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ITERATIVE SOLUTIONS FOR SOME FOURTH-ORDER PERIODIC BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, by using a new maximum principle and the Fredholm alternative, the monotone method in the presence of lower and upper solutions for the periodic problem

$$\begin{aligned} u^{(iv)}(x) &= f(x, u(x), u''(x)), \quad 0 < x < 2\pi, \\ u^{(i)}(0) &= u^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \end{aligned}$$

is developed, where $f : [0, 2\pi] \times R^2 \longrightarrow R$ is a Caratheodory function.

1. Introduction

In this paper, we shall employ the method of upper and lower solutions to study the existence of solutions of the fourth-order boundary value problem with periodic boundary condition

$$u^{(iv)}(x) = f(x, u(x), u''(x)), \quad 0 < x < 2\pi, \tag{1.1}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), i = 0, 1, 2, 3, \tag{1.2}$$

where $f : [0, 1] \times R^2 \longrightarrow R$ is a Caratheodory function.

The method of upper and lower solutions is extensively developed for lower-order equations with various kinds of boundary conditions. In general, there are two fundamental ways in the application. One is coupled with some growth restrictions on the nonlinearity such as Nagumo condition, see [5, 11]; the other is coupled with

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the monotone method, see [14, 6, 10, 12, 13]. While the former just can give the range of solutions, the later could get iterative sequences which converge uniformly to the extremal solutions of the problem considered. The later method has been very useful in the study of BVPs for higher-order functional equations. The reader can refer to [7] for an elegant exposition of the method. Recently, Bai[2], Ehme et al. [5], Ma et al. [10], and Pao [12] applied the monotone method to some fourth-order problems with non-periodic boundary conditions, and some excellent existence results were obtained.

However, to my best knowledge, only a few authors have studied fourth-order periodic boundary value problems (see [3, 6, 8, 9]). When $f = f(t,u)$, Cabada [3] has studied the problem by the method of upper and lower solutions and the monotone iterative technique. The method used in [8, 9] is a fixed-point theorem in a cone. In [6], Jiang et al. have studied Problem (1.1), (1.2) by transforming the fourth-order problem into the equivalent second-order problem and using Banach constriction principle. In this paper, we directly deal with the fourth-order problem by the use of a new maximum principle of the fourth-order equation and the Fredholm alternative. The method used here is different from [3, 6, 8, 9].

2. Definitions and Maximum Principle

Definition 1.1. A function $f : [0, 2\pi] \times R^2 \to R$ is said to be a Caratheodory function if it possesses the following properties:

(1) For all $(u, v) \in R^2$, the function $x \to f(x, u, v)$ is measurable on $[0, 2\pi]$.

(2) For almost all $x \in [0, 2\pi]$, the function $(u, v) \to f(x, u, v)$ is continuous on R^2.

(3) For any given $N > 0$, there exists $g_N(x)$, a Lebesgue integrable function defined on $[0, 2\pi]$ such that

$$|f(x, u, v)| \leq g_N(x) \quad \text{for a.e. } x \in [0, 2\pi],$$

whenever $|u|, |v| \leq N$.

Definition 1.2. Letting $\alpha \in W^{4,1}[0, 2\pi]$, we say α is an upper solution for the problem (1.1), (1.2) if α satisfies

$$\begin{cases} \alpha^{(iv)}(x) \geq f(x, \alpha(x), \alpha''(x)), \quad \text{for } x \in (0, 2\pi), \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) \leq \alpha'(2\pi), \\ \alpha''(0) = \alpha''(2\pi), \quad \alpha'''(0) \geq \alpha'''(2\pi). \end{cases} \tag{2.1}$$

Definition 1.3. Letting $\beta \in W^{4,1}[0, 2\pi]$, we say β is a lower solution for the problem (1.1), (1.2) if β satisfies

$$(2.2)\qquad \begin{cases} \beta^{(iv)}(x) \leq f(x, \beta(x), \beta''(x)), & \text{for } x \in (0, 2\pi), \\ \beta(0) = \beta(2\pi), \quad \beta'(0) \geq \beta'(2\pi), \\ \beta''(0) = \beta''(2\pi), \quad \beta'''(0) \leq \beta'''(2\pi). \end{cases}$$

We call a function $u \in W^{4,1}[0, 2\pi]$ a solution to Problem (1.1), (1.2), if it is an upper and a lower solution.

