\varepsilon_0 r\varepsilon_k\varepsilon_0 + \left(\sum_{\substack{i+j=k \\ i,j\geqslant 1}} \varepsilon_i r\varepsilon_j\varepsilon_0\right) + \varepsilon_k r\varepsilon_0 = r\varepsilon_k\varepsilon_0.$$

Observe that $\varepsilon_0 r\varepsilon_k\varepsilon_0 = r\varepsilon_k\varepsilon_0$ since $\varepsilon_0 \in \mathcal{S}_\ell(R)$, and

$$\varepsilon_i r\varepsilon_j\varepsilon_0 = \varepsilon_i r(\varepsilon_0\varepsilon_j\varepsilon_0) = (\varepsilon_i r\varepsilon_0)\varepsilon_j\varepsilon_0 = 0$$

for $1 \leqslant i \leqslant k-1$ by induction hypothesis. Consequently $\varepsilon_k r\varepsilon_0 = 0$. Thus $\varepsilon_i r\varepsilon_0 = 0$ for all $r \in R$, $i \geqslant 1$ by induction.

Now the system of equations $E(k)$ becomes

$$E'(k):\quad \varepsilon_0 r\varepsilon_k + \sum_{\substack{i+j=k \\ i,j\geqslant 1}} \varepsilon_i r\varepsilon_j = r\varepsilon_k \quad \text{for } k \geqslant 2.$$

Multiply the equation $E'(2)$ by ε_0 from left yields

$$\varepsilon_0 r\varepsilon_2 + \varepsilon_0\varepsilon_1 r\varepsilon_1 = \varepsilon_0 r\varepsilon_2,$$

and thus $\varepsilon_0\varepsilon_1 r\varepsilon_1 = 0$. Recall that $\varepsilon_0 r\varepsilon_1 = r\varepsilon_1$ from $E(1)$. It follows that

$$\varepsilon_1 r\varepsilon_1 = \varepsilon_1(\varepsilon_0 r\varepsilon_1) = (\varepsilon_0\varepsilon_1\varepsilon_0)r\varepsilon_1 = \varepsilon_0\varepsilon_1(\varepsilon_0 r\varepsilon_1) = \varepsilon_0\varepsilon_1 r\varepsilon_1 = 0.$$

Consequently, $\varepsilon_0 r\varepsilon_2 = r\varepsilon_2$ from $E'(2)$. Again, multiply $E'(3)$ by ε_0 from left yields

$$\varepsilon_0 r\varepsilon_3 + \varepsilon_0\varepsilon_1 r\varepsilon_2 + \varepsilon_0\varepsilon_2 r\varepsilon_1 = \varepsilon_0 r\varepsilon_3,$$

and thus $\varepsilon_0\varepsilon_1 r\varepsilon_2 + \varepsilon_0\varepsilon_2 r\varepsilon_1 = 0$. It follows that

$$\begin{aligned}
\varepsilon_1 r\varepsilon_2 + \varepsilon_2 r\varepsilon_1 &= \varepsilon_1(\varepsilon_0 r\varepsilon_2) + \varepsilon_2(\varepsilon_0 r\varepsilon_1) \\
&= (\varepsilon_0\varepsilon_1\varepsilon_0)r\varepsilon_2 + (\varepsilon_0\varepsilon_2\varepsilon_0)r\varepsilon_1 \\
&= \varepsilon_0\varepsilon_1(\varepsilon_0 r\varepsilon_2) + \varepsilon_0\varepsilon_2(\varepsilon_0 r\varepsilon_1) \\
&= \varepsilon_0\varepsilon_1 r\varepsilon_2 + \varepsilon_0\varepsilon_2 r\varepsilon_1 \\
&= 0.
\end{aligned}$$

Substitute this result back to the equation $E'(3)$, we get $\varepsilon_0 r\varepsilon_3 = r\varepsilon_3$. Assume that $\varepsilon_0 r\varepsilon_i = r\varepsilon_i$ for $i = 1, 2, \cdots, k-1$, and multiply $E'(k)$ by ε_0 from left yields

$$\varepsilon_0 r\varepsilon_k + \varepsilon_0 \left(\sum_{\substack{i+j=k \\ i,j\geqslant 1}} \varepsilon_i r\varepsilon_j \right) = \varepsilon_0 r\varepsilon_k.$$

A similar argument used above will show that $\sum_{\substack{i+j=k \\ i,j\geqslant 1}} \varepsilon_i r\varepsilon_j = 0$ by induction hypothesis and thus $\varepsilon_0 r\varepsilon_k = r\varepsilon_k$ for $k \geqslant 2$.

