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THE CONTACT NUMBER OF A PSEUDO-EUCLIDEAN SUBMANIFOLD

J. L. Cabrerizo, M. Fernández and J. S. Gómez

Abstract. In this paper we define the contact number of a pseudo-Riemannian submanifold into the pseudo-Euclidean space, and prove that this contact number is closely related to the notion of pseudo-isotropic submanifold. We give a classification of hypersurfaces into the pseudo-Euclidean space with contact number at least 3. A classification of the complete spacelike codimension-2 submanifolds of the Lorentz-Minkowski space with contact number at least 3 is also obtained.

1. Introduction

The concept of isotropic submanifold of a Riemannian manifold was introduced by B. O' Neill [11], who studied the general properties of such class of submanifold. These submanifolds can be considered as a generalization of the totally umbilical submanifolds, and constitute a distinguished family in submanifold theory. An interesting example [2] is provided by a G-equivariant isometric immersion ϕ of a rank one symmetric space M into an arbitrary Riemannian homogeneous space $\widetilde{M}$.

Recently, B.-Y. Chen and S.-J. Li introduced and studied the notion of contact number $\mathbf{c}_\sharp(M)$ of a Euclidean submanifold in [5], and they proved that the contact number is closely related with the notions of isotropic submanifolds and holomorphic curves. In particular, a surface in the Euclidean space $\mathbb{R}^4$ has contact number 3 if an only it is a non-planar holomorphic curve with respect to some orthogonal complex structure on $\mathbb{R}^4$. On the other hand, explicit examples of non-totally umbilical submanifold M of dimension n in a Euclidean space $\mathbb{R}^{4n}$ with contact number $\mathbf{c}_\sharp(M) = 4n - 2$ are exhibited in [4].

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In this paper we first define in Section 2 the contact number $\mathbf{c}_\sharp(M^n_s)$ of a pseudo-Riemannian submanifold M^n_s into the pseudo-Euclidean space $\mathbb{R}^{n+d}_\nu$, and then we show in Section 3 that the contact number is closely related to the notion of pseudo-isotropic submanifold. The essential step is the characterization property of pseudo-isotropy (Lemma 3.3), which reads as the one given by O'Neill [11] in the Riemannian case, whereas the proof needs to face up to a different background. Then, we prove that the contact number of any submanifold is at least 2, and it is at least 3 (respectively, 4) if and only if the submanifold is pseudo-isotropic (respectively, constant pseudo-isotropic).

Nevertheless, some remarkable differences for the contact number with respect to the Euclidean case are obtained. In particular, we show that any Lorentz submanifold of the Lorentz-Minkowski space with contact number at least 3, is totally umbilical. Another difference with respect the Euclidean case is obtained in Section 4, where we show that a complete 0-pseudo-isotropic submanifold M^n_s in the pseudo-Euclidean space $\mathbb{R}^{n+2}_{s+1}$ can be viewed as an expansion of $\mathbb{R}^n_s$ into $\mathbb{R}^{n+1}_{s,1}$, which has $\mathbf{c}_\sharp(M^n_s) = \infty$ but, in general, it is not totally umbilical. In particular, when $s = 0$ we show (Lemma 4.4) that any pseudo-isotropic non-totally umbilical submanifold M^n_0 of the Lorentz-Minkowski space $\mathbb{R}^{n+2}_1$ is 0-pseudo-isotropic. The notion of 0-pseudo-isotropic on codimension-2 submanifolds is related to the notion of (marginally) trapped surface 10, 13, since the mean curvature vector of such a submanifold satisfies $\langle H, H\rangle = 0$. A classification of complete codimension-2 spacelike submanifolds in the Lorentz-Minkowski space with contact number at least 3 is also obtained (Theorem 4.5).

2. Preliminaries and Basic Formulas

Let $\mathbb{R}^{n+d}_\nu$ be the $(n+d)$-dimensional pseudo-Euclidean space with metric tensor $\langle\cdot,\cdot\rangle$ of index ν given by

$$\langle\cdot,\cdot\rangle = \sum_{i=1}^{n+d-\nu} dx_i^2 - \sum_{i=n+d-\nu+1}^{n+d} dx_i^2$$

in terms of the natural coordinate system $(x_1, \ldots, x_{n+d})$ of the Euclidean $(n+d)$-dimensional space $\mathbb{R}^{n+d}$.

Throughout this paper M^n_s will denote an n-dimensional pseudo-Riemannian submanifold of index s $(0 \leq s \leq \nu)$ which lies into the pseudo-Euclidean space $\mathbb{R}^{n+d}_\nu$. The submanifolds are assumed to be connected and with dimension $n \geq 2$. Denote by ∇ and $\bar\nabla$ the Levi-Civita connections of M^n_s and $\mathbb{R}^{n+d}_\nu$, respectively, and let D stands for the normal connection of M^n_s in $\mathbb{R}^{n+d}_\nu$. Then, the Gauss and Weingarten formulas of M^n_s in $\mathbb{R}^{n+d}_\nu$ are given by

$$\bar\nabla_X Y = \nabla_X Y + h(X,Y), \tag{1}$$

and

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

for any two vector fields $X, Y \in \mathfrak{X}(M^n_s)$ and any normal vector field $\xi \in \mathfrak{X}^\perp(M^n_s)$, where h is the second fundamental form of M^n_s and A_ξ is the Weingarten endomorphism associated to ξ. The second fundamental form and the Weingarten endomorphism are related by $\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle$.

