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WEIGHTED COMPOSITION OPERATORS BETWEEN H^∞ AND α-BLOCH SPACES IN THE UNIT BALL

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Abstract. The boundedness and compactness of the weighted composition operator between H^∞ and the α-Bloch space $\mathcal{B}^\alpha$ on the unit ball are discussed in this paper.

1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, and let $d\nu$ denote the normalized Lebesgue area measure on the unit ball B such that $\nu(B) = 1$. Let $H(B)$ denote the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(B)$ the space of all bounded holomorphic functions on the unit ball.

For a holomorphic function f we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right).$$

For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known that

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z),$$

see, for example [12].

Let $\alpha > 0$. The α-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all holomorphic functions f on B such that

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$$b_\alpha(f) = \sup_{z\in B}(1-|z|^2)^\alpha\,|\Re f(z)| < \infty.$$

It is clear that $\mathcal{B}^\alpha$ is a normed space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$. It is well known (see, for example [12]) that $f \in \mathcal{B}^\alpha(B)$ if and only if

$$a_\alpha(f) = \sup_{z\in B}(1-|z|^2)^\alpha\,|\nabla f(z)| < \infty. \tag{1}$$

Moreover, in [1] it was shown that the quantities

$$\|f\|_{\mathcal{B}^\alpha} \quad \text{and} \quad |f(0)| + a_\alpha(f)$$

are equivalent.

Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which

$$(1-|z|^2)^\alpha|\Re f(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

This space is called the little α-Bloch space.

Let ψ be a holomorphic function on the open unit ball. Define a linear operator ψC_φ on $H(B)$, called weighted composition operator, by

$$(\psi C_\varphi f)(z) = \psi(z)\cdot(f\circ\varphi)(z), \tag{2}$$

where $f \in H(B)$. We can regard this operator as a generalization of a multiplication operator and a composition operator. It is interesting to provide a function theoretic characterization when ψ and φ induce a bounded or compact weighted composition operator on various spaces. The book [3] contains much information on this topic.

In [4], Ohno has characterized the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space $\mathcal{B}$ on the unit disk. In Theorem 1 of [4], Ohno gave the following Proposition: The operator $\psi C_\varphi : \mathcal{B} \to H^\infty$ is compact if and only if $\psi \in H^\infty$ and for every sequence $(z_n)_{n\in\mathbb{N}}$ in the unit disk U such that $\lim_{n\to\infty}|\varphi(z_n)| = 1$, $\lim_{n\to\infty}\psi(z_n) = 0$. However, in [5] we showed that this result is in fact wrong.

In the setting of the unit ball, some necessary and sufficient conditions for a composition operator to be compact on the Bloch space and the little Bloch space are given in [8]. In the setting of the unit polydisk, we have given some necessary and sufficient conditions for a weighted composition operator to be bounded or compact from H^∞ to the Bloch space in [5] (see, also papers [2] and [10]).

In this paper, we study the boundedness and compactness of the weighted composition operator between H^∞ and the α-Bloch space $\mathcal{B}^\alpha$, generalizing some results of [4]. Moreover, our method shows how one may improve Theorem 1 of [4].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then we say that $a \asymp b$.

2. The Boundedness and Compactness of $\psi C_\varphi : H^\infty \to \mathcal{B}^\alpha$

In this section, we will discuss the boundedness and compactness of weighted composition operators $\psi C_\varphi : \mathcal{B}^\alpha$ (*or* $\mathcal{B}_0^\alpha$) $\to H^\infty$.

The following lemma was proven in [9].

Lemma 1. *Let* $f \in \mathcal{B}^\alpha(B), 0 < \alpha < \infty$. *Then*

$$|f(z)| \le C \begin{cases} ||f||_{\mathcal{B}^\alpha} & , \quad \alpha \in (0,1) \\ ||f||_{\mathcal{B}^\alpha} \ln \dfrac{2}{1-|z|^2} & , \quad \alpha = 1 \\ \dfrac{||f||_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & , \quad \alpha > 1 \end{cases} ,$$

for some $C > 0$ *independent of* f.

The next lemma can be proved in a standard way (see, for example, Theorem 3.11 in [3]).

Lemma 2. *Let* X *and* Y *be* $\mathcal{B}^\alpha$ *or* H^∞. *Then the operator* $\psi C_\varphi : X \to Y$ *is compact if and only if* $\psi C_\varphi : X \to Y$ *is bounded and for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* X *which converges to zero uniformly on compact subsets of* B, $\psi C_\varphi f_k \to 0$ *in* Y *as* $k \to \infty$.

