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ON A SPECIAL GENERALIZED VANDERMONDE MATRIX AND ITS LU FACTORIZATION

Hsuan-Chu Li and Eng-Tjioe Tan

Abstract. We consider a special class of the generalized Vandermonde matrices and obtain an LU factorization for its member by giving closed-form formulae of the entries of L and U. Moreover, we express the matrices L and U as products of 1-banded (bidiagonal) matrices. Our result is applied to give the closed-form formula of the inverse of the considered matrix.

1. INTRODUCTION

Nonsingular Classical Vandermonde matrices are square $n \times n-$matrices of the form

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{n-1} & v_2^{n-1} & \cdots & v_n^{n-1} \end{bmatrix},$$

and solutions of the linear system of equations $Vx = b$ have been discussed in connection with numerical applications, such as approximation and interpolation problems, also in confluent problems (see for instance [1, 2, 9]); the solutions $x = V^{-1}b$ involve in finding explicit factorizations of V^{-1}, such as LU factorizations and 1-banded factorizations. In a recent paper [7], H. Oruç and G. M. Phillips obtained an explicit formula of the LU factorization of V and expressed the matrices L and U as a product of 1-banded matrices, and later Sheng-liang Yang [10] gave a simpler alternative approach and proofs of their results.

In the recent literature great interests have been revived on *generalized* Vandermonde matrices which also arise naturally in problems, among others, on differential

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equations and difference equations, and on their determinants, explicit factorizations and inverses(see e.g. [12], for treatments of various types of generalized Vandermonde matrices); another type (*totally positive* generalized Vandermonde matrices) has been studied by J. Demmel and P. Koev [3] and they gave explicit formulae for the entries of the bidiagonal factorization and the LDU factorization.

In this paper we consider a special class of another type of generalized Vandermonde matrix which was introduced by C. L. Liu ([6] p.90): It arises while solving a recurrence equation

$$(1.1) \qquad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_p a_{n-p}, \quad n \geq p. \quad (p \quad fixed)$$

where $c_1, c_2, \cdots, c_p$ are constants and $c_p \neq 0$. If the characteristic function of (1.1) has distinct roots $v_1, v_2, \cdots, v_q \in \mathbb{R}$ with multiplicities $u_1, u_2, \cdots, u_q$, respectively, and $\sum_{i=1}^{q} u_i = n$, then the corresponding generalized Vandermonde matrix has the form

$$\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ v_1 & v_1 & \cdots & v_1 & \cdots & v_q & v_q & \cdots & v_q \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ v_1^{n-1} & (n-1)v_1^{n-1} & \cdots & (n-1)^{u_1-1}v_1^{n-1} & \cdots & v_q^{n-1} & (n-1)v_q^{n-1} & \cdots & (n-1)^{u_q-1}v_q^{n-1} \end{bmatrix},$$

and its determinant has also been calculated as follows ([5, 8]):

$$(1.2). \ det\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = \left[\prod_{i=1}^{q}(\prod_{j=0}^{u_i-1} j!)\right] \times \left[\prod_{i=1}^{q} v_i^{\binom{u_i}{2}}\right] \times \left[\prod_{1 \leq i < j \leq q} (v_j - v_i)^{u_i u_j}\right]$$

A special class of generalized Vandermonde matrices which is the main subject of investigations in this paper will be denoted by $V_{\{2;1,n-1\}}$ and defined as follows: For $u_1 = 1, u_2 = n-1, q = 2$, $V_{\{2;1,n-1\}}$ is the transpose of $\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$, i.e.

$$V_{\{2;1,n-1\}} = \begin{bmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^{n-1} \\ 1 & v_2 & v_2^2 & \cdots & v_2^{n-1} \\ 0 & v_2 & 2v_2^2 & \cdots & (n-1)v_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_2 & 2^{n-2}v_2^2 & \cdots & (n-1)^{n-2}v_2^{n-1} \end{bmatrix}.$$

The above-mentioned type of generalized Vandermonde matrix $V_{\{q;u_1,u_2,\cdots,u_q\}}$ has also occured in the literature: It is an $n \times n$ generalized geometric progression(GP) block matrix, B, introduced by Y. Yang and H. Holtti[12], p.54, Definition 1. In [12], the authors' main results are a non-unique decomposition of the (GP) block matrix(Theorem 1, p.56) and its determinant formula, but not the determination of its inverse.

Related to his above-mentioned paper [10], Yang's another paper [11] treated the LU factorizations of two special classes of generalized Vandermonde matrices, different from ours. Yang's paper [11] prompts us to investigate the feasibility of the LU factorization and the 1-banded factorization of the above-defined matrix $V_{\{2;1,n-1\}}$. Our two main results are Theorem 2.1 and Theorem 3.1 on the LU and the bidiagonal(1-banded) factorizations of $V_{\{2;1,n-1\}}$.