In the following, we prove a new maximum principle for the operator

$$L : F \longrightarrow W^{4,1}[0, 2\pi]$$

defined by $Lu = u^{(iv)} - (a+b)u'' + abu$. Here $a, b \in R, a, b > 0$, $u \in F$ and

$$F = \{u \in W^{4,1}[0, 2\pi] \mid u(0) = u(2\pi), u'(0) \leq u'(2\pi), u''(0) = u''(2\pi), u'''(0) \geq u'''(2\pi)\}.$$

Lemma 2.1. [6]. *If* $u \in W^{2,1}[0, 2\pi]$ *satisfies*

$$u''(x) - Mu(x) \geq 0, \text{ for a.e. } x \in [0, 2\pi],$$
$$u(0) = u(2\pi), \quad u'(0) \geq u'(2\pi),$$

where $M > 0$, *then* $u(x) \leq 0$ *on* $[0, 2\pi]$.

The following maximum principle is fundamental to our main result.

Lemma 2.2. *If* $u \in F$ *satisfies* $Lu \geq 0$, *for a.e.* $x \in [0, 2\pi]$, *then* $u \geq 0$ *in* $[0, 2\pi]$.

Proof. Set $Au = u''$. Because $a, b > 0$, we have that

$$Lu = u^{(iv)} - (a+b)u'' + abu = (A-b)(A-a)u \geq 0.$$

Let $y = (A-a)u = u'' - au$, then

$$(A-b)y \geq 0,$$

i.e.,

$$y''(x) - by(x) \geq 0, \text{ for a.e. } x \in [0, 2\pi].$$

On the other hand, $a, b > 0$ and $u \in F$ yield that

$$y(0) = u''(0) - au(0) = u''(2\pi) - au(2\pi) = y(2\pi),$$
$$y'(0) = u'''(0) - au'(0) \geq u'''(2\pi) - au'(2\pi) = y'(2\pi).$$

Therefore, with the use of Lemma 2.1, we have that

$$y(x) \leq 0, \quad x \in [0, 2\pi],$$

i.e.,

$$u''(x) - au(x) \leq 0, \quad x \in [0, 2\pi].$$

Then, by the use of Lemma 2.1 again and the fact that $u(0) = u(2\pi), u'(0) \leq u'(2\pi)$, one has $u(x) \geq 0$ in $[0, 2\pi]$. The proof is complete. ■

The following conclusion about homogeneous two-parameter linear fourth-order periodic problem follows from Lemma 2.2 immediately.

Lemma 2.3. *Given $a, b > 0$, the problem*

$$u^{(iv)}(x) - (a+b)u''(x) + abu(x) = 0, \quad 0 < x < 2\pi, \tag{2.3}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{2.4}$$

has only a trivial solution.

Proof. It is clear that Problem (2.3), (2.4) has a trivial solution. On the other hand, suppose there are two solutions v_1 and v_2, then

$$L(v_1 - v_2) = 0,$$

$$(v_1 - v_2)^{(i)}(0) = (v_1 - v_2)^{(i)}(2\pi), i = 0, 1, 2, 3.$$

Therefore, with the use of Lemma 2.2, there is

$$v_1(x) \equiv v_2(x), \quad \text{for } x \in [0, 2\pi].$$

The proof is complete. ■

3. The Monotone Method

In this section, we develop the monotone method for the fourth-order periodic boundary value problem (1.1), (1.2).