The next lemma which follows is standard, but we will give a proof for the benefit of the reader.

Lemma 3. *If* $f \in H^\infty$, *then there exists a constant* C *such that* $\|f\|_{\mathcal{B}} \le C\|f\|_\infty$.

Proof. By Proposition 3.1.3 of [7], we have

$$f(z) = \int_B \frac{f(w)}{(1-\langle z,w\rangle)^{n+1}} d\nu(w).$$

From this and by [7, Proposition 1.4.10], we have that

$$\begin{aligned} |\Re f(z)| &= \left| \int_B \frac{(n+1)f(w)\langle z,w\rangle}{(1-\langle z,w\rangle)^{n+2}} d\nu(w) \right| \\ &\le C \int_B \frac{\|f\|_\infty}{|1-\langle z,w\rangle|^{n+2}} d\nu(w) \le C \frac{\|f\|_\infty}{1-|z|^2}. \end{aligned}$$

From this and since $|f(0)| \le \|f\|_\infty$, we can obtain

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in B} |\Re f(z)|(1-|z|^2) \le C\|f\|_\infty.$$

Theorem 1. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. Then the following statements are equivalent:*

(1) $\psi C_\varphi : \mathcal{B}_0 \to H^\infty$ *is a bounded operator;*

(2) $\psi C_\varphi : \mathcal{B} \to H^\infty$ *is a bounded operator;*

(3)

$$K := \sup_{z\in B} |\psi(z)| \ln \frac{2}{1-|\varphi(z)|^2} < \infty. \tag{3}$$

(4) *Moreover, if $\psi C_\varphi : \mathcal{B} \to H^\infty$ is bounded, then*

$$\|\psi C_\varphi\|_{\mathcal{B}\to H^\infty} \asymp \sup_{z\in B} |\psi(z)| \ln \frac{2}{1-|\varphi(z)|^2}. \tag{4}$$

Proof. *(2)* $\Rightarrow$ *(1)* is obvious.

(1) $\Rightarrow$ *(3)*. Suppose $\psi C_\varphi : \mathcal{B}_0 \to H^\infty$ is a bounded operator. For $\lambda \in B$, put

$$f(z) = \ln \frac{2}{1-\langle z, \varphi(\lambda)\rangle}. \tag{5}$$

Since $f(0) = \ln 2$ and

$$\begin{aligned}(1-|z|^2)|\Re f(z)| &\le (1-|z|^2)\,|\nabla f(z)| = (1-|z|^2)\left|\frac{\varphi(\lambda)}{1-\langle z, \varphi(\lambda)\rangle}\right| \\ &\le \frac{(1-|z|^2)}{|1-\langle z, \varphi(\lambda)\rangle|} \le 2,\end{aligned}$$

we get that $||f||_{\mathcal{B}} \le 2 + \ln 2$.

On the other hand, we have

$$(1-|z|^2)|\Re f(z)| \le \frac{(1-|z|^2)}{|1-\langle z, \varphi(\lambda)\rangle|} \le \frac{(1-|z|^2)}{1-|\varphi(\lambda)|} \to 0,$$

as $|z| \to 1$, hence $f \in \mathcal{B}_0$. Thus

$$(2+\ln 2)\|\psi C_\varphi\|_{\mathcal{B}\to H^\infty} \ge \|\psi C_\varphi f\|_\infty = \sup_{z\in B} |\psi(z) f(\varphi(z))| \ge |\psi(\lambda)| \ln \frac{2}{1-|\varphi(\lambda)|^2}.$$

Therefore

$$\sup_{z\in B} |\psi(z)| \ln \frac{2}{1-|\varphi(z)|^2} \le (2+\ln 2)\|\psi C_\varphi\|_{\mathcal{B}\to H^\infty} < \infty. \tag{6}$$

