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ON THE STABILITY OF AN n-DIMENSIONAL FUNCTIONAL EQUATION ORIGINATING FROM QUADRATIC FORMS

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Abstract. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of an n-dimensional functional equation

$$f\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\Big) + \sum_{1\leq i<j\leq n} f(x_i - x_j, y_i - y_j) = n\sum_{i=1}^{n} f(x_i, y_i), \ \ (n \geq 2).$$

1. Introduction

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive functions was solved by Hyers [10] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was

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given by Rassias [20]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 6, 8, 12, 15, 17, 18]). The terminology *generalized Hyers-Ulam-Rassias stability* originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [11, 13, 21].

Quadratic functional equation was used to characterize inner product spaces [1, 2, 14]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is related to a symmetric biadditive function [1, 19]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [1, 19]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}\Big(f(x+y) - f(x-y)\Big). \tag{2}$$

A generalized Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [22]). Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In [7], Czerwik proved the generalized Hyers-Ulam-Rassias stability of the quadratic functional equation (1). Grabiec [9] has generalized these results mentioned above. Jun and Lee [16] proved the generalized Hyers-Ulam-Rassias stability of the Pexiderized quadratic equation (1).

Let E_1 and E_2 be real vector spaces. A mapping $f : E_1 \to E_2$ is called a *quadratic form* if there exist $a, b, c \in \mathbb{R}$ such that

$$f(x,y) = ax^2 + bxy + cy^2$$

for all $x, y \in E_1$.

Bae and Park [3] introduced the following 2-variable quadratic functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w) \tag{3}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for the functional equation (3). In fact, they proved the following theorem:

Theorem 1.1. [3]. *Let X, Y be real vector spaces. A mapping $f : X \times X \to Y$ satisfies* (3) *if and only if there exist two symmetric bi-additive mappings $S_1, S_2 : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x, y) = S_1(x, x) + B(x, y) + S_2(y, y)$$

for all $x, y \in X$.

It is clear that when $X = Y = \mathbb{R}$, the quadratic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := ax^2 + bxy + cy^2$ is a solution of (3). In this case, we have

$$\begin{aligned}
&\sum_{1 \le i < j \le n} f(x_i - x_j, y_i - y_j) \\
&= \frac{1}{2} \sum_{1 \le i, j \le n} f(x_i - x_j, y_i - y_j) \\
&= n \sum_{i=1}^{n} f(x_i, y_i) - \sum_{1 \le i, j \le n} (a x_i x_j + c y_i y_j) - b \sum_{1 \le i, j \le n} x_i y_j
\end{aligned}$$

and

$$f\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\Big) = \sum_{i=1}^{n} f(x_i, y_i) + a \sum_{\substack{1 \le i, j \le n \\ i \ne j}} x_i x_j + b \sum_{\substack{1 \le i, j \le n \\ i \ne j}} x_i y_j + c \sum_{\substack{1 \le i, j \le n \\ i \ne j}} y_i y_j$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. Therefore, f satisfies the functional equation

$$f\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\Big) + \sum_{1 \le i < j \le n} f(x_i - x_j, y_i - y_j) = n \sum_{i=1}^{n} f(x_i, y_i) \quad (n \ge 2).$$

In this paper, we establish the general solution and the generalized Hyers-Ulam-Rassias stability problem for the functional equation

$$f\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\Big) + \sum_{1 \le i < j \le n} f(x_i - x_j, y_i - y_j) = n \sum_{i=1}^{n} f(x_i, y_i) \quad (n \ge 2). \tag{4}$$

2. Solution of Functional Equation (4)

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be real vector spaces. We here present the general solution of (4).

Lemma 2.1 *Let a function $f : E_1 \to E_2$ satisfies the functional equation* (3)*. Then*

(i) $f(0,0) = 0.$

(ii) $f(\lambda x, \lambda y) = \lambda^2 f(x,y)$ *for all* $x, y \in E_1$ *and all* $\lambda \in \mathbb{Q}$*. In particular,* $f(-x,-y) = f(x,y)$ *for all* $x, y \in E_1$*.*

Proof. Putting $x = y = z = w = 0$ in (3), we get $f(0,0) = 0$.