For given $a, b > 0$ and $f : [0, 2\pi] \times R^2 \longrightarrow R$, let

$$f_1(x, u, v) = f(x, u, v) - (a+b)u + abv. \tag{3.1}$$

Then (1.1) equal to

$$Lu = f_1(x, u, u''). \tag{3.2}$$

It is clear that if α, β are upper and lower solutions of the problem (1.1), (1.2) respectively, then α, β are upper and lower solutions of the problem (3.2), (1.2) respectively, too.

For $\beta \le \alpha,\ \beta'' + b(\alpha - \beta) \ge \alpha''$, denote

$$D_\alpha^\beta = \left\{(u, v) \mid \beta \le u \le \alpha, \alpha'' - b(\alpha - \beta) \le v \le \beta'' + b(\alpha - \beta)\right\}.$$

Theorem 3.1. *If there exist α and β, upper and lower solutions, respectively, for the problem* (1.1), (1.2) *which satisfy*

$$\beta \le \alpha \quad \text{and} \quad \beta'' + b(\alpha - \beta) \ge \alpha''. \tag{3.3}$$

If $f : [0, 2\pi] \times R^2 \longrightarrow R$ is a Caratheodory function and satisfies

$$f(x, u_2, v) - f(x, u_1, v) \ge -(a + b)(u_2 - u_1) \tag{3.4}$$

on D_α^β, for a.e. $x \in [0, 2\pi], u_1 \le u_2$;

$$f(x, u, v_2) - f(x, u, v_1) \le ab(v_2 - v_1) \tag{3.5}$$

on D_α^β, for a.e. $x \in [0, 2\pi], v_2 + b(\alpha - \beta) \ge v_1$, where $a \ge b > 0$, then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, non-increasing and non-decreasing respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\beta, \alpha]$ of the problem (1.1), (1.2).

Proof. Consider the problem

$$u^{(iv)}(x) - (a+b)u''(x) + abu(x) = f_1(x, \eta(x), \eta''(x)), \ \text{for} \ x \in (0, 2\pi), \tag{3.6}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{3.7}$$

with $\eta \in W^{2,1}[0, 2\pi]$.

Since $a \ge b > 0$, with Lemma 2.3 Problem (2.3), (2.4) has only a trivial solution. By Fredholm alternative, $L : F \to L^1[0, 2\pi]$ is invertible, namely the problem (3.6), (3.7) has a unique solution u. Define $T : W^{2,1}[0, 2\pi] \longrightarrow W^{4,1}[0, 2\pi]$ by

$$T\eta = u. \tag{3.8}$$

Now, we divide the proof into three steps.

Step 1. We show

$$TC \subseteq C. \tag{3.9}$$

Here, $C = \{\eta \in W^{2,1}[0, 2\pi] \mid \beta \le \eta \le \alpha, \alpha'' - b(\alpha - \beta) \le \eta'' \le \beta'' + b(\alpha - \beta)\}$ is a nonempty bounded closed subset in $W^{2,1}[0, 2\pi]$.

In fact, for $\zeta \in C$, set $\omega = T\zeta$. By the definition of α, β and C, combining (3.1), (3.4),and (3.5), we have that for a.e. $x \in [0, 2\pi]$

$$\begin{aligned} &(\alpha - \omega)^{(iv)}(x) - (a + b)(\alpha - \omega)''(x) + ab(\alpha - \omega)(x) \\ &\ge f_1(x, \alpha(x), \alpha''(x)) - f_1(x, \zeta(x), \zeta''(x)) \\ &= f(x, \alpha(x), \alpha''(x)) - f(x, \zeta(x), \zeta''(x)) \\ &\quad - (a+b)(\alpha - \zeta)''(x) + ab(\alpha - \zeta)(x) \\ &\ge 0, \end{aligned} \tag{3.10}$$

$$(\alpha - \omega)(0) = (\alpha - \omega)(2\pi), \quad (\alpha - \omega)'(0) \le (\alpha - \omega)'(2\pi), \tag{3.11}$$

$$(\alpha - \omega)''(0) = (\alpha - \omega)''(2\pi), (\alpha - \omega)'''(0) \ge (\alpha - \omega)'''(2\pi). \tag{3.12}$$

With the use of Lemma 2.2, we obtain that $\alpha \ge \omega$. Analogously, there holds $\omega \ge \beta$.