As an application of our result Theorem 3.1, we give in our last Section 4 the closed-form formula of $V^{-1}_{\{2;1,n-1\}}$ as product of triangular matrices; such closed-form formulae are possible thank to the fact that the explicit formula of the inverse of a tridiagonal matrix has been calculated in the literature (see e.g.[4]).

2. The LU Factorization of $\mathbf{V_{\{2;1,n-1\}}}$

The main goal of this section is to obtain an explicit formula of the LU factorization of the special generalized Vandermonde matrix $V_{\{2;1,n-1\}}$. Our first main result is the following theorem:

Theorem 2.1. *$V_{\{2;1,n-1\}}$ can be factorized as $V_{\{2;1,n-1\}} = L_nU_n$, where $L_n = [L_n(i,j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]$ is an upper triangular matrix, whose entries are defined as follows:*

$$L_n(i,j) = 0, i < j;\ L_n(i,i) = \frac{a_{i,i}v_2 - a_{i-1,i-1}v_1}{v_2 - v_1} = 1, i \geq 1;$$
$$L_n(2,1) = 1;\ L_n(i,1) = 0, i \geq 3;$$
$$L_n(i,j) = \frac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1},\ a_{i,j} = (j-1)a_{i-1,j} + a_{i-1,j-1}, i \geq 3, j \geq 2, i > j,$$

and

$$U_n(i,j) = 0, i > j;\ U_n(1,1) = 1;\ U_n(1,j) = v_1^{j-1}, j \geq 2;$$
$$U_n(i,j) = (i-2)!v_2^{i-2}(v_2 - v_1)[\sum_{m=0}^{j-i} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}], j \geq i \geq 2.$$

where $a_{i,i} = 1, i \geq 0;\ a_{i,1} = 0, i \geq 2;\ a_{i,2} = 1, i \geq 3;\ a_{i+1,i} = \sum_{k=1}^{i-1} k, i \geq 2.$

Proof. We use mathematical induction on n, the size of $V_{\{2;1,n-1\}}$.

(1) The case $n = 2$:

$$L_2U_2 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & v_1\\ 0 & v_2 - v_1\end{bmatrix} = \begin{bmatrix}1 & v_1\\ 1 & v_2\end{bmatrix} = V_{\{2;1,1\}}.$$

(2) The case $k \Rightarrow k+1$ with $k > 1$: Assume $V_{\{2;1,k-1\}} = L_kU_k$ holds, we want to prove $V_{\{2;1,k\}} = L_{k+1}U_{k+1}$, i.e. $\forall_{1\le d\le k+1}$,

$$V_{\{2;1,k\}}(k+1,d) = \sum_{f=1}^{k+1} L_{k+1}(k+1,f)U_{k+1}(f,d), \tag{2.1.1}$$

and

$$V_{\{2;1,k\}}(d,k+1) = \sum_{f=1}^{k+1} L_{k+1}(d,f)U_{k+1}(f,k+1). \tag{2.1.1$'$}$$

First, it is easy to verify that $(2.1.1)$ holds for $d = 1, 2$, by definitions.

Next we show $(2.1.1)$ for $3 \le d \le k$. To this end, we start with our assumption:

$$V_{\{2;1,k-1\}}(k,d) = \sum_{f=1}^{k} L_k(k,f)U_k(f,d), 3 \le d \le k.$$

We have by definitions:

$$\begin{aligned}(d-1)^{k-2}v_2^{d-1} &= V_{\{2;1,k-1\}}(k,d)\\ &= v_2 \times (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2})\\ &\quad + \sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\tbinom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\ &\quad - v_2v_1[\sum_{m=0}^{d-3}\tbinom{1+m}{1}v_2^m v_1^{(d-3)-m}]\\ &\quad - \sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\tbinom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\ &\quad + a_{k,d} \times (d-2)!v_2^{d-1} - a_{k-1,d-1} \times (d-2)!v_2^{d-2}v_1.\end{aligned}$$