(3) $\Rightarrow$ *(2)*. Assume that (3) holds. For any $f \in \mathcal{B}$, by Lemma 1, we have that

$$
\begin{aligned}
|(\psi C_\varphi f)(z)| &= |\psi(z)||f(\varphi(z))| \\
&\leq C|\psi(z)| \ln \frac{2}{1-|\varphi(z)|^2} \|f\|_{\mathcal{B}} \leq CK\|f\|_{\mathcal{B}},
\end{aligned} \tag{7}
$$

for any $z \in B$. Taking the supremum in (7) over $z \in B$, it follows that

$$\|\psi C_\varphi f\|_\infty \leq CK\|f\|_{\mathcal{B}}. \tag{8}$$

Thus $\psi C_\varphi : \mathcal{B} \to H^\infty$ is bounded. By (8), we get

$$\|\psi C_\varphi\|_{\mathcal{B}\to H^\infty} = \sup_{\|f\|_{\mathcal{B}}\leq 1} \|\psi C_\varphi f\|_\infty \leq \sup_{\|f\|_{\mathcal{B}}\leq 1} CK\|f\|_{\mathcal{B}} \leq CK. \tag{9}$$

Combining (6) and (9), we obtain (4). This completes the proof of the theorem.

Theorem 2. *Let $\alpha \in (0,1)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. Then, the following statements are equivalent:*

(1) $\psi C_\varphi : \mathcal{B}^\alpha_0 \to H^\infty$ *is a bounded operator;*
(2) $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ *is a bounded operator;*
(3) $\psi \in H^\infty$.

Moreover, if $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is bounded, then

$$\|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty} \asymp \|\psi\|_\infty. \tag{10}$$

Proof. *(2)* $\Rightarrow$ *(1)* is obvious.

(1) $\Rightarrow$ *(3)*. Suppose $\psi C_\varphi : \mathcal{B}^\alpha_0 \to H^\infty$ is bounded. Choose $f(z) = 1$, then $f \in \mathcal{B}^\alpha_0$ and $\|f\|_{\mathcal{B}^\alpha} \leq 1$. Thus

$$\|\psi\|_\infty = \|\psi C_\varphi f\|_\infty \leq \|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty}\|f\|_{\mathcal{B}^\alpha} \leq \|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty}. \tag{11}$$

Hence $\psi \in H^\infty$.

(3) $\Rightarrow$ *(2)*. Suppose $\psi \in H^\infty$, then for any $f \in \mathcal{B}^\alpha$, by Lemma 1, we have

$$|(\psi C_\varphi f)(z)| = |\psi(z)||f(\varphi(z))| \leq C|\psi(z)|\|f\|_{\mathcal{B}^\alpha} \leq C\|\psi\|_\infty\|f\|_{\mathcal{B}^\alpha}, \tag{12}$$

where C depends only on α. Taking the supremum in (12) over B we obtain

$$\|\psi C_\varphi f\|_\infty \leq C\|\psi\|_\infty\|f\|_{\mathcal{B}^\alpha},$$

from which the boundedness of $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ follows.

From this and (11), we get

$$\|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty} \asymp \|\psi\|_\infty,$$

finishing the proof of the theorem.

Theorem 3. *Let $\alpha > 1$, $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. Then, the following statements are equivalent:*

(1) *$\psi C_\varphi : \mathcal{B}_0^\alpha \to H^\infty$ is a bounded operator;*

(2) *$\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is a bounded operator;*

(3)

$$M_1 := \sup_{z\in B} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty. \tag{13}$$

Furthermore, if $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is bounded, then

$$\|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty} \asymp \sup_{z\in B} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}. \tag{14}$$

Proof. *(2) ⇒ (1)* is obvious.

(1) ⇒ (3). Suppose $\psi C_\varphi : \mathcal{B}_0^\alpha \to H^\infty$ is bounded. For $\lambda \in B$, let

$$f(z) = \frac{1}{(1-\langle z, \varphi(\lambda)\rangle)^{\alpha-1}}.$$

It is clear that $f \in \mathcal{B}^\alpha$ and that $\|f\|_{\mathcal{B}^\alpha} \le 2^\alpha(\alpha-1)+1$. Moreover,

$$(1-|z|^2)^\alpha|\nabla f(z)| \le 2^\alpha(\alpha-1)\frac{(1-|z|^2)^\alpha}{(1-|\varphi(\lambda)|)^\alpha} \to 0$$

as $z \to \partial B$. This implies that $f \in \mathcal{B}_0^\alpha$. Similar to the proof of "*(1) ⇒ (3)*" in Theorem 1, we have

$$\begin{aligned}\sup_{z\in B} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} &\le \sup_{z\in B} |\psi(z) f(\varphi(z))| \\ &\le (2^\alpha(\alpha-1)+1)\|\psi C_\varphi\|_{\mathcal{B}^\alpha\to H^\infty} < \infty,\end{aligned} \tag{15}$$

hence (13) holds.