To prove (ii), let $x = y$ and $z = w$ in (3). Then $f(2x, 2y) = 4f(x,y)$ for all $x, y \in E_1$. Letting $x = 2y$ and $z = 2w$ in (3), we obtain that $f(3y, 3w) = 9f(y,w)$ for all $y, w \in E_1$. By induction we lead to $f(mx, my) = m^2 f(x,y)$ for all $x, y \in E_1$ and all nonnegative integers m. Letting $y = 2x$ and $w = 2z$ in (3), we obtain $f(-x,-z) = f(x,z)$ for all $x, z \in E_1$. Let k be a negative integer. Then

$$f(kx, ky) = f(-kx, -ky) = (-k)^2 f(x,y) = k^2 f(x,y)$$

for all $x, y \in E_1$. So

$$f(mx, my) = m^2 f(x,y) \tag{5}$$

for all $x, y \in E_1$ and all $m \in \mathbb{Z}$. Let $m \in \mathbb{Z} \setminus \{0\}$. Replacing x and y by $\frac{x}{m}$ and $\frac{y}{m}$ in (5), respectively, we get

$$f(\frac{x}{m}, \frac{y}{m}) = \frac{1}{m^2} f(x,y) \tag{6}$$

for all $x, y \in E_1$. Therefore,

$$f(\lambda x, \lambda y) = \lambda^2 f(x,y) \tag{7}$$

for all $x, y \in E_1$ and all $\lambda \in \mathbb{Q}$. ■

Lemma 2.2 *A function $f : E_1 \to E_2$ satisfies the functional equation* (3) *if and only if $f : E_1 \to E_2$ satisfies the functional equation*

$$f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) + \sum_{1 \le i < j \le 3} f(x_i - x_j, y_i - y_j) = 3 \sum_{i=1}^{3} f(x_i, y_i) \tag{8}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$*.*

Proof. Let

$$f(x_1 + x_2, y_1 + y_2) + f(x_1 - x_2, y_1 - y_2) = 2f(x_1, y_1) + 2f(x_2, y_2) \tag{9}$$

for all $x_1, x_2, y_1, y_2 \in E_1$. Replacing x_2 and y_2 by $x_2 + x_3$ and $y_2 + y_3$ in (9), respectively, we get that

$$\begin{aligned} f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+ f(x_1 - x_2 - x_3, y_1 - y_2 - y_3) \\ &= 2f(x_1, y_1) + 2f(x_2 + x_3, y_2 + y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. Hence we have

$$\begin{aligned} f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+ f(x_1 - x_2 - x_3, y_1 - y_2 - y_3) \\ &\quad + f(x_1 - x_2 + x_3, y_1 - y_2 + y_3) \\ &= 2f(x_1, y_1) + 2f(x_2 + x_3, y_2 + y_3) \\ &\quad + f(x_1 - x_2 + x_3, y_1 - y_2 + y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. It follows from (9) that

$$\begin{aligned} f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+ 2f(x_1 - x_2, y_1 - y_2) \\ &\quad + 2f(x_2 - x_3, y_2 - y_3) + 2f(x_3, y_3) \\ &= 2f(x_1, y_1) + f(x_1 - x_2 + x_3, y_1 - y_2 + y_3) \\ &\quad + 2\Big[f(x_2 + x_3, y_2 + y_3) + f(x_2 - x_3, y_2 - y_3)\Big] \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. Once again using (9), we infer that

$$\begin{aligned} f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+ 2f(x_1 - x_2, y_1 - y_2) + 2f(x_2 - x_3, y_2 - y_3) \\ &= 2f(x_1, y_1) + 4f(x_2, y_2) + 2f(x_3, y_3) \\ &\quad + f(x_1 - x_2 + x_3, y_1 - y_2 + y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. Therefore, we have

$$\begin{aligned} 2f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+ 2f(x_1 - x_2, y_1 - y_2) + 2f(x_2 - x_3, y_2 - y_3) \\ &= 2f(x_1, y_1) + 4f(x_2, y_2) + 2f(x_3, y_3) \\ &\quad + f(x_1 - x_2 + x_3, y_1 - y_2 + y_3) \\ &\quad + f(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. Once again (9) implies that

$$\begin{aligned} 2f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+2f(x_1 - x_2, y_1 - y_2) + 2f(x_2 - x_3, y_2 - y_3) \\ &= 2f(x_1, y_1) + 6f(x_2, y_2) + 2f(x_3, y_3) \\ &\quad +2f(x_1 + x_3, y_1 + y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. By using (9), we have

$$\begin{aligned} 2f\Big(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\Big) &+2f(x_1 - x_2, y_1 - y_2) \\ &\quad +2f(x_1 - x_3, y_1 - y_3) + 2f(x_2 - x_3, y_2 - y_3) \\ &= 2f(x_1 + x_3, y_1 + y_3) + 2f(x_1 - x_3, y_1 - y_3) \\ &\quad +2f(x_1, y_1) + 6f(x_2, y_2) + 2f(x_3, y_3) \\ &= 6f(x_1, y_1) + 6f(x_2, y_2) + 6f(x_3, y_3) \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in E_1$. It proves that f satisfies the functional equation (8).