Next, by partially differentiating to v_2 on both sides, we obtain

$$\begin{aligned}(d-1)^{k-1}v_2^{d-2} &= (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2})\\ &\quad + v_2 \times [(d-2)v_2^{d-3} + (d-3)v_2^{d-4}v_1 + \cdots + 2v_2v_1^{d-4} + v_1^{d-3}]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{f=3}^{d-1} a_{k,f}(f-1)!v_2^{f-2}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\
&+\sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}m v_2^{m-1} v_1^{(d-f)-m}]\\
&-v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}] - v_2 v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}m v_2^{m-1} v_1^{(d-3)-m}]\\
&-\sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!(f-2)v_2^{f-3}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\
&-\sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}m v_2^{m-1} v_1^{(d-f)-m}]\\
&+a_{k,d}\times(d-1)!v_2^{d-2} - a_{k-1,d-1}\times(d-2)!(d-2)v_2^{d-3}v_1,
\end{aligned}
$$

and, by multiplying v_2 to both sides, we yield the left hand side of (2.1.1):

$$
\begin{aligned}
&(d-1)^{k-1}v_2^{d-1}\\
&= V_{\{2;1,k\}}(k+1,d) = (v_2^{d-1} + v_2^{d-2}v_1 + \cdots + v_2^2 v_1^{d-3} + v_2 v_1^{d-2})\\
&+[(d-2)v_2^{d-1} + (d-3)v_2^{d-2}v_1 + \cdots + 2v_2^3 v_1^{d-4} + v_2^2 v_1^{d-3}]\\
&+\sum_{f=3}^{d-1} a_{k,f}(f-1)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\
&+\sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}m v_2^{m} v_1^{(d-f)-m}]\\
&-v_2 v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}] - v_2 v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}m v_2^{m} v_1^{(d-3)-m}]\\
&-\sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!(f-2)v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]\\
&-\sum_{f=4}^{d-1} a_{d-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}m v_2^{m} v_1^{(d-f)-m}]\\
&+a_{k,d}\times(d-1)!v_2^{d-1} - a_{k-1,d-1}\times(d-2)!(d-2)v_2^{d-2}v_1.
\end{aligned}
\tag{2.1.2}
$$

On the other hand, the right hand side of (2.1.1) is

$$
\sum_{f=1}^{k+1} L_{k+1}(k+1,f)U_{k+1}(f,d)
$$

$$
\begin{aligned}
&= v_2 \times (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2 v_1^{d-3} + v_1^{d-2}) \\
&\quad + \sum_{f=3}^{d-1} a_{k,f}(f-1)! v_2^{f-1} [\sum_{m=0}^{d-f} \tbinom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] \\
&\quad + v_2^2 [\sum_{m=0}^{d-3} \tbinom{1+m}{1} v_2^m v_1^{(d-3)-m}] \\
&\quad + \sum_{f=4}^{d-1} a_{k,f-1}(f-2)! v_2^{f-1} [\sum_{m=0}^{d-f} \tbinom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] \\
&\quad - v_2 v_1 [\sum_{m=0}^{d-3} \tbinom{1+m}{1} v_2^m v_1^{(d-3)-m}] \\
&\quad - \sum_{f=4}^{d-1} (f-2) a_{k-1,f-1}(f-2)! v_2^{f-2} v_1 [\sum_{m=0}^{d-f} \tbinom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] \\
&\quad - 2! v_2^2 v_1 [\sum_{m=0}^{d-4} \tbinom{2+m}{2} v_2^m v_1^{(d-4)-m}] \\
&\quad - \sum_{f=5}^{d-1} a_{k-1,f-2}(f-2)! v_2^{f-2} v_1 [\sum_{m=0}^{d-f} \tbinom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] \\
&\quad + a_{k,d} \times (d-1)! v_2^{d-1} + a_{k,d-1} \times (d-2)! v_2^{d-1} - a_{k-1,d-1} \\
&\quad \times (d-2)(d-2)! v_2^{d-2} v_1 - a_{k-1,d-2} \times (d-2)! v_2^{d-2} v_1.
\end{aligned}
\tag{2.1.3}
$$

It remains to prove the equations $(2.1.2) = (2.1.3)$.
To this end, we compare the ten terms $(2.1.2.i), 1 \le i \le 10$, of equation $(2.1.2)$ with those twelve terms $(2.1.3.j), 1 \le j \le 12$, of equation $(2.1.3)$. Apparently, by direct calculations, the following eight terms are equal: $(2.1.2.1) = (2.1.3.1), (2.1.2.2) = (2.1.3.3), (2.1.2.3) = (2.1.3.2), (2.1.2.5) = (2.1.3.5), (2.1.2.6) = (2.1.3.7), (2.1.2.7) = (2.1.3.6), (2.1.2.9) = (2.1.3.9), (2.1.2.10) = (2.1.3.11)$.