By the proof of Lemma 2.2, combining (3.10) and (3.12), we have that

$$(\alpha - \omega)''(x) - b(\alpha - \omega)(x) \le 0, \quad x \in [0, 2\pi],$$

hence, for $x \in [0, 2\pi]$,

$$\omega''(x) + b(\alpha - \beta)(x) \ge \omega''(x) + b(\alpha - \omega)(x) \ge \alpha''(x),$$

i.e.,

$$\omega''(x) \ge \alpha''(x) - b(\alpha - \beta)(x), \text{ for } x \in [0, 2\pi].$$

Analogously,

$$\omega''(x) \le \beta''(x) + b(\alpha - \beta)(x), \text{ for } x \in [0, 2\pi].$$

Thus, (3.9) holds.

Step 2. Let $u_1 = T\eta_1, u_2 = T\eta_2$, where $\eta_1, \eta_2 \in C$ satisfy $\eta_1 \le \eta_2$ and $\eta_1'' + b(\alpha - \beta) \ge \eta_2''$. We show

$$u_1 \le u_2, \quad u_1'' + b(\alpha - \beta) \ge u_2''. \tag{3.13}$$

In fact, by (3.4), (3.5), and the definition of u_1, u_2,

$$\begin{aligned} L(u_2 - u_1)(x) &= f_1(x, \eta_2(x), \eta_2''(x)) - f_1(x, \eta_1(x), \eta_1''(x)) \ge 0, \\ (u_2 - u_1)(0) &= (u_2 - u_1)(2\pi) = 0, \\ (u_2 - u_1)''(0) &= (u_2 - u_1)''(2\pi) = 0. \end{aligned}$$

With the use of Lemma 2.2, we get that $u_1 \leq u_2$. Similar to Step 1, we can easily prove $u_1'' + b(\alpha - \beta) \geq u_2''$. Thus, (3.13) holds.

Step 3. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence:

$$\alpha_0 = \alpha, \beta_0 = \beta, \alpha_n = T\alpha_{n-1}, \beta_n = T\beta_{n-1}, n = 1, 2, \ldots.$$

From the results of Step 1 and Step 2, we have that

$$\beta = \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_1 \leq \alpha_0 = \alpha, \tag{3.14}$$

$$\beta'' = \beta_0'', \alpha'' = \alpha_0'', \alpha'' - b(\alpha - \beta) \leq \alpha_n'', \beta_n'' \leq \beta'' + b(\alpha - \beta). \tag{3.15}$$

Moreover, from the definition of T (see (3.8)), we get

$$\alpha_n^{(iv)}(x) - (a+b)\alpha_n''(x) + ab\alpha_n(x) = f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)).$$

i.e.,

$$\begin{aligned} \alpha_n^{(iv)}(x) &= f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)) + (a+b)\alpha_n''(x) - ab\alpha_n(x) \\ &\leq f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)) \\ &\quad +(a+b)\left[\beta''(x) + b(\alpha(x) - \beta(x))\right] - ab\beta(x), \end{aligned} \tag{3.16}$$

$$\alpha_n^{(i)}(0) = \alpha_n^{(i)}(2\pi), \quad i = 0, 1, 2, 3. \tag{3.17}$$

Analogously,

$$\begin{aligned} \beta_n^{(iv)}(x) &= f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) + (a+b)\beta_n''(x) - ab\beta_n(x) \\ &\leq f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) \\ &\quad +(a+b)\left[\beta''(x) + b(\alpha(x) - \beta(x))\right] - ab\beta(x), \end{aligned} \tag{3.18}$$

$$\beta_n^{(i)}(0) = \beta_n^{(i)}(2\pi) \quad i = 0, 1, 2, 3. \tag{3.19}$$

From $(3.14), (3.15)$, we have that there exists $C_1 > 0$ depending only on α and β (but not on n or x) such that

$$|\alpha_n(x)| \leq C_1, |\alpha_n''(x)| \leq C_1.$$

Which together with (3.16) and the fact that $f(t, u, v)$ is a Caratheodory function yields there exists a Lebesgue integrable function $g_{C_1}(x)$ defined on $[0, 2\pi]$ such that

$$|\alpha_n^{(iv)}(x)| \leq g_{C_1}(x), \quad \text{for a.e.} \quad x \in [0, 2\pi]. \tag{3.20}$$