Conversely, let $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in R[[x]]$ such that conditions (1), (2), (3) hold. To show $\varepsilon(x) \in \mathcal{S}_\ell(R[[x]])$, it suffices to show that $(\varepsilon(x) - 1)r\varepsilon(x) = 0$ or $\varepsilon(x)r\varepsilon(x) = r\varepsilon(x)$ for all $r \in R$ by Lemma 2 and Lemma 1. Observe that

$$\sum_{i+j=k} \varepsilon_i r\varepsilon_j = \varepsilon_0 r\varepsilon_k + \left(\sum_{\substack{i+j=k \\ i,j\geqslant 1}} \varepsilon_i r\varepsilon_j \right) + \varepsilon_k r\varepsilon_0 = r\varepsilon_k, \quad \text{for } k \geqslant 1,$$

and thus

$$\varepsilon(x)r\varepsilon(x) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \varepsilon_i r\varepsilon_j \right) x^k = \sum_{k=0}^{\infty} r\varepsilon_k x^k = r\varepsilon(x).$$

Consequently, $\varepsilon(x) \in \mathcal{S}_\ell(R[[x]])$. ■

Corollary 4. [4, Proposition 2.4(iv)] *Let R be a ring with unity and $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i x^i \in \mathcal{S}_\ell(R[[x]])$. Then $\varepsilon(x)R[[x]] = \varepsilon_0 R[[x]]$.*

Proof. Observe that

$$\varepsilon_0 \cdot \varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_0 \varepsilon_i x^i = \sum_{i=0}^{\infty} \varepsilon_i x^i = \varepsilon(x) \text{ and}$$
$$\varepsilon(x) \cdot \varepsilon_0 = \sum_{i=0}^{\infty} \varepsilon_i \varepsilon_0 x^i = \varepsilon_0,$$

by Proposition 3. Thus $\varepsilon(x)f(x) = \varepsilon_0(\varepsilon(x)f(x))$ and $\varepsilon_0 f(x) = \varepsilon(x)\varepsilon_0 f(x)$ for all $f(x) \in R[[x]]$. Consequently, $\varepsilon(x)R[[x]] = \varepsilon_0 R[[x]]$. ■

3. Generalized Countable Join

Let R be a ring with unity and $E = \{e_0, e_1, e_2, \cdots\}$ a countable subset of $\mathcal{S}_r(R)$. We say E has a *generalized countable join* e if, given $a \in R$, there exists $e \in \mathcal{S}_r(R)$ such that

(1) $e_ie = e_i$ for all $i \in \mathbb{N}$;

(2) if $e_ia = e_i$ for all $i \in \mathbb{N}$, then $ea = e$.

Note that if there exists an element $e \in R$ satisfies conditions (1) and (2) above, then $e \in \mathcal{S}_r(R)$. Indeed, the condition (1): $e_ie = e_i$ for all $i \in \mathbb{N}$ implies $ee = e$ by (2) and so e is an idempotent. Further, let $a \in R$ be arbitrary. Then the element $d = e - ea + eae$ is an idempotent in R and $e_id = e_i$ for all $i \in \mathbb{N}$. Thus $ed = e$ by (2). Note that $ed = e(e - ea + eae) = d$. Consequently, $e = d = e - ea + eae$ or $ea = eae$. Thus $e \in \mathcal{S}_r(R)$.

Note that a generalized countable join e, if it exists, is indeed a join if $\mathcal{S}_r(R)$ is a lattice. Recall that when R is an abelian ring (i.e., every idempotent is central), then the set $B(R) = \mathcal{S}_r(R)$ of all idempotents in R is a Boolean algebra where $e \leqslant d$ means $ed = e$. Let e be a join of $E = \{e_0, e_1, e_2, \cdots\}$ in $B(R)$ where R is a reduced PP ring. That is e satisfies (1) $e_ie = e_i$ for all $i \in \mathbb{N}$; (2$'$) if $e_id = e_i$ for all $i \in \mathbb{N}$ and any $d \in B(R)$, then $ed = e$. Given an arbitrary $a \in R$, then $1 - a = pu$ for some central idempotent $p \in R$ and some $u \in R$ such that $r\mathrm{Ann}_R(u) = 0 = \ell\mathrm{Ann}_R(u)$ [12, Proposition 2]. Observe that if $e_ia = e_i$ for all $i \in \mathbb{N}$, then $e_i(1 - a) = e_ipu = 0$. It follows that $e_ip = 0$ for all $i \in \mathbb{N}$ since $\ell\mathrm{Ann}_R(u) = 0$. Thus $ep = 0$ or $e(1 - a) = epu = 0$. Therefore $ea = e$ and e is a generalized countable join of E. In other words, a generalized countable join is a join and vice versa in the class of reduced PP rings.