For the rest, it remains to check term by term, using direct calculations that $(2.1.2.4) = (2.1.3.4) + (2.1.3.10)$ and $(2.1.2.8) = (2.1.3.8) + (2.1.3.12)$:

$$
\begin{aligned}
&(2.1.2.4) \\
&= \sum_{f=3}^{d-1} a_{k,f}(f-2)! v_2^{f-1} [\sum_{m=0}^{d-f} \tbinom{f-2+m}{f-2} m v_2^m v_1^{(d-f)-m}] \\
&= \sum_{f=3}^{d-2} a_{k,f}(f-1)! v_2^{f} \left[\sum_{m=1}^{d-f} \frac{(f-2+m)!}{(f-1)!(m-1)!} v_2^{m-1} v_1^{[d-(f+1)]-(m-1)} \right] \\
&\quad + a_{k,d-1}(d-2)! v_2^{d-1}.
\end{aligned}
\tag{2.1.2.4'}
$$

Now, the crucial step is to show that $(2.1.2.4') = (2.1.3.4) + (2.1.3.10)$: Let $g = m - 1$, then

$$
\begin{aligned}
&(2.1.2.4') \\
&= \sum_{f=3}^{d-2} a_{k,f}(f-1)! v_2^f [\sum_{g=0}^{d-(f+1)} \binom{f-1+g}{f-1} v_2^g v_1^{[d-(f+1)]-g}] + a_{k,d-1}(d-2)! v_2^{d-1} \\
&= \sum_{f=4}^{d-1} a_{k,f-1}(f-2)! v_2^{f-1} [\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] + a_{k,d-1}(d-2)! v_2^{d-1} \\
&= (2.1.3.4) + (2.1.3.10).
\end{aligned}
$$

Similarly, we also have $(2.1.2.8) = (2.1.3.8) + (2.1.3.12)$. This completes the proof of the equality $(2.1.2) = (2.1.3)$.

To prove the equality $(2.1.1')$, it can easily be shown that $(2.1.1')$ holds for $d = 1, 2, 3$, by definitions.

Using the same technique as in the case of equation $(2.1.1)$, we can obtain that $(2.1.1')$ holds for $4 \le d \le k+1$, proving the equation $(2.1.1')$ and completing the proof of Theorem 2.1.

Example. To illustrate our result, we give an example for $n = 7$: $V_{\{2;1,6\}} = L_7 U_7$, where

$$
V_{\{2;1,6\}} = \begin{bmatrix}
1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 & v_1^6 \\
1 & v_2 & v_2^2 & v_2^3 & v_2^4 & v_2^5 & v_2^6 \\
0 & v_2 & 2v_2^2 & 3v_2^3 & 4v_2^4 & 5v_2^5 & 6v_2^6 \\
0 & v_2 & 4v_2^2 & 9v_2^3 & 16v_2^4 & 25v_2^5 & 36v_2^6 \\
0 & v_2 & 8v_2^2 & 27v_2^3 & 64v_2^4 & 125v_2^5 & 216v_2^6 \\
0 & v_2 & 16v_2^2 & 81v_2^3 & 256v_2^4 & 625v_2^5 & 1296v_2^6 \\
0 & v_2 & 32v_2^2 & 243v_2^3 & 1024v_2^4 & 3125v_2^5 & 7776v_2^6
\end{bmatrix},
$$

$$
L_7 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{3v_2-v_1}{v_2-v_1} & 1 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{7v_2-v_1}{v_2-v_1} & \frac{6v_2-3v_1}{v_2-v_1} & 1 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{15v_2-v_1}{v_2-v_1} & \frac{25v_2-7v_1}{v_2-v_1} & \frac{10v_2-6v_1}{v_2-v_1} & 1 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{31v_2-v_1}{v_2-v_1} & \frac{90v_2-15v_1}{v_2-v_1} & \frac{65v_2-25v_1}{v_2-v_1} & \frac{15v_2-10v_1}{v_2-v_1} & 1
\end{bmatrix},
$$