The converse is proved in the following theorem. ■

Theorem 2.3. *A function $f : E_1 \to E_2$ satisfies the functional equation* (3) *if and only if $f : E_1 \to E_2$ satisfies the functional equation* (4).

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (4). Putting $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = 0$ in (4), we get

$$f(0,0) + \frac{n^2 - n}{2} f(0,0) = n^2 f(0,0).$$

Therefore, $f(0,0) = 0$. Now putting $x_3 = \cdots = x_n = 0$ and $y_3 = \cdots = y_n = 0$ in (4), we infer that

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) + f(x_1 - x_2, y_1 - y_2) + (n-2)f(x_1, y_1) + (n-2)f(x_2, y_2) \\ = nf(x_1, y_1) + nf(x_2, y_2) \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in E_1$. Therefore, f satisfies the functional equation (3).

Conversely, let $n \geq 2$ and let f satisfy the functional equation (3). We use induction on n to prove (4). Our main assertion is true for $n = 2$. Now we make the induction hypothesis that (4) is true for some $n \geq 2$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in$

E_1. Let $x_{n+1}, y_{n+1} \in E_1$. Replacing x_n and y_n by $x_n + x_{n+1}$ and $y_n + y_{n+1}$ in (4), respectively, we get that

$$\begin{aligned} &f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + \sum_{1 \le i < j \le n-1} f(x_i - x_j, y_i - y_j) \\ &+ \sum_{i=1}^{n-1} f(x_i - x_n - x_{n+1}, y_i - y_n - y_{n+1}) \\ = &\, n \sum_{i=1}^{n-1} f(x_i, y_i) + n f(x_n + x_{n+1}, y_n + y_{n+1}) \end{aligned}$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Hence we have

$$\begin{aligned} &f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + \sum_{1 \le i < j \le n-1} f(x_i - x_j, y_i - y_j) \\ &+ \sum_{i=1}^{n-1} \Big[f(x_i - x_n - x_{n+1}, y_i - y_n - y_{n+1}) + f(x_i - x_n + x_{n+1}, y_i - y_n + y_{n+1})\Big] \\ = &\, n \sum_{i=1}^{n-1} f(x_i, y_i) + \sum_{i=1}^{n-1} f(x_i - x_n + x_{n+1}, y_i - y_n + y_{n+1}) \\ &+ n f(x_n + x_{n+1}, y_n + y_{n+1}) \end{aligned}$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. It follows from (3) that

$$\begin{aligned} &f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + \sum_{1 \le i < j \le n-1} f(x_i - x_j, y_i - y_j) \\ &+ 2 \sum_{i=1}^{n-1} \Big[f(x_i - x_n, y_i - y_n) + f(x_{n+1}, y_{n+1})\Big] \\ = &\, n \sum_{i=1}^{n-1} f(x_i, y_i) + \sum_{i=1}^{n-1} f(x_i - x_n + x_{n+1}, y_i - y_n + y_{n+1}) \\ &+ n f(x_n + x_{n+1}, y_n + y_{n+1}) \end{aligned}$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Therefore,

$$
\begin{aligned}
& f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} \Big[f(x_i - x_n, y_i - y_n) + f(x_{n+1}, y_{n+1})\Big] \\
& + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1}) \\
= {} & n\sum_{i=1}^{n-1} f(x_i, y_i) + \sum_{i=1}^{n-1} f(x_i - x_n + x_{n+1}, y_i - y_n + y_{n+1}) \\
& + nf(x_n + x_{n+1}, y_n + y_{n+1}) + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1})
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Once again using (3), we get

$$
\begin{aligned}
& f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} \Big[f(x_i - x_n, y_i - y_n) + f(x_{n+1}, y_{n+1})\Big] \\
& + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1}) \\
= {} & n\sum_{i=1}^{n-1} f(x_i, y_i) + nf(x_n + x_{n+1}, y_n + y_{n+1}) \\
& + 2\sum_{i=1}^{n-1} \Big[f(x_i + x_{n+1}, y_i + y_{n+1}) + f(x_n, y_n)\Big]
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Hence we have