Be aware that $(\mathcal{S}_r(R), \leqslant)$ is not partially ordered by defining $d \leqslant e$ when $de = d$ in an arbitrary ring R. This relation is reflexive, transitive but not antisymmetric. However, let $a, b \in \mathcal{S}_r(R)$ and define $a \sim b$ if $a = ab$ and $b = ba$. Then $\sim$ is an equivalence relation on $\mathcal{S}_r(R)$ and $(\mathcal{S}_r(R)/\sim, \leqslant)$ is a partially ordered set. In the case when $(\mathcal{S}_r(R)/\sim, \leqslant)$ is a complete lattice, then a generalized countable join exists for any subset of $\mathcal{S}_r(R)$. In particular when R is a Boolean ring or a reduced PP ring, then the generalized countable join is indeed a join in R.

In [20, Definition 2], Liu defined the notion of generalized join for a countable set of idempotents. Explicitely, let $\{e_0, e_1, \cdots\}$ be a countable family of idempotents of R. The set $\{e_0, e_1, \cdots\}$ is said to have a *generalized join* e if there exists $e = e^2$ such that

(i) $e_iR(1 - e) = 0$;

(ii) if d is an idempotent and $e_iR(1 - d) = 0$ then $eR(1 - d) = 0$.

Observe that

$$e_ir(1 - e) = e_ire_i(1 - e) = e_ir(e_i - e_ie),$$

when $e_i \in \mathcal{S}_r(R)$. Thus $e_i = e_ie$ if and only if $e_ir(1 - e) = 0$ for all $r \in R$ when $e_i \in \mathcal{S}_r(R)$ for all $i \in \mathbb{N}$. Now, let $E = \{e_0, e_1, e_2, \cdots\} \subseteq \mathcal{S}_r(R)$ and e a generalized countable join of E. To show e is a generalized join (in the sense of

Liu), it remains to show condition (ii) holds. Let f be an idempotent in R such that $e_iR(1-f)=0$. Then, in particular, $e_i(1-f)=0$ for all $i\in\mathbb{N}$. Thus $e(1-f)=0$ by hypothesis. It follows that $er(1-f)=ere(1-f)=0$ and thus $eR(1-f)=0$. Therefore, e is a generalized join of E. Thus, in the content of right semicentral idempotents, a generalized countable join is a generalized join in the sense of Liu.

Conversely, let $e\in\mathcal{S}_r(R)$ be a generalized join (in the sense of Liu) of the set $E=\{e_0,e_1,e_2,\cdots\}\subseteq\mathcal{S}_r(R)$. Observe that condition (ii) is equivalent to

(ii') if d is an idempotent and $e_id=e_i$ then $ed=e$.

Let $a\in R$ be arbitrary such that $e_ia=e_i$ for all $i\in\mathbb{N}$. Then condition (ii') and a similar argument used in the case of reduced PP rings implies that $ea=e$. Thus e is a generalized countable join. Therefore, in the content of right semicentral idempotents, Liu's generalized join is equivalent to generalized countable join.

4. Main Result

If X is a nonempty subset of R, then denote the right annihilator of X in R as $r\mathrm{Ann}_R(X)=\{a\in R\mid Xa=0\}$ and the left annihilator $\ell\mathrm{Ann}_R(X)=\{a\in R\mid aX=0\}$. In the proof of next result, it is often to deal with the right annihilator in the ring R or in the ring $R[[x]]$. To simplify the notation, $r\mathrm{Ann}_{R[[x]]}(X)$ will be denoted $r\mathrm{Ann}(X)$ and the subscript R will be kept for $r\mathrm{Ann}_R(X)$.