The covariant derivative $\widetilde{\nabla} h$ of h is defined by

$$(\widetilde{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{3}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M^n_s)$. Sometimes we write $(\widetilde{\nabla}_X h)(Y, Z)$ as $(\widetilde{\nabla} h)(Y, Z, X)$. If we set $\widetilde{\nabla}^0 h = h$, in general, the kth ($k \geq 1$) covariant derivative $\widetilde{\nabla}^k h$ of h [12] is given by

$$\begin{aligned}(\widetilde{\nabla}^k h)(X_1, \ldots, X_{k+2}) &= D_{X_{k+2}}((\widetilde{\nabla}^{k-1} h)(X_1, \ldots, X_{k+1})) \\ &\quad - \sum_{i=1}^{k+1} (\widetilde{\nabla}^{k-1} h)(X_1, \ldots, \nabla_{X_{k+2}} X_i, \ldots, X_{k+1}).\end{aligned} \tag{4}$$

It is clear that $(\widetilde{\nabla}^k h)$ is a normal-bundle-valued tensor field of type $(0, k+2)$. We simply denote $(\widetilde{\nabla}^k h)(X, \ldots, X)$ by $(\widetilde{\nabla}^k h)(X^{k+2})$.

The equations of Gauss and Codazzi are given, respectively, by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle \tag{5}$$

$$(\widetilde{\nabla}_X h)(Y, Z) = (\widetilde{\nabla}_Y h)(X, Z), \tag{6}$$

for vector fields $X, Y, Z, W \in \mathfrak{X}(M^n_s)$.

3. The Contact Number and Pseudo-isotropic Submanifols

For a given point $p \in M^n_s$ and any unit tangent vector $u \in U_p M^n_s$ (i.e., $\langle u, u\rangle = \pm 1$), there is a unique unit speed *geodesic* γ_u (spacelike or timelike) in M^n_s through p satisfying $\gamma_u(0) = p$, $\gamma_u'(0) = u$. For the same pair (p, u), we define the *normal section* β_u at (p, u) as follows. Let $E_\delta(p, u)$ be the affine $(d+1)$-subspace in $\mathbb{R}^{n+d}_\nu$ through p spanned by u and the normal space $T^\perp_p M^n_s$ at p, where the index $\delta = \nu - s$ if u is spacelike or $\delta = \nu - s + 1$ if u is timelike. Then the intersection of M^n_s and $E_\delta(p, u)$ gives rise to a unit speed curve $\beta_u(s)$ defined on an open interval containing 0 with $\beta_u(0) = p$ and $\beta_u'(0) = u$.

The geodesic γ_u and the normal section β_u at (p, u) are said to be in *contact of order* $k \in \mathbb{N}$ if $\gamma_u^{(i)}(0) = \beta_u^{(i)}(0)$ for $i = 1, \ldots, k$, where $\gamma_u^{(i)}$ and $\beta_u^{(i)}$ denote the ith derivatives of γ_u and β_u in $\mathbb{R}_\nu^{n+d}$, respectively, with respect to their arclength functions.

Definition 3.1. Let M_s^n be a pseudo-Riemannian submanifold of the pseudo-Euclidean space $\mathbb{R}_\nu^{n+d}$. Then M_s^n is said to be *in contact of order* k if, for each $p \in M_s^n$ and unit tangent vector $u \in U_pM_s^n$, the geodesic γ_u and the normal section β_u at (p, u) are in contact of order k. The *contact number* $\mathbf{c}_\sharp(M_s^n)$ of M_s^n is defined to be the largest natural number k such that M_s^n is in contact of order k and but not of order $k+1$. If the submanifold M_s^n is in contact of order k for every natural number k, the contact number is defined to be ∞.

We recall the following definition [8]: M_s^n is called *pseudo-isotropic at* $p \in M_s^n$ if $\langle h(u, u), h(u, u)\rangle = \lambda(p)$ does not depends on the choice of the unit tangent vector $u \in T_pM_s^n$, and M_s^n is said to be *pseudo-isotropic* if M_s^n is pseudo-isotropic at each point of M_s^n. If $\lambda(p)$ is also independent of $p \in M_s^n$, then M_s^n is said to be *constant pseudo-isotropic* (denoted by λ-pseudo-isotropic).

Example 3.1. Every totally umbilical pseudo-Riemannian submanifold is pseudo-isotropic. Thus, for example [12], pseudo-Riemannian spheres

$$\begin{aligned} \mathbb{S}_\nu^n(r) &= \{x \in \mathbb{R}_\nu^{n+1} : \langle x, x\rangle = r^2\}, \\ \mathbb{H}_\nu^n(r) &= \{x \in \mathbb{R}_{\nu+1}^{n+1} : \langle x, x\rangle = -r^2\}, \end{aligned}$$

and (non-degenerate) n-planes into a pseudo-Euclidean space are constant pseudo-isotropic submanifolds.