and

$$U_7=\begin{bmatrix}
1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^6 & v_1^5 \\
0 & v_2-v_1 & (v_2-v_1)(v_2+v_1) & (v_2-v_1)(v_2^2+v_2v_1+v_1^2) & (v_2-v_1)(v_2^3+v_2^2v_1+v_2v_1^2+v_1^3) & (v_2-v_1)(v_2^5+v_2^4v_1+v_2^3v_1^2+v_2^2v_1^3+v_2v_1^4+v_1^5) & (v_2-v_1)(v_2^4+v_2^3v_1+v_2^2v_1^2+v_2v_1^3+v_1^4) \\
0 & 0 & v_2(v_2-v_1) & v_2(v_2-v_1)(2v_2+v_1) & v_2(v_2-v_1)(3v_2^2+2v_2v_1+v_1^2) & v_2(v_2-v_1)(5v_2^4+4v_2^3v_1+3v_2^2v_1^2+2v_2v_1^3+v_1^4) & v_2(v_2-v_1)(4v_2^3+3v_2^2v_1+2v_2v_1^2+v_1^3) \\
0 & 0 & 0 & 2v_2^2(v_2-v_1) & 2v_2^2(v_2-v_1)(3v_2+v_1) & 2v_2^2(v_2-v_1)(10v_2^3+6v_2^2v_1+3v_2v_1^2+v_1^3) & 2v_2^2(v_2-v_1)(6v_2^2+3v_2v_1+v_1^2) \\
0 & 0 & 0 & 0 & 6v_2^3(v_2-v_1) & 6v_2^3(v_2-v_1)(10v_2^2+4v_2v_1+v_1^2) & 6v_2^3(v_2-v_1)(4v_2+v_1) \\
0 & 0 & 0 & 0 & 0 & 24v_2^4(v_2-v_1)(5v_2+v_1) & 24v_2^4(v_2-v_1) \\
0 & 0 & 0 & 0 & 0 & 120v_2^5(v_2-v_1) & 0
\end{bmatrix}.$$

3. Factorization of $\mathbf{V_{\{2;1,n-1\}}}$ Into 1-Banded (bidiagonal) Matrices

Motivated by previous results by several authors (see [3,7,10]), we now formulate our second main result on the 1-banded factorizations of the special generalized Vandermonde matrix $V_{\{2;1,n-1\}}$.

Theorem 3.1. *$V_{\{2;1,n-1\}}$ can be factorized into $n-2$ 1-lower banded matrices and $n-1$ 1-upper banded matrices such that*

$$\forall\, n\geq 3,\ V_{\{2;1,n-1\}}=L_n^{(1)}L_n^{(2)}\cdots L_n^{(n-2)}U_n^{(n-1)}U_n^{(n-2)}\cdots U_n^{(1)},$$

where $\forall\ 1\leq l\leq n-3$,

$$L_n^{(l)}(i,j)=\begin{cases}1, & i=j;\\ j-(n-l-1), & i=j+1, n-l\leq j\leq n-1;\\ 0, & otherwise.\end{cases}$$

$$L_n^{(n-2)}(i,j)=\begin{cases}1, & i=j;\\ 1, & i=2, j=1;\\ \frac{(j-1)v_2}{v_2-v_1}, & i=j+1, j\geq 2;\\ 0 & otherwise.\end{cases}$$

and

$$U_2^{(1)}=\begin{bmatrix}1 & v_1\\ 0 & v_2-v_1\end{bmatrix},$$

$$\forall\, 1 \le l \le n-2,\ U_n^{(l)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_{n-1}^{(l)} \end{bmatrix},$$

where $\mathbf{0}$ *denotes the appropriate zero row vector,*

$$U_n^{(n-1)}(i,j) = \begin{cases} 1, & i=j=1; \\ v_1, & i=1, j=2; \\ v_2 - v_1, & i=j=2; \\ (j-2)v_2, & i=j\ge 3; \\ v_2, & i=j-1, j\ge 3; \\ 0, & otherwise. \end{cases}$$

Hence

$$L_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-2)},$$
$$U_n = U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)}.$$

Before proving Theorem 3.1, we need the following lemma:

Lemma. *For* $n \ge 4$*,*

$$L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-3)} = \widetilde{L}_n,$$

where

$$\widetilde{L}_n(i,j) = \begin{cases} 1, & i=j; \\ a_{i-1,j-1}, & i>j\ge 3; \\ 0, & otherwise. \end{cases},$$

where $a_{i,j}$ *are defined as in Theorem 2.1.*

Proof. We use mathematical induction on n.

(1) The case $n=4$:

$$L_4^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{3,2} & 1 \end{bmatrix} = \widetilde{L}_4.$$

(2) The case $k \Rightarrow k+1$ with $k > 4$:

Assume $L_k^{(1)} L_k^{(2)} \cdots L_k^{(k-3)} = \widetilde{L}_k$ holds, we want to prove $L_{k+1}^{(1)} L_{k+1}^{(2)} \cdots L_{k+1}^{(k-2)} = \widetilde{L}_{k+1}$. Observe that

$$\widetilde{L}'_{k+1} := L_{k+1}^{(1)} L_{k+1}^{(2)} \cdots L_{k+1}^{(k-2)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & \widetilde{L}_k \end{bmatrix} L_{k+1}^{(k-2)},$$

since the matrices $L_n^{(l)}, 1 \leq l \leq n-3$, in Theorem 3.1 are all of the form $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & M \end{bmatrix}$.