Theorem 5. *Let R be a ring with unity. Then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and every countable subset of $\mathcal{S}_r(R)$ has a generalized countable join.*

Proof. If $R[[x]]$ is right p.q.-Baer then R is right p.q.-Baer by [7, Proposition 2.5]. It remains to show that every countable subset of $\mathcal{S}_r(R)$ has a generalized countable join.

Let $E=\{e_0,e_1,e_2,\cdots\}\subseteq\mathcal{S}_r(R)$ and $\varepsilon(x)=\sum_{i=0}^{\infty}e_ix^i\in R[[x]]$. Since $R[[x]]$ is right p.q.-Baer, there exists $\eta(x)=\sum_{j=0}^{\infty}\eta_ix^i\in\mathcal{S}_\ell(R[[x]])$ such that

$$r\mathrm{Ann}(\varepsilon(x)R[[x]])=\eta(x)R[[x]]=\eta_0R[[x]]$$

by Corollary 4. Since $r\mathrm{Ann}(\varepsilon(x)R[[x]])=r\mathrm{Ann}(\varepsilon(x)R)$ by Lemma 1, we have

$$0=\varepsilon(x)r\eta_0=\sum_{i=0}^{\infty}(e_ir\eta_0)x^i,\quad\text{for any } r\in R.$$

Thus $e_ir\eta_0=0$ for all $i\in\mathbb{N}$, $r\in R$. We will show that $1-\eta_0$ is a generalized countable join for E. Since $\eta(x)\in\mathcal{S}_\ell(R[[x]])$, it follows that $1-\eta_0\in\mathcal{S}_r(R)$ by Proposition 3 and Lemma 2. Furthermore, $e_ir\eta_0=0$ for all $i\in\mathbb{N}$, $r\in R$ implies that $e_i\eta_0 = 0$ or $e_i(1-\eta_0) = e_i$ for all $i \in \mathbb{N}$. Now let $a \in R$ such that $e_i a = e_i$ for all $i \in \mathbb{N}$. Then $e_i(1-a) = 0$ for all $i \in \mathbb{N}$. Since $e_i \in \mathcal{S}_r(R)$, it follows that

$$e_i r(1-a) = e_i r e_i(1-a) = 0$$

and so $\varepsilon(x)r(1-a) = 0$ for all $r \in R$. Thus $1-a \in r\mathrm{Ann}(\varepsilon(x)R) = \eta_0 R[[x]]$. In particular $\eta_0(1-a) = 1-a$. Consequently, $(1-\eta_0)a = 1-\eta_0$. Thus $1-\eta_0$ is a generalized countable join of E.

Conversely, assume the ring R is right p.q.-Baer and every countable subset of $\mathcal{S}_r(R)$ has a generalized countable join. Let $f(x) = \sum_{i=0}^{\infty} f_i x^i \in R[[x]]$. Since R is right p.q.-Baer, there exists $e_i \in \mathcal{S}_\ell(R)$ for all $i \in \mathbb{N}$ such that $r\mathrm{Ann}_R(f_i R) = e_i R$. Thus $1-e_i \in \mathcal{S}_r(R)$ by Lemma 2. By hypothesis, the set $\{1-e_i \mid i \in \mathbb{N}\}$ has a generalized countable join $e \in \mathcal{S}_r(R)$. It follows that

$$(1-e_i)e = 1-e_i \ \text{ or } \ e_i(1-e) = 1-e \ \text{ for all } \ i \in \mathbb{N}.$$

Let $a \in R$ be arbitrary. Then

$$f(x)a(1-e) = \sum_{i=0}^{\infty} f_i a(1-e)x^i.$$

Since $1-e = e_i(1-e) \in \mathcal{S}_\ell(R)$ for all $i \in \mathbb{N}$, the coefficient of each terms in the expansion of $f(x)a(1-e)$ becomes

$$f_i a(1-e) = f_i a e_i(1-e) \in f_i R e_i R = 0.$$

Thus $f(x)a(1-e) = 0$ for all $a \in R$. Consequently, $(1-e)R[[x]] \subseteq r\mathrm{Ann}(f(x)R[[x]])$ by Lemma 1.