(3) ⇒ (2). Assume that (13) holds. Then, by Lemma 1, for every $f \in \mathcal{B}^\alpha$ and $z \in B$, we obtain

$$\begin{aligned}|(\psi C_\varphi f)(z)| &= |\psi(z)||f(\varphi(z))| \\ &\le C|\psi(z)|(1-|\varphi(z)|^2)^{1-\alpha}\|f\|_{\mathcal{B}^\alpha} \le CM_1\|f\|_{\mathcal{B}^\alpha},\end{aligned}$$

and consequently

$$\|\psi C_\varphi f\|_\infty \le CM_1\|f\|_{\mathcal{B}^\alpha}. \tag{16}$$

Thus $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is bounded.

Similar to the proof of Theorem 1, combining (15) and (16), we have

$$\|\psi C_\varphi\|_{\mathcal{B}^\alpha \to H^\infty} \asymp \sup_{z\in B} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

Next, we will discuss the compactness of the operator $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ or $\psi C_\varphi : \mathcal{B}_0^\alpha \to H^\infty$.

Theorem 4. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. Then, the following statements are equivalent:*

(1) $\psi C_\varphi : \mathcal{B}_0 \to H^\infty$ *is a compact operator;*

(2) $\psi C_\varphi : \mathcal{B} \to H^\infty$ *is a compact operator;*

(3) $\psi \in H^\infty$ *and*

$$\lim_{|\varphi(z)|\to 1} |\psi(z)| \ln \frac{2}{1-|\varphi(z)|^2} = 0. \tag{17}$$

Proof. *(2)* $\Rightarrow$ *(1)* is obvious.

(1) $\Rightarrow$ *(3)*. Suppose $\psi C_\varphi : \mathcal{B}_0 \to H^\infty$ is compact. We have that $\psi = \psi C_\varphi 1 \in H^\infty$. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Let

$$g_k(z) = \left[\ln \frac{2}{1-|\varphi(z_k)|^2}\right]^{-1} \left[\ln \frac{2}{1-\langle z, \varphi(z_k)\rangle}\right]^2. \tag{18}$$

For any $z \in B$,

$$\Re g_k(z) = 2\left[\ln \frac{2}{1-|\varphi(z_k)|^2}\right]^{-1} \left(\ln \frac{2}{1-\langle z, \varphi(z_k)\rangle}\right) \frac{\langle z, \varphi(z_k)\rangle}{1-\langle z, \varphi(z_k)\rangle}.$$

Thus for any $z \in B$,

$$\begin{aligned}(1-|z|^2)|\Re g_k(z)| &\le 2(1-|z|^2)\left|\frac{\ln \frac{2}{1-\langle z,\varphi(z_k)\rangle}}{\ln \frac{2}{1-|\varphi(z_k)|^2}}\right| \frac{1}{1-|z|} \\ &\le 4\frac{C + \ln \frac{2}{1-|\varphi(z_k)|}}{\ln \frac{2}{1-|\varphi(z_k)|^2}} \le C\end{aligned}$$

On the other hand,

$$|g_k(0)| \le \left(\ln\frac{2}{1-|\varphi(z_k)|^2}\right)^{-1}(\ln 2)^2 \le \ln 2.$$

Thus $\|g_k\|_{\mathcal{B}} \le M$, where M is a constant independent of k. It is obvious that $g_k \in H(\overline{B})$, thus $g_k \in \mathcal{B}_0$ for every $k \in \mathbb{N}$. Since for $|z| = r < 1$, we have

$$|g_k(z)| = \frac{\left|\ln\frac{2}{1-\langle z,\varphi(z_k)\rangle}\right|^2}{\ln\frac{2}{1-|\varphi(z_k)|^2}} \le \frac{\left(\ln\frac{2}{1-r}+C\right)^2}{\ln\frac{2}{1-|\varphi(z_k)|^2}} \to 0 \quad (k\to\infty),$$

that is, $g_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. Since ψC_φ is compact, by Lemma 2, we have $\lim_{k\to\infty}\|\psi C_\varphi g_k\|_\infty = 0$. On the other hand, the following estimate holds

$$\|\psi C_\varphi g_k\|_\infty = \sup_{z\in B}|\psi(z)||g_k(\varphi(z))| \ge |\psi(z_k)|\ln\frac{2}{1-|\varphi(z_k)|^2}.$$

Thus

$$\lim_{k\to\infty}|\psi(z_k)|\ln\frac{2}{1-|\varphi(z_k)|^2} = 0,$$

which implies (17).