Example 3.2. [1] Expansions of $\mathbb{R}_s^n$ into $\mathbb{R}_{s,1}^{n+1}$. Let $f\colon \mathbb{R}_s^n \to \mathbb{R}$ be a smooth function. Define the space $\mathbb{R}_{s,1}^{n+1}$ as $\mathbb{R}^{n+1}$ equipped with the degenerate metric tensor given by the matrix

$$\begin{pmatrix} I_{n-s} & & \\ & -I_s & \\ & & 0 \end{pmatrix}.$$

The isometric immersion

$$\psi : \mathbb{R}_s^n \to \mathbb{R}_{s+1}^{n+2}, \quad \psi(x) = (f(x), x, f(x)),$$

is a 0-pseudo-isotropic immersion which is full in $\mathbb{R}_{s,1}^{n+1}$ (if f is not linear), i.e., the imagen $\psi(\mathbb{R}_s^n)$ is contained in no affine hyperplane of $\mathbb{R}_{s,1}^{n+1}$. If we denote by

$(x_1, \dots, x_n)$ the canonical coordinates of $\mathbb{R}^n$, then the second fundamental form h becomes

$$h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}, 0, \dots, 0, \frac{\partial^2 f}{\partial x_i \partial x_j}\right), \, i, j = 1, ..., n.$$

If ψ is an isometric immersion with parallel second fundamental form, the geodesics are mapped to parabolas (or line segments), so f must be a quadratic polynomial. Up to an isometry of $\mathbb{R}^{n+1}_{s,1}$, $f(x_1, \dots, x_n) = \sum_{i=1}^{n} a_i x_i^2$, so that $\psi(\mathbb{R}^n_s)$ is an elliptic or hyperbolic paraboloid, or an orthogonal cylinder over one of these.

Notice that $M^n_s = \psi(\mathbb{R}^n_s)$ is totally geodesic if and only if f is an affine function, and it is totally umbilical [9] if and only if f is given by

$$f(x_1, \dots, x_n) = a\left(\sum_{i=1}^{n-s} x_i^2 - \sum_{i=n-s+1}^{n} x_i^2\right) + \sum_{i=1}^{n} b_i x_i + c,$$

with $a, b_1, \dots, b_n, c \in \mathbb{R}$. Furthermore, if ψ is non-totally geodesic, the first normal space $Im(h)$ at each point is entirely constituted by null vectors and $dim(Im(h)) = 1$. As a consequence, the mean curvature vector field satisfies $\langle H, H\rangle = 0$.

Lemma 3.3. *A pseudo-Riemannian submanifold M^n_s in $\mathbb{R}^{n+d}_\nu$ is pseudo-isotropic if and only if we have*

(7) $$\langle h(u,u), h(u,v)\rangle = 0,$$

for any orthonormal vectors u, v tangents to M^n_s at each point. Furthermore, if M^n_s is a pseudo-isotropic submanifold, we have

(8) $$\langle h(u,u), h(v,v)\rangle + 2\,\langle h(u,v), h(u,v)\rangle = \lambda(p)\,\langle u,u\rangle\,\langle v,v\rangle,$$

(9) $$\langle h(u,u), h(v,w)\rangle + 2\,\langle h(u,v), h(u,w)\rangle = 0,$$

for any orthogonal vectors u, v, w tangents to M^n_s at each point.

Proof. The set $\Sigma = U_p M^n_s = \{u \in T_p M^n_s : |\langle u,u\rangle| = 1\}$ is a $(n-1)-$dimensional submanifold of $\mathbb{R}^{n+d}$ and $T_u\Sigma = \{v \in T_p M^n_s : \langle u, v\rangle = 0\}$. Now, define the function $f\colon \Sigma \to \mathbb{R}$ by $f(w) = \langle h(w,w), h(w,w)\rangle$. Then, for any $v \in T_u\Sigma$ we have

(10) $$(df)_u(v) = 4\,\langle h(u,u), h(u,v)\rangle.$$

If M^n_s is pseudo-isotropic, then f is constant, and therefore from Eq. (10) we get $\langle h(u,u), h(u,v)\rangle = 0$.

Conversely, assume that $\langle h(u,u), h(u,v)\rangle = 0$ for any orthonormal pair $u, v \in U_pM_s^n$. Then, f is the constant function on each connected component of Σ. We have two cases.

Case (i). $s \in \{0, n\}$. Then Σ is connected and $\langle h(u,u), h(u,v)\rangle = 0$ for $u \in U_pM_s^n = \Sigma$, $v \in T_pM_s^n$ with $\langle u, v\rangle = 0$. Thus, f is constant on Σ and M_s^n is pseudo-isotropic.

Case (ii). Assume $0 < s < n$. Let Σ^+ (respectively, Σ^-) be a connected component of $\{u \in T_pM_s^n : \langle u,u\rangle = 1\}$ (respectively, $\{u \in T_pM_s^n : \langle u,u\rangle = -1\}$). Then f is constant on Σ^+ and Σ^-. Let $\lambda^+(p)$ (respectively, $\lambda^-(p)$) denote the constant value of f on Σ^+ (respectively, Σ^-). Since $\langle h(v,v), h(v,v)\rangle = \langle v,v\rangle^2\ \lambda^-(p)$ for any timelike $v \in T_pM_s^n$, then for any $w \in T_pM_s^n$ we have $w(\langle h(v,v), h(v,v)\rangle) = 4\lambda^-(p)\ \langle v,v\rangle\langle v,w\rangle$. But we have also

$$w(\langle h(v,v), h(v,v)\rangle) = 4\langle h(v,v), h(v,w)\rangle,$$

and then

$$\langle h(v,v), h(v,w)\rangle = \lambda^-(p)\ \langle v,v\rangle\langle v,w\rangle. \tag{11}$$