On the other hand, $\alpha_n''(0) = \alpha_n''(1)$ yields there are $x_n \in (0, 2\pi)$ such that $\alpha_n'''(x_n) = 0$, thus there exists $C_2 > 0$ such that

$$|\alpha_n'''(x)| \le C_2.$$

Therefore, $\{\alpha_n(x)\}$ is bounded in C. Similarly, $\{\beta_n(x)\}$ is bounded in C, too.

It then follows by a standard argument (see e.g. [7]) that

$$\lim_{n\to\infty} \beta_n(t) := \beta^*(t) \quad \text{and} \quad \lim_{n\to\infty} \alpha_n(t) := \alpha^*(t)$$

uniformly and monotonically on $[0, 2\pi]$. And $\alpha^*(t), \beta^*(t)$ are maximal and minimal fixed points in the segment $[\beta, \alpha]$ to T, then, $\alpha^*(t), \beta^*(t)$ are both solutions to Problem (1.1), (1.2). ∎

An Example. As an application of the main result, we study the existence of periodic solutions of the fourth-order equations

$$u^{(iv)} - pu'' - a(x)u + b(x)u^3 = 0, x \in R \tag{3.21}$$

and

$$u^{(iv)} - pu'' + a(x)u - b(x)u^3 = 0, x \in R \tag{3.22}$$

where p is a positive constant, and $a(x)$ and $b(x)$ are continuous positive 2π-periodic functions on R. Equations of this type are known as extended Fisher-Kolmogorov equations which have been proposed as a model for phase transitions and other bistable phenomena [14]. We assume that there are positive constants a_1, a_2, b_1 and b_2 such that

$$0 < a_1 \le a(x) \le a_2, \quad 0 < b_1 \le b(x) \le b_2.$$

Consider the following boundary value problem

$$u^{(iv)} - pu'' - a(x)u + b(x)u^3 = 0, \quad x \in [0, 2\pi] \tag{3.23}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, 3. \tag{3.24}$$

Given positive constants C_1, C_2, it is easy to check that $\alpha = C_1, \beta = C_2 \sin\frac{x}{2}$ are upper and lower solutions of (3.23), (3.24), respectively, as well as

$$C_1^2 b_1 - a_2 \ge 0 \tag{3.25}$$

and

$$C_2^2 b_2 - a_1 + \frac{1}{16} + \frac{p}{4} \le 0. \tag{3.26}$$

And to guarantee the assumption (3.3) of Theorem 3.1 holds, we assume that

$$C_2 \leq C_1, \quad bC_1 - C_2(\frac{1}{4} + b) \geq 0; \tag{3.27}$$

to guarantee the assumption (3.4) of Theorem 3.1 holds, we assume that

$$a_1 - 3b_2C_1^2 \geq -(a + b). \tag{3.28}$$

Because $f(x, u, v) = pv + a(x)u - b(x)u^3$, let $p = ab$, one has assumption (3.5) of Theorem 3.1 holds. To sum up, as well as $a_1 - \frac{1}{16} - \frac{p}{4} > 0$, we can always choose $b > 0$ sufficiently small such that there are sufficiently large $C_1 > 0$ and sufficiently small $C_2 > 0$ such that all assumptions of Theorem 3.1 are fulfilled. Hence the problem (3.23), (3.24) has two solutions u_1, u_2, which satisfies

$$C_2 \sin \frac{x}{2} \leq u_1 \leq u_2 \leq C_1.$$

On the other hand, the previous conclusion, for example, see [3, 6, 8, 9, 14], can't be used to the example.

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