(3)⇒(2). Suppose $\psi \in H^\infty$ and condition (17) hold, then it is easy to see that

$$\sup_{z\in B}|\psi(z)|\ln\frac{2}{1-|\varphi(z)|^2} < \infty.$$

By Theorem 1, $\psi C_\varphi : \mathcal{B} \to H^\infty$ is bounded. Assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence and $f_k \to 0$ uniformly on compact subsets of B. Denote $K = \sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}}$. For any $\epsilon > 0$, by (17), there exists a $\delta \in (0,1)$ such that if $\delta < |\varphi(z)| < 1$,

$$|\psi(z)|\ln\frac{2}{1-|\varphi(z)|^2} < \epsilon.$$

Thus if $|\varphi(z)| > \delta$, for every $k \in \mathbb{N}$, we have

$$|\psi(z)||f_k(\varphi(z))| \le C\|f_k\|_{\mathcal{B}}|\psi(z)|\ln\frac{2}{1-|\varphi(z)|^2} \le CK\epsilon. \tag{19}$$

On the other hand, since $f_k \to 0$ uniformly on the compact $\{w : |w| \le \delta\}$ as $k \to \infty$, there exists a k_0 such that $|f_k(\varphi(z))| < \epsilon$ if $|\varphi(z)| \le \delta$ and $k \ge k_0$. Hence for $|\varphi(z)| \le \delta$ and $k \ge k_0$, we have

$$|\psi(z)||f_k(\varphi(z))| \le \|\psi\|_\infty\epsilon.$$

This and (19) imply that $\lim_{k\to\infty} \|\psi C_\varphi f_k\|_\infty = 0$. By Lemma 2, it follows that $\psi C_\varphi : \mathcal{B} \to H^\infty$ is a compact operator, as desired.

Theorem 5. *Let $\alpha > 1$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. Then, the following statements are equivalent:*

(1) *$\psi C_\varphi : \mathcal{B}_0^\alpha \to H^\infty$ is a compact operator;*

(2) *$\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is a compact operator;*

(3) *$\psi \in H^\infty$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0. \tag{20}$$

Proof. *(2)⇒(1)* is obvious.

(3)⇒(2) Assume that $\psi \in H^\infty$ and condition (20) holds, then

$$\sup_{z\in B} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty.$$

By Theorem 3, $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ is bounded. Now assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence and $f_k \to 0$ uniformly on compact subsets of B. Denote $K_1 = \sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha}$. From (20) we have that, for every $\epsilon > 0$, there is a $\delta \in (0, 1)$ such that if $\delta < |\varphi(z)| < 1$,

$$\frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < \epsilon.$$

This shows that if $|\varphi(z)| > \delta$, for any $k \in \mathbb{N}$, we have

$$|\psi(z)||f_k(\varphi(z))| \le C\|f_k\|_{\mathcal{B}^\alpha} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \le CK_1\epsilon. \tag{21}$$

The rest of the proof is similar to the corresponding proof of Theorem 4 and will be omitted.

(1)⇒(3). Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of B such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Choose

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\langle z, \varphi(z_k)\rangle)^\alpha}.$$

It is easy to see that $f_k \in \mathcal{B}_0^\alpha$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} \le C$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. Since ψC_φ is compact, by Lemma 2, we have $\lim_{k\to\infty} \|\psi C_\varphi f_k\|_\infty = 0$. From this and since

$$\|\psi C_\varphi f_k\|_\infty = \sup_{z\in B} |\psi(z)||f_k(\varphi(z))| \ge \frac{|\psi(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha-1}},$$

we obtain

$$\lim_{k\to\infty} \frac{|\psi(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha-1}} = 0,$$

which implies (20).

Similar to the proof of Theorem 4, the following theorem can be obtained.

Theorem 6. *Let* $0 < \alpha < 1$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ *be a holomorphic self-map of* B, $\psi \in H^\infty(B)$ *and* $\lim_{|\varphi(z)|\to 1} |\psi(z)| = 0$. *Then,* $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ *is a compact operator.*

Remark 1. Note that if $\|\varphi\|_\infty < 1$, then similar to the proof of Theorem 5, it can be proved that the compactness of the operator $\psi C_\varphi : \mathcal{B}^\alpha \to H^\infty$ implies that $\lim_{|\varphi(z)|\to 1} |\psi(z)| = 0$. However, if $\|\varphi\|_\infty = 1$, we do not know, at the moment, if this is true.