$$
\begin{aligned}
& f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) \\
& + 2(n-1)f(x_{n+1}, y_{n+1}) + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) \\
& + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1}) + 2\sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1}) \\
= {} & n\sum_{i=1}^{n-1} f(x_i, y_i) + nf(x_n + x_{n+1}, y_n + y_{n+1}) + 2(n-1)f(x_n, y_n) \\
& + 2\sum_{i=1}^{n-1} \Big[f(x_i + x_{n+1}, y_i + y_{n+1}) + f(x_i - x_{n+1}, y_i - y_{n+1})\Big]
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. By simple computation, it follows from (3) that

$$\begin{aligned}
&f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) \\
&\quad + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1}) + 2\sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1}) \\
&= (n+4)\sum_{i=1}^{n-1} f(x_i, y_i) + nf(x_n + x_{n+1}, y_n + y_{n+1}) \\
&\quad + 2(n-1)f(x_n, y_n) + 2(n-1)f(x_{n+1}, y_{n+1})
\end{aligned}$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Hence we have

$$\begin{aligned}
&f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) \\
&\quad + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) + 2\sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1}) \\
&\quad + \sum_{i=1}^{n-1} f(x_i + x_n + x_{n+1}, y_i + y_n + y_{n+1}) + nf(x_n - x_{n+1}, y_n - y_{n+1}) \\
&= (n+4)\sum_{i=1}^{n-1} f(x_i, y_i) + 2(n-1)f(x_n, y_n) + 2(n-1)f(x_{n+1}, y_{n+1}) \\
&\quad + n\Big[f(x_n + x_{n+1}, y_n + y_{n+1}) + f(x_n - x_{n+1}, y_n - y_{n+1})\Big]
\end{aligned}$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Using (3) and (8) to get that

$$\begin{aligned}
&f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + 2\sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) \\
&\quad + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) + 2\sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1}) \\
&\quad + 3\sum_{i=1}^{n-1} f(x_i, y_i) + 3(n-1)f(x_n, y_n) + 3(n-1)f(x_{n+1}, y_{n+1}) \\
&\quad - \sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) - \sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&- (n-1)f(x_n - x_{n+1}, y_n - y_{n+1}) + nf(x_n - x_{n+1}, y_n - y_{n+1}) \\
&= (n+4)\sum_{i=1}^{n-1} f(x_i, y_i) + (4n-2)f(x_n, y_n) + (4n-2)f(x_{n+1}, y_{n+1})
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. Therefore, we have

$$
\begin{aligned}
&f\Big(\sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i\Big) + \sum_{i=1}^{n-1} f(x_i - x_n, y_i - y_n) + \sum_{1\le i<j\le n-1} f(x_i - x_j, y_i - y_j) \\
&+ \sum_{i=1}^{n-1} f(x_i - x_{n+1}, y_i - y_{n+1}) + f(x_n - x_{n+1}, y_n - y_{n+1}) \\
&= (n+1)\sum_{i=1}^{n+1} f(x_i, y_i)
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in E_1$. The last equation means (4). ■

Corollary 2.4. *Let $2 \le m < n$. A function $f : E_1 \to E_2$ satisfies the functional equation (4) if and only if $f : E_1 \to E_2$ satisfies the functional equation*

$$
f\Big(\sum_{i=1}^{m} x_i, \sum_{i=1}^{m} y_i\Big) + \sum_{1\le i<j\le m} f(x_i - x_j, y_i - y_j) = m\sum_{i=1}^{n} f(x_i, y_i) \tag{10}
$$

for all $x_1, \ldots, x_m, y_1, \ldots, y_m \in E_1$.

3. Generalized Hyers-ulam-rassias Stability of Equation (4)

From now on, let X and Y be a real vector space and a real Banach space, respectively. In this section, using an idea of Găvruta [8] we prove the stability of Eq. (4) in the spirit of Hyers, Ulam and Rassias. Thus we find the condition that there exists a true 2-variable quadratic mapping near a approximately 2-variable quadratic mapping. For convenience, for a given function $f : X \times X \to Y$ we use the following abbreviation:

$$
\begin{aligned}
&Df(x_1, \ldots, x_n, y_1, \ldots, y_n) \\
&= f\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\Big) + \sum_{1\le i<j\le n} f(x_i - x_j, y_i - y_j) - n\sum_{i=1}^{n} f(x_i, y_i)
\end{aligned}
$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.