To show $\widetilde{L}'_{k+1} = \widetilde{L}_{k+1}$, we calculate $\forall\, i > j \geq 3$,

$$\widetilde{L}'_{k+1}(i,j) = \widetilde{L}_k(i-1,j-1)L_{k+1}^{(k-2)}(j,j) + \widetilde{L}_k(i-1,j)L_{k+1}^{(k-2)}(j+1,j)$$

$$= a_{i-2,j-2} \times 1 + a_{i-2,j-1} \times (j-2) = a_{i-1,j-1},$$

i.e.

$$\widetilde{L}'_{k+1}(i,j) = \begin{cases} 1, & i = j; \\ a_{i-1,j-1}, & i > j \geq 3; \\ 0, & otherwise. \end{cases},$$

yielding $\widetilde{L}'_{k+1} = \widetilde{L}_{k+1}$.

Thus by induction, we complete the proof of this lemma.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. First, we show that $L_n = L_n^{(1)}L_n^{(2)}\cdots L_n^{(n-3)}L_n^{(n-2)}$. Let $L'_n = L_n^{(1)}L_n^{(2)}\cdots L_n^{(n-3)}L_n^{(n-2)} = \widetilde{L}_n L_n^{(n-2)}$(by Lemma), then $\forall\, i \geq 3$,

$$L'_n(i,2) = \widetilde{L}_n(i,2)L_n^{(n-2)}(2,2) + \widetilde{L}_n(i,3)L_n^{(n-2)}(3,2) = \frac{v_2}{v_2 - v_1},$$

and $\forall\, i \geq 3,\ j \geq 2,\ i > j$,

$$L'_n(i,j) = \widetilde{L}_n(i,j)L_n^{(n-2)}(j,j) + \widetilde{L}_n(i,j+1)L_n^{(n-2)}(j+1,j) = \frac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1},$$

so

$$L'_n(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ 1, & i = 2, j = 1; \\ 0, & j = 1, i \geq 3; \\ \dfrac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1}, & i \geq 3, j \geq 2, i > j. \end{cases},$$

thus $L'_n = L_n$.

Next, we claim that $U_n = U_n^{(n-1)}U_n^{(n-2)}\cdots U_n^{(1)}$; we use mathematical induction on n.

(1) The case $n = 2$:

$$U_2^{(1)} = \begin{bmatrix} 1 & v_1 \\ 0 & v_2 - v_1 \end{bmatrix} = U_2.$$

(2) The case $k \Rightarrow k+1$ with $k > 4$:

Assume $U_k = U_k^{(k-1)} U_k^{(k-2)} \cdots U_k^{(1)}$ holds, we claim that $U_{k+1} = U_{k+1}^{(k)} U_{k+1}^{(k-1)} \cdots U_{k+1}^{(1)}$.

Observe that

$$\begin{aligned}
U'_{k+1} &:= U_{k+1}^{(k)} U_{k+1}^{(k-1)} \cdots U_{k+1}^{(1)} \\
&= U_{(k+1)}^{(k)} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(k-1)} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(k-2)} \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(1)} \end{bmatrix} \\
&= U_{(k+1)}^{(k)} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k \end{bmatrix},
\end{aligned}$$

by definition and inductive hypothesis. To prove $U'_{k+1} = U_{k+1}$, we calculate $\forall\ k+1 \geq j \geq i \geq 3$,

$$\begin{aligned}
U'_{k+1}(i,j) &= U_{k+1}^{(k)}(i,i) U_k(i-1, j-1) + U_{k+1}^{(k)}(i, i+1) U_k(i, j-1) \\
&= (i-2) v_2 (i-3)! v_2^{i-3} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-3+m}{i-3} v_2^m v_1^{(j-i)-m}] \\
&+ v_2 (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i-1} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i-1)-m}] \\
&= (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-3+m}{i-3} v_2^m v_1^{(j-i)-m} + \sum_{g=1}^{j-i} \tbinom{i-3+g}{i-2} v_2^g v_1^{(j-i)-g}] \\
&= (i-2)! v_2^{i-2} (v_2 - v_1) [v_1^{j-i} + \sum_{m=1}^{j-i} \tbinom{i-3+m}{m} (\frac{i-2+m}{i-2}) v_2^m v_1^{(j-i)-m}] \\
&= (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}],
\end{aligned}$$

hence

$$U'_{k+1}(i,j) = \begin{cases} 0, & i > j; \\ 1, & i = j = 1; \\ v_1^{j-1}, & i = 1, j \geq 2; \\ (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}], & j \geq i \geq 2, \end{cases},$$

thus $U'_{k+1} = U_{k+1}$.