3. The Boundedness and Compactness of $\psi C_\varphi : H^\infty \to \mathcal{B}^\alpha$

In this section, we characterize the boundedness and compactness of the operator $\psi C_\varphi : H^\infty \to \mathcal{B}^\alpha$. For simplicity of notation, we restrict ourselves to the case of $\alpha = 1$. We will begin by introducing some preliminary notation.

Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B, denote

$$D\varphi(z) = \begin{pmatrix} \frac{\partial \varphi_1(z)}{\partial z_1} & \cdots & \frac{\partial \varphi_1(z)}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial \varphi_n(z)}{\partial z_1} & \cdots & \frac{\partial \varphi_n(z)}{\partial z_n} \end{pmatrix}$$

and $D\varphi(z)^T$ be the transpose of the matrix $D\varphi(z)$ (see [11]). Here

$$|D\varphi(z)| = \left(\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l(z)}{\partial z_k} \right|^2 \right)^{1/2}.$$

Theorem 7. *Let* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *be a holomorphic self-map of* B *and* $\psi \in H(B)$. *If*

(a) $\psi \in \mathcal{B}$

(b)

$$\sup_{z\in B} \frac{(1-|z|^2)}{1-|\varphi(z)|^2} |\psi(z)|\, |D\varphi(z)| < \infty, \tag{22}$$

then, $\psi C_\varphi : H^\infty \to \mathcal{B}$ *is bounded.*

Conversely, if $\psi C_\varphi : H^\infty \to \mathcal{B}$ *is bounded, then*

(c) $\psi \in \mathcal{B}$

(d)

$$\sup_{z\in B} \frac{|\psi(z)|(1-|z|^2)}{1-|\varphi(z)|^2}|D\varphi(z)^T\overline{\varphi(z)^T}| < \infty. \tag{23}$$

Proof. Suppose that (a) and (b) hold. For a function $f \in H^\infty(B)$, we have

$$\begin{aligned}
&|\nabla(\psi C_\varphi f)|(1-|z|^2)\\
\le\ & (1-|z|^2)|\nabla\psi(z)||f(\varphi(z))| + |\psi(z)||\nabla(f\circ\varphi)(z))|(1-|z|^2)\\
=\ & (1-|z|^2)|\nabla\psi(z)||f(\varphi(z))| + |\psi(z)|(1-|z|^2)\Big(\sum_{k=1}^n\Big|\sum_{l=1}^n \frac{\partial f}{\partial\zeta_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\Big|^2\Big)^{1/2}\\
\le\ & (1-|z|^2)|\nabla\psi(z)||f(\varphi(z))|\\
&+|\psi(z)|(1-|z|^2)\Big(\sum_{k=1}^n\sum_{l=1}^n\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|^2\Big)^{1/2}\Big(\sum_{l=1}^n\Big|\frac{\partial f}{\partial\zeta_l}(\varphi(z))\Big|^2\Big)^{1/2}\\
\le\ & (1-|z|^2)|\nabla\psi(z)||f(\varphi(z))| + |\psi(z)|(1-|z|^2)|D\varphi(z)|\,|(\nabla f)(\varphi(z))|\\
\le\ & C\|\psi\|_{\mathcal{B}}\|f\|_\infty + C\|f\|_{\mathcal{B}}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\psi(z)||D\varphi(z)|.
\end{aligned}$$

By Lemma 3 we know that $\|f\|_{\mathcal{B}} \le C\|f\|_\infty$ for every $f \in H^\infty(B)$. This along with conditions (a) and (b) show that the operator $\psi C_\varphi : H^\infty(B) \to \mathcal{B}(B)$ is bounded.

Conversely, suppose that $\psi C_\varphi : H^\infty(B) \to \mathcal{B}(B)$ is bounded, i.e. there exists a constant C such that

$$\|\psi C_\varphi f\|_{\mathcal{B}} \le C\|f\|_\infty \tag{24}$$

for all $f \in H^\infty(B)$. Taking $f(z) \equiv 1$ and $f(z) = z_l$, $l = 1, \ldots, n$, it follows that $\psi \in \mathcal{B}(B)$ and $\psi\varphi_l \in \mathcal{B}(B)$.