Let x be a null vector and take a sequence $\{v^k\}_{k\in\mathbb{N}}$ of timelike vectors which converges to x. Then, by continuity and Eq. (11) we get $\langle h(x,x), h(x,w)\rangle = \lambda^-(p)\ \langle x,x\rangle\langle x,w\rangle\ = 0$, for any $w \in T_pM_s^n$. In particular,

$$\langle h(x,x), h(x,y)\rangle = 0, \tag{12}$$

for any pair of null vectors x, y. Consider two orthonormal vectors $u \in \Sigma^+$ and $v \in \Sigma^-$. Then, $x = u + v$ and $y = u - v$ are null vectors, and from Eq. (12) we have

$$0 = \langle h(x,x), h(x,y)\rangle = \lambda^+(p) - \lambda^-(p).$$

Therefore, $\lambda^+(p) = \lambda^-(p) = \lambda(p)$, and M_s^n is pseudo-isotropic.

On the other hand, notice that for any spacelike or timelike $v \in T_pM_s^n$ we have

$$\langle h(v,v), h(v,v)\rangle = \langle v,v\rangle^2\ \lambda(p), \tag{13}$$

and hence, by continuity, the same equation is satisfied for null vectors. Now, for any $p \in M_s^n$ define on $T_pM_s^n$ the multilinear function

$$F(x,y,z,t) = \langle h(x,y), h(z,t)\rangle - \lambda(p)\ \langle x,y\rangle\langle z,t\rangle.$$

rom Eq. (13), for any $x \in T_pM_s^n$ we have $B(x) = F(x,x,x,x) = 0$, and therefore $B(x+y) + B(x-y) = 0$. This equation gives

$$F(x,x,y,y) + 2\ F(x,y,x,y) = 0, \tag{14}$$

for any $x, y \in T_pM^n_s$. If we take $x = u$, $y = v$ orthogonal vectors, then we obtain Eq. (8). Substitution of $x + y$ by y in Eq. (14) gives $F(x, x, x, y) = 0$. Assume $n \geq 3$ and take u, v, w orthogonal vectors. If we set $x = u$ and $y = v + w$ in Eq. (14) we obtain

$$F(u, u, v, w) + 2F(u, v, u, w) = 0,$$

and this is Eq. (9). ∎

Remark 3.4. The notion of pseudo-isotropic (indefinite) submanifolds can be sharped. In fact, it is enough to deal with timelike tangent vectors as the following argument shows. Assume

$$\langle h(u, u), h(u, u)\rangle = \lambda \langle u, u\rangle^2$$

holds for any timelike $u \in T_pM^n_s$. Let v be any tangent vector (of any causal character) in $T_pM^n_s$ and consider the curve u_t, given by $u_t := u + tv$, where u is a (fixed) timelike tangent vector in $T_pM^n_s$. By continuity, there exists $\epsilon > 0$ such that $\langle u_t, u_t\rangle < 0$ for any $t \in]-\epsilon, \epsilon[$, and therefore

$$\langle h(u_t, u_t), h(u_t, u_t)\rangle = \lambda \langle u_t, u_t\rangle^2,$$

which is an equality between two polynomials of degree 4. Therefore, their coefficients of same degree coincide and, in particular, it gives

$$\langle h(v, v), h(v, v)\rangle = \lambda \langle v, v\rangle^2,$$

for any $v \in T_pM^n_s$. Similarly, we can prove that a pseudo-Riemannian submanifol is pseudo-isotropic if $\langle h(u, u), h(u, u)\rangle = \lambda \langle u, u\rangle^2$ holds for any spacelike tangent vector u at a point $p \in M^n_s$. Moreover, formulas (8) and (9) in Lemma 3.3 can be also obtained.

Corollary 3.5. *Let $M^n_s \subseteq \mathbb{R}^{n+1}_\nu$ be a pseudo-isotropic hypersurface. Then, M^n_s is totally umbilical.*

Proof. First, assume M^n_s has a definite induced metric. For any $u \in U_pM^n_s$ we have $\langle h(u, u), h(u, u)\rangle = \lambda(p)$, and hence $\langle A_{h(u,u)}u, u\rangle = \lambda(p)$. By Eq. (7), $\langle A_{h(u,u)}u, v\rangle = 0$ for any $v \in U_pM^n_s$ orthogonal to u, and therefore $A_{h(u,u)}u = \lambda(p)\langle u, u\rangle u$. Since $\langle u, u\rangle$ does not change the sign, M^n_s is totally umbilical.

Finally, suppose M^n_s has an indefinite induced metric. From Eq. (12) $\langle h(x, x), h(x, x)\rangle = 0$ for any null vector $x \in T_pM^n_s$. As $T^\perp_p M^n_s$ is 1-dimensional, this means $h(x, x) = 0$, and then [6] M^n_s is totally umbilical. ∎

As for constant isotropic submanifolds in the Euclidean space [5], we have the following characterization of the constant pseudo-isotropic submanifolds in the semi-Euclidean space.