On the other hand, let $g(x) = \sum_{j=0}^{\infty} g_j x^j \in r\mathrm{Ann}(f(x)R[[x]])$. Then $f(x)Rg(x) = 0$ for all $r \in R$. Thus we have a system of equations

$$E(k)\colon \quad \sum_{i+j=k} f_i r g_j = 0 \ \text{ for all } k \in \mathbb{N}, r \in R$$

by Lemma 1. From equation $E(0)$: $f_0 r g_0 = 0$, it follows that $g_0 \in r\mathrm{Ann}_R(f_0 R) = e_0 R$ and thus $e_0 g_0 = g_0$. Since r is arbitrary, we may replace r as se_0 for arbitrary $s \in R$ into the equation $E(1)$: $f_0 r g_1 + f_1 r g_0 = 0$ and get

$$f_0 s e_0 g_1 + f_1 s e_0 g_0 = 0.$$

Observe that $f_0 s e_0 g_1 \in f_0 R e_0 R = 0$ and thus $f_1 s g_0 = f_1 s e_0 g_0 = 0$. It follows that $g_0 \in r\mathrm{Ann}_R(f_1 R) = e_1 R$. Consequently, $e_1 g_0 = g_0$ and $f_0 r g_1 = 0$ from equation $E(1)$. Thus $g_1 \in r\mathrm{Ann}_R(f_0R) = e_0R$ and $e_0g_1 = g_1$. Inductively, assume $e_ig_j = g_j$ for $0 \leqslant i + j \leqslant k - 1$. Observe that

$$f_ise_0e_1 \cdots e_{k-1}g_j = f_ise_ie_0e_1 \cdots e_{k-1}g_j \in f_iRe_iR = 0$$

for $0 \leqslant i \leqslant k - 1$ and that

$$f_kse_0e_1 \cdots e_{k-1}g_0 = f_ksg_0$$

by induction hypothesis. If we replace r by $se_0e_1 \cdots e_{k-1}$ in $E(k)$ for arbitrary $s \in R$, then

$$0 = \sum_{i+j=k} f_ise_0e_1 \cdots e_{k-1}g_j = f_ksg_0.$$

Thus $g_0 \in r\mathrm{Ann}_R(f_kR) = e_kR$ or $e_kg_0 = g_0$. Consequently, the equation $E(k)$ becomes

$$E'(k)\colon \quad \sum_{i=0}^{k-1} f_irg_{k-j} = 0 \ \text{ for all } k \in \mathbb{N}, r \in R.$$

Replace r as $se_0e_1 \cdots e_{k-2}$ into $E'(k)$, we get

$$0 = \sum_{i=0}^{k-1} f_ise_0e_1 \cdots e_{k-2}g_{k-j} = f_{k-1}sg_1.$$

Therefore $g_1 \in r\mathrm{Ann}_R(f_{k-1}R) = e_{k-1}R$ or $e_{k-1}g_1 = g_1$. Continue this process, we get $e_ig_j = g_j$ when $i + j = k$. Thus $e_ig_j = g_j$ for $i + j \in \mathbb{N}$ by induction.

Consequently, $(1 - e_i)g_j = 0$ or $(1 - e_i)(1 - g_j) = 1 - e_i$ for all $i, j \in \mathbb{N}$. Thus $e(1 - g_j) = e$ or $(1 - e)g_j = g_j$, for all $j \in \mathbb{N}$ by hypothesis. It follows that $g(x) = \sum_{j=0}^{\infty} g_jx^j = \sum_{j=0}^{\infty}(1 - e)g_jx^j = (1 - e)g(x) \in (1 - e)R[[x]]$. Thus $r\mathrm{Ann}(f(x)R[[x]]) \subseteq (1 - e)R[[x]]$, and $R[[x]]$ is right p.q.-Baer. ■

Since Liu's generalized join is equivalent to generalized countable join in the set of right semicentral idempotents $\mathcal{S}_r(R)$. The following result is immediated from Theorem 5.

Corollary 6. [20, Theorem 3]. *Let R be a ring such that $\mathcal{S}_\ell(R) \subseteq B(R)$. Then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join.*

Corollary 7. [12, Theorem 3]. *If R is a ring then $R[[x]]$ is a reduced PP ring if and only if R is a reduced PP ring and any countable family of idempotents in R has a join in $B(R)$.*

Proof. Since R is a reduced PP ring if and only if R is a reduced p.q.-Baer ring [9, Proposition 1.14(iii)] and a join in $B(R)$ is equivalent to a generalized countable join in $B(R)$ when R is a reduced PP ring, the assertion follows immediately from Theorem 5. ■

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