For fixed $\lambda \in B$, we define the test function

$$f(z) = \frac{1-|\varphi(\lambda)|^2}{1-\langle z, \varphi(\lambda)\rangle}.$$

It is easy to see that $f \in H^\infty(B)$ and $\|f\|_\infty \le 2$. Therefore we have

$$
\begin{aligned}
&2\|\psi C_\varphi\|_{H^\infty\to\mathcal{B}} \ge \|\psi C_\varphi f\|_{\mathcal{B}}\\
&\ge \sup_{z\in B}(1-|z|^2)|\nabla\psi(z)f(\varphi(z))+\psi(z)\nabla(f\circ\varphi)(z))|\\
&\ge (1-|\lambda|^2)\Big|\nabla\psi(\lambda)f(\varphi(\lambda))+\psi(\lambda)\nabla(f\circ\varphi)(\lambda))\Big|\\
&\ge (1-|\lambda|^2)\big|\psi(\lambda)\nabla(f\circ\varphi)(\lambda))\big|-(1-|\lambda|^2)\big|\nabla\psi(\lambda)f(\varphi(\lambda))\big|\\
&=(1-|\lambda|^2)|\psi(\lambda)|\Big(\sum_{k=1}^n\Big|\sum_{l=1}^n\frac{\partial f}{\partial\zeta_l}(\varphi(\lambda))\frac{\partial\varphi_l}{\partial z_k}(\lambda)\Big|^2\Big)^{1/2}-(1-|\lambda|^2)|\nabla\psi(\lambda)|\\
&=(1-|\lambda|^2)|\psi(\lambda)|\Big(\sum_{k=1}^n\Big|\sum_{l=1}^n\frac{\overline{\varphi_l(\lambda)}}{1-|\varphi(\lambda)|^2}\frac{\partial\varphi_l}{\partial z_k}(\lambda)\Big|^2\Big)^{1/2}-(1-|\lambda|^2)|\nabla\psi(\lambda)|\\
&=\frac{|\psi(\lambda)|(1-|\lambda|^2)}{1-|\varphi(\lambda)|^2}|D\varphi(z)^T\overline{\varphi(\lambda)^T}|-|\nabla\psi(\lambda)|(1-|\lambda|^2).
\end{aligned}
\tag{25}
$$

Since $\psi \in \mathcal{B}(B)$, we obtain

$$
\sup_{\lambda\in B}\frac{|\psi(\lambda)|(1-|\lambda|^2)}{1-|\varphi(\lambda)|^2}|D\varphi(\lambda)^T\overline{\varphi(\lambda)^T}|<\infty. \tag{26}
$$

Which completes the proof of the theorem.

Theorem 8. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. If*

(a) $\lim_{|z|\to 1}(1-|z|^2)|\nabla\psi(z)| = 0$;

(b)

$$
\lim_{|z|\to 1}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\psi(z)||D\varphi(z)|=0, \tag{27}
$$

then, $\psi C_\varphi : H^\infty \to \mathcal{B}$ is compact.

Conversely, if $\psi C_\varphi : H^\infty \to \mathcal{B}$ is compact, then

(c) $\lim_{|\varphi(z)|\to 1}(1-|z|^2)|\nabla\psi(z)| = 0$;

(d)

$$
\lim_{|\varphi(z)|\to 1}\frac{|\psi(z)|(1-|z|^2)}{1-|\varphi(z)|^2}|D\varphi(z)^T\overline{\varphi(z)^T}|=0. \tag{28}
$$

Proof. Suppose that conditions (a) and (b) hold. Then it is clear that $\psi C_\varphi : H^\infty \to \mathcal{B}$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in H^∞ such that $\sup_{k\in\mathbb{N}}\|f_k\|_\infty \le$

L and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. By the assumptions, for any $\epsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$(1 - |z|^2)|\nabla\psi(z)| < \epsilon \tag{29}$$

and

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\psi(z)||D\varphi(z)| < \epsilon. \tag{30}$$

whenever $\delta < |z| < 1$. Let $K = \{w \in B : |w| \le \delta\}$. Note that K is a compact subset of B. Then, by employing (29), (30) and Lemma 3, we have that