So by induction, we conclude that $U_n = U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)}$, and the proof of Theorem 3.1. is completed.

Example. For $n = 5$, we have $V_{\{2;1,4\}} = L_5 U_5$, then $L_5 = L_5^{(1)} L_5^{(2)} L_5^{(3)}$, where

$$L_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \; L_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix},$$

$$L_5^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2 - v_1} & 1 & 0 & 0 \\ 0 & 0 & \frac{2v_2}{v_2 - v_1} & 1 & 0 \\ 0 & 0 & 0 & \frac{3v_2}{v_2 - v_1} & 1 \end{bmatrix}.$$

And $U_5 = U_5^{(4)} U_5^{(3)} U_5^{(2)} U_5^{(1)}$, where

$$U_5^{(4)} = \begin{bmatrix} 1 & v_1 & 0 & 0 & 0 \\ 0 & v_2 - v_1 & v_2 & 0 & 0 \\ 0 & 0 & v_2 & v_2 & 0 \\ 0 & 0 & 0 & 2v_2 & v_2 \\ 0 & 0 & 0 & 0 & 3v_2 \end{bmatrix}, \; U_5^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & v_1 & 0 & 0 \\ 0 & 0 & v_2 - v_1 & v_2 & 0 \\ 0 & 0 & 0 & v_2 & v_2 \\ 0 & 0 & 0 & 0 & 2v_2 \end{bmatrix},$$

$$U_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & v_1 & 0 \\ 0 & 0 & 0 & v_2 - v_1 & v_2 \\ 0 & 0 & 0 & 0 & v_2 \end{bmatrix}, \; U_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 0 & v_2 - v_1 \end{bmatrix}.$$

4. Application to the Closed-form Formula of $V_{\{2;1,n-1\}}^{-1}$

As a first application, the calculation of the determinant of $V_{\{2;1,n-1\}}$ is obvious, by Theorem 2.1: It is equal to the product of the entries of the main diagonal of U_n which is equal to the product of the products of the main diagonals of $U_n^{(l)}, 1 \leq l \leq n-1$, and this coincides with the corresponding formula (1.2). Furthermore, using our result of 1-banded factorizations we can get, as an application of Theorem 3.1, the closed-form formula of the inverse of $V_{\{2;1,n-1\}}$. This is feasible, since the explicit formula of the inverse of a tridiagonal matrix had been calculated in the literature (see e.g. a recent result [4], P.713, by M. El-Mikkawy and A. Karawia). For convenience's sake we reproduce the result on the inversion of a general tridiagonal matrix in the following: Let $T = (t_{ij})_{1 \leq i,j \leq n}$ be a general tridiagonal matrix, i.e.

$$T = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & \cdots & 0 \\ b_2 & d_2 & a_2 & \cdots & \cdots & 0 \\ 0 & b_3 & d_3 & a_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & b_n & d_n \end{bmatrix}$$

in which $t_{ij} = 0$ for $|i-j| \geq 2$, the inverse matrix $T^{-1} = (u_{ij})_{1 \leq i,j \leq n}$ of the matrix T is given by:

$$\begin{aligned} u_{11} &= (d_1 - \frac{b_2 a_1 \beta_3}{\beta_2})^{-1}, \\ u_{nn} &= (d_n - \frac{b_n a_{n-1} \alpha_{n-2}}{\alpha_{n-1}})^{-1}, \\ u_{ii} &= (d_i - \frac{b_i a_{i-1} \alpha_{i-2}}{\alpha_{i-1}} - \frac{b_{i+1} a_i \beta_{i+2}}{\beta_{i+1}})^{-1}, i = 2, 3, \cdots, n-1, \\ u_{ij} &= \begin{cases} (-1)^{j-i} (\prod_{k=1}^{j-i} a_{j-k}) \frac{\alpha_{i-1}}{\alpha_{j-1}} u_{jj}, & i < j; \\ (-1)^{i-j} (\prod_{k=1}^{i-j} b_{j+k}) \frac{\beta_{i+1}}{\beta_{j+1}} u_{jj}, & i > j. \end{cases}, \end{aligned}$$

where

$$\alpha_i = \begin{cases} 1, & i = 0; \\ d_1, & i = 1; \\ d_i \alpha_{i-1} - b_i a_{i-1} \alpha_{i-2}, & i = 2, 3, \cdots, n. \end{cases},$$

and

$$\beta_i = \begin{cases} 1, & i = n+1; \\ d_n, & i = n; \\ d_i \beta_{i+1} - b_{i+1} a_i \beta_{i+2}, & i = n-1, n-2, \cdots, 1. \end{cases}.$$

Now let us state our result in the following where the proof is based on direct calculations and omitted.