Lemma 3.6. *Let M_s^n be a pseudo-isotropic submanifold of $\mathbb{R}_\nu^{n+d}$. The following assertions are equivalent:*

(1) *M_s^n is constant pseudo-isotropic.*

(2) *$\langle A_{(\tilde{\nabla} h)(u^3)} u, v\rangle = 0$, for any orthonormal vectors u, v tangent to M_s^n at each point.*

(3) *$A_{(\tilde{\nabla} h)(u^3)} u = 0$ for any tangent vector u to M_s^n at each point.*

Now, from Lemmas 3.3 and 3.6 it can be proved as in the Euclidean case [5], the following two theorems which relate the contact number $\mathbf{c}_\sharp(M_s^n)$, the pseudo-isotropic property and the Weingarten endomorphism of the submanifold M_s^n.

Theorem 3.7. *Let M_s^n be a pseudo-Riemannian submanifold in $\mathbb{R}_\nu^{n+d}$. Then, we have*

(1) *The contact number $\mathbf{c}_\sharp(M_s^n)$ of M_s^n satisfies $\mathbf{c}_\sharp(M_s^n) \geq 2$.*

(2) *M_s^n is pseudo-isotropic if and only if $\mathbf{c}_\sharp(M_s^n) \geq 3$.*

(3) *M_s^n is constant pseudo-isotropic if and only if $\mathbf{c}_\sharp(M_s^n) \geq 4$.*

Theorem 3.8. *A pseudo-Riemannian submanifold M_s^n into a pseudo-Euclidean space $\mathbb{R}_\nu^{n+d}$ is in contact of order k $(k \geq 3)$ if and only if each $u \in UM_s^n$ is eigenvector of $A_{(\tilde{\nabla}^j h)(u^{j+2})}$ for $j = 0, 1, \ldots, k-3$.*

As an consequence of Theorem 3.8 we have the following.

Corollary 3.9. *A pseudo-isotropic submanifold with parallel second fundamental form in pseudo-Euclidean space satisfies $\mathbf{c}_\sharp(M_s^n) = \infty$.*

Remark 3.10. Notice that the pseudo-isotropic condition in the Corollary is essential. In fact, the right cylinder C in the Lorentz-Minkowski space $\mathbb{L}^3 = \mathbb{R}_1^3$ is a non-pseudo-isotropic surface with parallel second fundamental form, and $\mathbf{c}_\sharp(C) = 2$.

Corollary 3.11. *A totally umbilical submanifold into the pseudo-Euclidean space satisfies $\mathbf{c}_\sharp(M_s^n) = \infty$.*

For hypersurfaces of the pseudo-Euclidean space we have the following classification.

Theorem 3.12. *Let M_s^n be a pseudo-Riemannian hypersurface of $\mathbb{R}_\nu^{n+1}$. Then one of the following cases holds.*

(1) *$\mathbf{c}_\sharp(M_s^n) = 2$.*

(2) *$\mathbf{c}_\sharp(M_s^n) = \infty$ and M_s^n is an open portion of a hyperplane.*

(3) $\mathbf{c}_\sharp(M^n_s) = \infty$ *and* M^n_s *is an open portion of a pseudo-Riemannian hypersphere.*

Proof. When the contact number $\mathbf{c}_\sharp(M^n_s) \geq 3$, from Theorem 3.7 the hypersurface is pseudo-isotropic, and Corollaries 3.5 and 3.1 imply that the hypersurface is totally umbilical and its contact number is $\mathbf{c}_\sharp(M^n_s) = \infty$. Moreover, M^n_s is an open portion of an hyperplane or an hypersphere [12]. ∎

With respect to the contact number, the behavior of the Lorentzian submanifolds in the Lorentz-Minkowski space $\mathbb{R}^{n+d}_1$ is similar to that of the pseudo-Riemanian hypersurfaces of $\mathbb{R}^{n+1}_\nu$. In fact, we have the following.

Theorem 3.13. *Let* M^n_1 *be a Lorentzian submanifold of* $\mathbb{R}^{n+d}_1$*. Then one of the following cases happens.*

(1) $\mathbf{c}_\sharp(M^n_1) = 2.$

(2) $\mathbf{c}_\sharp(M^n_1) = \infty$ *and* M^n_1 *is an open portion of a Lorentzian* n*-plane.*

(3) $\mathbf{c}_\sharp(M^n_1) = \infty$ *and* M^n_1 *is an open portion of a De Sitter space* $\mathbb{S}^n_1$ *contained in a Lorentzian* $(n+1) - plane.$

Proof. If $\mathbf{c}_\sharp(M^n_1) \geq 3$, from Theorem 3.7 M^n_1 is pseudo-isotropic. Since $T^\perp_p M^n_1$ is a spacelike subspace of $\mathbb{R}^{n+d}_1$, then from Eq. (12) we have $h(x,x) = 0$ for any null vector x, and then M^n_1 is totally umbilical [6]. Corollary 3.11 applies to give $\mathbf{c}_\sharp(M^n_1) = \infty$, and with a similar argument than [3], it is proved that any totally umbilical Lorentz submanifold in the Lorentz-Minkowski space is an open portion of a Lorentzian n-plane or De Sitter space. ∎

With similar arguments as in the proof of the last theorem, the following result can be stated.