$$\begin{aligned}
&\|\psi C_\varphi f_k\|_{\mathcal{B}} \\
&= \sup_{z\in B} |\nabla(\psi C_\varphi f_k)|(1 - |z|^2) + |\psi(0) f_k(\varphi(0))| \\
&\le \sup_{z\in B}(1 - |z|^2)|\nabla\psi(z)||f_k(\varphi(z))| \\
&\quad + \sup_{z\in B} |\psi(z)||\nabla(f_k \circ \varphi)(z)|(1 - |z|^2) + |\psi(0) f_k(\varphi(0))| \\
&\le \sup_{z\in K}(1 - |z|^2)|\nabla\psi(z)||f_k(\varphi(z))| + \sup_{\delta<|z|<1} (1 - |z|^2)|\nabla\psi(z)||f_k(\varphi(z))| \\
&\quad + \sup_{z\in K}(1 - |\varphi(z)|^2)|\nabla f_k(\varphi(z))|\frac{(1 - |z|^2)}{1 - |\varphi(z)|^2}|\psi(z)||D\varphi(z)| \\
&\quad + \sup_{\delta<|z|<1}(1-|\varphi(z)|^2)|\nabla f_k(\varphi(z))|\frac{(1-|z|^2)}{1-|\varphi(z)|^2}|\psi(z)||D\varphi(z)|+|\psi(0) f_k(\varphi(0))| \\
&\le \sup_{w\in\varphi(K)} |f_k(w)|\|\psi\|_{\mathcal{B}} + \sup_{w\in\varphi(K)} M(1 - |w|^2)|\nabla f_k(w)| \\
&\quad +|\psi(0) f_k(\varphi(0))| + C\epsilon,
\end{aligned} \tag{31}$$

where

$$M = \sup_{z\in B} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\psi(z)||D\varphi(z)|.$$

Note that M is finite in view of (27). Cauchy's estimate gives that $|\nabla f_k(w)| \to 0$ as $k \to \infty$ on compacta, in particular on $\varphi(K)$. Hence, letting $k \to \infty$ in (31) we obtain

$$\lim_{k\to\infty} \|\psi C_\varphi f_k\|_{\mathcal{B}} = 0.$$

From this and applying Lemma 2 the result follows.

Now, suppose that $\psi C_\varphi : H^\infty \to \mathcal{B}$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Let

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{1 - \langle z, \varphi(z_k)\rangle}.$$

Then $f_k \in H^\infty$, $\sup_{k\in\mathbb{N}} \|f_k\|_\infty \le 2$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since $\psi C_\varphi : H^\infty \to \mathcal{B}$ is compact, we have

$$\lim_{k\to\infty} \|\psi C_\varphi f_k\|_{\mathcal{B}} = 0.$$

Therefore, similar to the proof of Theorem 7, we obtain

$$\|\psi C_\varphi f_k\|_{\mathcal{B}} \ge \Big| \frac{1-|z_k|^2}{1-|\varphi(z_k)|^2} |\psi(z_k)||D\varphi(z_k)^T \overline{\varphi(z_k)^T}| - (1-|z_k|^2)|\nabla\psi(z_k)| \Big|.$$

Hence

$$\begin{aligned} &\lim_{|\varphi(z_k)|\to 1} (1-|z_k|^2)|\nabla\psi(z_k)| \\ &= \lim_{|\varphi(z_k)|\to 1} \frac{1-|z_k|^2}{1-|\varphi(z_k)|^2} |\psi(z_k)||D\varphi(z_k)^T \overline{\varphi(z_k)^T}|, \end{aligned} \tag{32}$$

if one of these two limits exists.

Next for a sequence $(z_k)_{k\in\mathbb{N}}$ in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$, we take

$$g_k(z) = \frac{1-|\varphi(z_k)|^2}{1-\langle z, \varphi(z_k)\rangle} - \left(\frac{1-|\varphi(z_k)|^2}{1-\langle z, \varphi(z_k)\rangle} \right)^{1/2}, \quad k \in \mathbb{N}.$$

We notice that g_k is a sequence in H^∞ and g_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Note also that $g_k(\varphi(z_k)) = 0$ and

$$\nabla g_k(\varphi(z_k)) = \frac{\overline{\varphi(z_k)}}{2(1-|\varphi(z_k)|^2)}.$$

Similar to (25), we obtain

$$\frac{1-|z_k|^2}{2(1-|\varphi(z_k)|^2)} |\psi(z_k)||D\varphi(z_k)^T \overline{\varphi(z_k)^T}| \le \|\psi C_\varphi g_k\|_{\mathcal{B}} \to 0,$$

as $k \to \infty$. Therefore we get the condition (d) and so by (32), we obtain (c).

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