Theorem 4.1. $\forall\, n \geq 3$,

$$V_{\{2;1,n-1\}}^{-1} = (U_n^{(1)})^{-1} (U_n^{(2)})^{-1} \cdots (U_n^{(n-1)})^{-1} (L_n^{(n-2)})^{-1} (L_n^{(n-3)})^{-1} \cdots (L_n^{(1)})^{-1},$$

where

$$(U_2^{(1)})^{-1} = \begin{bmatrix} 1 & -\frac{v_1}{v_2 - v_1} \\ 0 & \frac{1}{v_2 - v_1} \end{bmatrix},$$

$$\forall\, 1 \le l \le n-2,\ (U_n^{(l)})^{-1} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & (U_{n-1}^{(l)})^{-1} \end{bmatrix},$$

where $\mathbf{0}$ denotes the appropriate zero row vector,

$$(U_n^{(n-1)})^{-1}(i,j) = \begin{cases} 0, & i > j; \\ 1, & i = j = 1; \\ (-1)^{j-1}\dfrac{v_1}{(j-2)!(v_2-v_1)}, & i = 1, j \ge 2; \\ (-1)^{j-2}\dfrac{1}{(j-2)!(v_2-v_1)}, & i = 2, j \ge 2; \\ (-1)^{j-i}\dfrac{(i-3)!}{(j-2)!v_2}, & i \ge 3, i \le j. \end{cases}$$

and

$$(L_n^{(n-2)})^{-1}(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ -1, & i = 2, j = 1, ; \\ (-1)^{i-1}(i-2)!\frac{v_2^{i-2}}{(v_2-v_1)^{i-2}}, & i \ge 3, j = 1; \\ (-1)^{i-j}\dfrac{(i-2)!}{(j-2)!}\dfrac{v_2^{i-j}}{(v_2-v_1)^{i-j}}, & i > j \ge 2, i \ge 3. \end{cases}$$

$\forall\, 1 \le l \le n-3$,

$$(L_n^{(l)})^{-1}(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ 0, & j < n-l; \\ (-1)^{i-j}\dfrac{(i-n+l)!}{(j-n+l)!}, & j \ge n-l. \end{cases}$$

Example. For $n = 5$, in Example in Section 3, we have $V_{\{2;1,4\}}^{-1} = U_5^{-1}L_5^{-1}$, then $U_5^{-1} = (U_5^{(1)})^{-1}(U_5^{(2)})^{-1}(U_5^{(3)})^{-1}(U_5^{(4)})^{-1}$ and $L_5^{-1} = (L_5^{(3)})^{-1}(L_5^{(2)})^{-1}(L_5^{(1)})^{-1}$. By Theorem 4.1, we obtain

$$(U_5^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} \end{bmatrix}, \; (U_5^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & \frac{1}{v_2} \end{bmatrix},$$

$$(U_5^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} \\ 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} \\ 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} \\ 0 & 0 & 0 & 0 & \frac{1}{2v_2} \end{bmatrix},$$

$$(U_5^{(4)})^{-1} = \begin{bmatrix} 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} \\ 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} \\ 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} \\ 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} \\ 0 & 0 & 0 & 0 & \frac{1}{3v_2} \end{bmatrix},$$

$$(L_5^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \frac{v_2}{v_2-v_1} & -\frac{v_2}{v_2-v_1} & 1 & 0 & 0 \\ -\frac{2v_2^2}{(v_2-v_1)^2} & \frac{2v_2^2}{(v_2-v_1)^2} & -\frac{2v_2}{v_2-v_1} & 1 & 0 \\ \frac{6v_2^3}{(v_2-v_1)^3} & -\frac{6v_2^3}{(v_2-v_1)^3} & \frac{6v_2^2}{(v_2-v_1)^2} & -\frac{3v_2}{v_2-v_1} & 1 \end{bmatrix},$$

$$(L_5^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix}, \; (L_5^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

In conclusion, this example showed that our Theorem 4.1 provided us with the explicit inversion formula for $V_{\{2;1,n-1\}}$.

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Hsuan-Chu Li
Department of Mathematical Sciences,
National Chengchi University,
Taipei 11623, Taiwan
and
Center of General Education,
Jen-Teh Junior College of Medicine,
Nursing and Management,
Miaoli, Taiwan
E-mail: k0401001@ms4.kntech.com.tw

Eng-Tjioe Tan*
Department of Mathematical Sciences,
National Chengchi University,
Taipei 11623, Taiwan
E-mail: engt@nccu.edu.tw