Theorem 3.14. *Let* M^n_s *be a pseudo-isotropic submanifold of the pseudo-Euclidean space* $\mathbb{R}^{n+d}_\nu$ *with indefinite induced metric, and* $\nu = s$*, or* $\nu = s + d$*. Then,* M^n_s *is totally umbilical.*

4. Codimension-2 Submanifolds

Lemma 4.1. *Let* M^n_s *be a* 0*-pseudo-isotropic submanifold of* $\mathbb{R}^{n+2}_{s+1}$*. Then, at each non-totally geodesic point the first normal space* $Im(h)$ *is entirely constituted by null vectors and* $dim(Im(h)) = 1$*.*

Proof. If M^n_s is a submanifold of $\mathbb{R}^{n+2}_{s+1}$, the normal space at $p \in M^n_s$ is a Lorentzian plane. Let p be a non-totally geodesic point and take a unit vector e_1 at

p such that $h(e_1, e_1) \neq 0$. Since $\langle h(e_1, e_1), h(e_1, e_1)\rangle = 0$, then $h(e_1, e_1)$ is a null vector. We can find $e_2, \ldots, e_n$ such that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis at p. By Eq. (7), (8) and (9) we can prove that $h(e_i, e_j)$ and $h(e_1, e_1)$ are linearly dependent for any $i, j \in \{1, \ldots, n\}$, and the result follows. ■

Lemma 4.2. *Let M_s^n be a 0-pseudo-isotropic submanifold of $\mathbb{R}_{s+1}^{n+2}$. Then* $\mathbf{c}_\sharp(M_s^n) = \infty$.

Proof. If M_s^n is totally umbilical, by Corollary 3.11, $\mathbf{c}_\sharp(M_s^n) = \infty$. Assume M_s^n is non-totally umbilical. Let $p \in M_s^n$ be a non-totally geodesic point. By Lemma 4.1, $Im(h)$ is a null straight line. Now we prove $(\widetilde{\nabla}^j h)(u^{j+2}) \in Im(h)$ for any non-totally geodesic $p \in M_s^n$ and $u \in U_pM_s^n$. For this end, take $u \in U_pM_s^n$. If $h(u, u) \neq 0$, by Lemma 3.6 we have

$$\langle (\widetilde{\nabla} h)(u^3), h(u, u)\rangle = 0,$$

and then $(\widetilde{\nabla} h)(u^3) \in Im(h)$. If $h(u, u) = 0$, take a unit v with $h(v, v) \neq 0$. Let X, Y be local orthonormal extensions of u, v respectively, and satisfying $\nabla_w X = \nabla_w Y = 0$ for every $w \in T_pM_s^n$. Since $\langle h(X, X), h(Y, Y)\rangle = 0$, then by Eq. (3) we obtain

$$\begin{aligned} 0 &= u(\langle h(X, X), h(Y, Y)\rangle) \\ &= \langle D_u h(X, X), h(v, v)\rangle + \langle h(u, u), D_u h(Y, Y)\rangle \\ &= \langle (\widetilde{\nabla} h)(u^3), h(v, v)\rangle + \langle h(u, u), (\widetilde{\nabla} h)(v, v, u)\rangle \\ &= \langle (\widetilde{\nabla} h)(u^3), h(v, v)\rangle. \end{aligned}$$

Thus $(\widetilde{\nabla} h)(u^3) \in Im(h)$ for any $u \in U_pM_s^n$.

By the induction method, suppose $(\widetilde{\nabla}^\ell h)(u^{\ell+2}) \in Im(h)$ for any non-totally geodesic point p, $u \in U_pM_s^n$ and $\ell = 0, 1, 2, \ldots, j-1$. Take orthonormal $u, v \in U_pM_s^n$ and assume $h(v, v) \neq 0$. Let $\mathcal{U}$ be the (open) subset of non-totally geodesic points of the submanifold $M_s^n \subseteq \mathbb{R}_{s+1}^{n+2}$. It is clear that $\mathcal{U} \neq \emptyset$. Now we extend u, v to local vector fields X, Y respectively, defined on the open neighborhood $O_p \subseteq \mathcal{U}$ and satisfying $\nabla_w X = \nabla_w Y = 0$ for any $w \in T_pM_s^n$. Then,

$$\langle (\widetilde{\nabla}^{j-1} h)(X^{j+1}), h(Y, Y)\rangle = 0.$$

Therefore, Eq. (3) and (4) yield

$$\begin{aligned} 0 &= u(\langle (\widetilde{\nabla}^{j-1} h)(X^{j+1}), h(Y, Y)\rangle) \\ &= \langle D_u(\widetilde{\nabla}^{j-1} h)(X^{j+1}), h(v, v)\rangle + \langle (\widetilde{\nabla}^{j-1} h)(u^{j+1}), D_u h(Y, Y)\rangle \end{aligned}$$

$$= \langle (\widetilde{\nabla}^j h)(u^{j+2}), h(v,v)\rangle + \langle (\widetilde{\nabla}^{j-1} h)(u^{j+1}), (\widetilde{\nabla} h)(v^2, u)\rangle$$
$$= \langle (\widetilde{\nabla}^j h)(u^{j+2}), h(v,v)\rangle.$$

Hence $(\widetilde{\nabla}^j h)(u^{j+2}) \in Im(h)$, and in particular $\langle (\widetilde{\nabla}^j h)(u^{j+2}), h(u,v)\rangle = 0$. Thus, by Theorem 3.8, $\mathbf{c}_\sharp(M^n_s) = \infty$. ■

Theorem 4.3. *Let M^n_s be a complete 0-pseudo-isotropic submanifold in a pseudo-Euclidean space $\mathbb{R}^{n+2}_{s+1}$. Then, M^n_s is an expansion of $\mathbb{R}^n_s$ into $\mathbb{R}^{n+1}_{s,1}$ with $\mathbf{c}_\sharp(M^n_s) = \infty$.*

Proof. From the proof of Lemma 4.2, for any $u \in U_pM^n_s$ we have that the vectors $(\widetilde{\nabla} h)(u^3) \in Im(h)$ and $h(u,u)$ are linearly dependents. This joined to Codazzi Eq. (6) can be used to show that $(\widetilde{\nabla} h)(u,v,w) \in Im(h)$ for any vectors u, v, w. Thus [1, 7], the codimension can be reduced in such a way that M^n_s is contained in a degenerate hyperplane of $\mathbb{R}^{n+2}_{s+1}$. As this hyperplane is isometric to $\mathbb{R}^{n+1}_{s,1}$, M^n_s is imbedded in $\mathbb{R}^{n+1}_{s,1}$. Let $\pi\colon \mathbb{R}^{n+1}_{s,1} \to \mathbb{R}^n_s$ be the projection map on the first n coordinates. Then $\pi(M^n_s)$ is an open subset of the pseudo-Euclidean space $\mathbb{R}^n_s$ [9]. In consequence, by completeness, there exists a smooth function $f\colon \mathbb{R}^n_s \to \mathbb{R}$ such that M^n_s (viewed in $\mathbb{R}^{n+1}_{s,1}$) can be realized as the set of points $(x, f(x))$. Finally, it suffices to note that the map $\mathbb{R}^{n+1}_{s,1} \hookrightarrow \mathbb{R}^{n+2}_{s+1}$ given by $(y_1, \ldots, y_{n+1}) \mapsto (y_{n+1}, y_1, \ldots, y_{n+1})$ is an isometric embedding. ■

Lemma 4.4. *Let M^n_0 be a non-totally umbilical submanifold of $\mathbb{R}^{n+2}_1$ with $\mathbf{c}_\sharp(M^n_0) \geq 3$. Then, M^n_0 is 0-pseudo-isotropic.*

Proof. Set

$$\mathcal{V} = \{p \in M^n_0 : h(u,v) \neq 0 \text{ for some orthonormal vectors } u, v \in T_pM^n_0\}.$$

Clearly, $\mathcal{V}$ is a non-empty open subset which is non-totally umbilical at every point. Take $u, v \in T_pM^n_0$ such that $h(u,v) \neq 0$ and set $e_1 = u$, $e_n = v$. We extend e_1, e_n to an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ at p. Since $\mathbf{c}_\sharp(M^n_0) \geq 3$, from Theorem 3.7 there exists $\lambda = \lambda(p) \in \mathbb{R}$ such that

$$\langle h(u,u), h(u,u)\rangle = \lambda \tag{15}$$

for any $u \in U_pM^n_0$. Now we prove that $\lambda = 0$.

Case (i). Suppose $\lambda > 0$. Since $\langle h(e_1,e_1), h(e_1,e_1)\rangle = \lambda$, and the pseudo-isotropy condition of Eq. (7) yields $\langle h(e_1,e_1), h(e_1,e_n)\rangle = 0$, we have that $h(e_1,e_1) \neq 0$ is spacelike and $h(e_1,e_n)$ is timelike. Let us write

$$h(e_1,e_1) = \delta\, e_{n+1}, \tag{16}$$

(17) $$h(e_1, e_n) = \mu\ e_{n+2},$$

where $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}\}$ is an orthonormal frame of $\mathbb{R}^{n+2}_1$ (e_{n+2} is a unit timelike vector), and $\delta = \sqrt{\lambda} > 0$, $\mu = \sqrt{-\langle h(e_1, e_n), h(e_1, e_n)\rangle} > 0$. On the other hand, from Eq. (7) we have also that $\langle h(e_1, e_n), h(e_n, e_n)\rangle = 0$. Thus Eq. (15) and Eq. (17) give $h(e_n, e_n) = \pm\delta\ e_{n+1}$. If $h(e_n, e_n) = \delta\ e_{n+1}$, then from Eq. (8) we have

(18) $$\langle h(e_1, e_1), h(e_n, e_n)\rangle + 2\langle h(e_1, e_n), h(e_1, e_n)\rangle = \lambda.$$

Thus Eq. (16) and (17) yield $\langle h(e_1, e_n), h(e_1, e_n)\rangle = 0$, which is a contradiction because $\mu > 0$. The case $h(e_n, e_n) = -\delta\ e_{n+1}$ yields a contradiction because Eq. (18) says $\mu^2 = -\lambda$.

Case (ii). Assume $\lambda < 0$. As in *Case (i)*, we can reach a contradiction by writing $h(e_1, e_1) = \delta\ e_{n+2}$, $h(e_1, e_n) = \mu\ e_{n+1}$, where now $\delta = \sqrt{-\lambda}$, $\mu = \sqrt{\langle h(e_1, e_n), h(e_1, e_n)\rangle}$.

Now, let $p \in \overline{\mathcal{V}}$ be an umbilical point in the boundary of $\mathcal{V}$ and take a sequence $\{p_N\}_{N\in\mathbb{N}}$ in M^n_0 which converges to p so that every p_N is not umbilical. Now let X be a local unitary vector field around p and consider the function $\langle h(X, X), h(X, X)\rangle$. By continuity we have

$$\lambda(p) = \lim_N \langle h(X(p_N), X(p_N)), h(X(p_N), X(p_N))\rangle = \lim_N \lambda(p_N) = 0.$$

If $\mathcal{V}$ is dense the resul follows. If $\mathcal{V}$ is not dense, the set $M^n_0 - \overline{\mathcal{V}}$ is an open submanifold of M^n_0 which is totally umbilical, and thus constant pseudo-isotropic. Let $\lambda \in \mathbb{R}$ be the pseudo-isotropy constant. Now we take a sequence of umbilical points converging to a point in the boundary of $\mathcal{V}$, we obtain $\lambda = 0$. ■

Theorem 4.5. *Let M^n_0 be a complete submanifold of $\mathbb{R}^{n+2}_1$. Then one of the following cases holds.*

1. $\mathbf{c}_\sharp(M^n_0) = 2$.
2. $\mathbf{c}_\sharp(M^n_0) = \infty$ *and M^n_0 is a spacelike n-plane.*
3. $\mathbf{c}_\sharp(M^n_0) = \infty$ *and M^n_0 is a Riemannian n-sphere contained in a spacelike $(n+1)$-plane of $\mathbb{R}^{n+2}_1$.*
4. $\mathbf{c}_\sharp(M^n_0) = \infty$ *and M^n_0 is a hyperbolic n-plane contained in a timelike $(n+1)$-plane.*
5. $\mathbf{c}_\sharp(M^n_0) = \infty$ *and M^n_0 is a totally umbilical and non-totally geodesic expansion of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}_{0,1}$.*
6. $\mathbf{c}_\sharp(M^n_0) = \infty$ *and M^n_0 in non-totally umbilical expansion of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}_{0,1}$.*

Proof. Assume $\mathbf{c}_\sharp(M_0^n) \geq 3$. Then, by Lemma 4.2 and Lemma 4.4, $\mathbf{c}_\sharp(M_0^n) = \infty$ and hence there exists a constant $\lambda \in \mathbb{R}$ such that $\langle h(u,u), h(u,u)\rangle = \lambda$ for any $u \in UM_0^n$. We distinguish two cases.

Case (a). Suppose M_0^n is totally umbilical. Then $h(X,Y) = \langle X, Y\rangle H$ for any $X, Y \in \mathfrak{X}(M_0^n)$, where H is the mean curvature vector field of M_0^n. Thus $\langle H, H\rangle = \lambda$. Now we have three possibilities for λ.

$(a1)$ $\lambda = 0$. If M_0^n is totally geodesic then it satisfies assertion *(2)* in the Theorem, and if M_0^n is not totally geodesic then M_0^n is as indicated on assertion *(5)*.

$(a2)$ $\lambda > 0$. Since M_0^n has parallel second fundamental form, the codimension can be reduced so that M_0^n lies in a $(n+1)$-dimensional spacelike plane Π. On the other hand, it is easy to check that $P + (\xi/\lambda) = c$, where P is the vector position vector for points of M_0^n and $c \in \mathbb{R}_1^{n+2}$. Therefore $\langle P-c, P-c\rangle = 1/\lambda$, and then M_0^n lives in the De Sitter space $\mathbb{S}_1^{n+1}(c, 1/\sqrt{\lambda})$. Thus M_0^n lies in the intersection $\Pi \cap \mathbb{S}_1^{n+1}(c, 1/\sqrt{\lambda})$, which is isometric to an n-dimensional sphere $\mathbb{S}^n$, and this means that M_0^n satisfies assertion *(3)*.

$(a3)$ If $\lambda < 0$, M_0^n lives in an $(n+1)$-dimensional Lorentzian plane Π. But we have also that $P - \frac{\xi}{\lambda} = c \in \mathbb{R}_1^{n+2}$, and therefore M_0^n lies also in an $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}\left(c, 1/\sqrt{-\lambda}\right)$. Thus, assertion *(4)* follows.

Case (b). Assume M_0^n is not totally umbilical. Then by Theorem 4.3 and Lemma 4.6 the assertion *(6)* is fulfilled. ■

Corollary 4.6. *Let M_0^n be a complete non-totally umbilical pseudo-isotropic submanifold in the Lorentz-Minkowski space $\mathbb{R}_1^{n+2}$. Then, M_0^n is a non-totally umbilical expansion of $\mathbb{R}^n$ into $\mathbb{R}_{0,1}^{n+1}$.*

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J. L. Cabrerizo, M. Fernández and J. S. Gómez
Departamento de Geometríay Topología,
Universidad de Sevilla,
Apdo. Correos 1160,
41080-Sevilla, Spain
E-mail: jaraiz@us.es
mafernan@us.es
jsalvador@us.es