Theorem 3.1. *Let $\varphi : X^{2n} \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x,y) := \sum_{i=1}^{\infty} n^{2i}\varphi\Big(\underbrace{\frac{x}{n^i},\ldots,\frac{x}{n^i}}_{n \text{ times}},\underbrace{\frac{y}{n^i},\ldots,\frac{y}{n^i}}_{n \text{ times}}\Big) < \infty \tag{11}$$

and

$$\lim_{n\to\infty} n^{2i}\varphi\Big(\frac{x_1}{n^i},\ldots,\frac{x_n}{n^i},\frac{y_1}{n^i},\ldots,\frac{y_n}{n^i}\Big) = 0 \tag{12}$$

for all $x, y, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Suppose that a function $f : X \times X \to Y$ satisfies the inequality

$$\|Df(x_1,\ldots,x_n,y_1,\ldots,y_n)\| \le \varphi(x_1,\ldots,x_n,y_1,\ldots,y_n) \tag{13}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Then there exists a unique 2-variable quadratic mapping $T : X \times X \to Y$ such that

$$\|T(x,y) - f(x,y)\| \le \frac{1}{n^2}\widetilde{\varphi}(x,y) \tag{14}$$

for all $x, y \in X$. The mapping T is given by

$$T(x,y) = \lim_{k\to\infty} n^{2k} f\Big(\frac{x}{n^k},\frac{y}{n^k}\Big) \tag{15}$$

for all $x, y \in X$.

Proof. It follows from (11) that $\varphi(0,\ldots,0) = 0$, and therefore (13) implies that $f(0,0) = 0$.

Letting $x_1 = \cdots = x_n = x$ and $y_1 = \cdots = y_n = y$ in (13), we have

$$\|f(nx,ny) - n^2 f(x,y)\| \le \varphi(\underbrace{x,\ldots,x}_{n \text{ times}},\underbrace{y,\ldots,y}_{n \text{ times}}) \tag{16}$$

for all $x, y \in X$. Replacing x by $\frac{x}{n}$ and y by $\frac{y}{n}$ in (16), respectively, we get

$$\|f(x,y) - n^2 f(\frac{x}{n},\frac{y}{n})\| \le \varphi(\underbrace{\frac{x}{n},\ldots,\frac{x}{n}}_{n \text{ times}},\underbrace{\frac{y}{n},\ldots,\frac{y}{n}}_{n \text{ times}}) \tag{17}$$

for all $x, y \in X$. Replacing x by $\frac{x}{n}$ and y by $\frac{y}{n}$ in (17), respectively, and multiplying by n^2, we get

$$\|n^2 f(\frac{x}{n},\frac{y}{n}) - n^4 f(\frac{x}{n^2},\frac{y}{n^2}))\| \le n^2\varphi(\underbrace{\frac{x}{n^2},\ldots,\frac{x}{n^2}}_{n \text{ times}},\underbrace{\frac{y}{n^2},\ldots,\frac{y}{n^2}}_{n \text{ times}})$$

for all $x, y \in X$. Using the induction on m, we obtain that

$$\left\|n^{2m+2}f(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}) - n^{2m}f(\frac{x}{n^m}, \frac{y}{n^m})\right\| \leq n^{2m}\varphi(\underbrace{\frac{x}{n^{m+1}}, \dots, \frac{x}{n^{m+1}}}_{n \text{ times}}, \underbrace{\frac{y}{n^{m+1}}, \dots, \frac{y}{n^{m+1}}}_{n \text{ times}}) \tag{18}$$

for all $x, y \in X$. Therefore, we have

$$\begin{aligned} &\left\|\sum_{i=k}^{m}\left(n^{2i+2}f\left(\frac{x}{n^{i+1}}, \frac{y}{n^{i+1}}\right) - n^{2i}f\left(\frac{x}{n^i}, \frac{y}{n^i}\right)\right)\right\| \\ &\leq \sum_{i=k}^{m}\left\|n^{2i+2}f(\frac{x}{n^{i+1}}, \frac{y}{n^{i+1}}) - n^{2i}f(\frac{x}{n^i}, \frac{y}{n^i})\right\| \\ &\leq \sum_{i=k}^{m} n^{2i}\varphi(\underbrace{\frac{x}{n^{i+1}}, \dots, \frac{x}{n^{i+1}}}_{n \text{ times}}, \underbrace{\frac{y}{n^{i+1}}, \dots, \frac{y}{n^{i+1}}}_{n \text{ times}}) \end{aligned}$$

for all $x, y \in X$ and all integers $m \geq k \geq 0$. Hence

$$\left\|n^{2m+2}f(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}) - n^{2k}f(\frac{x}{n^k}, \frac{y}{n^k})\right\| \leq \sum_{i=k}^{m} n^{2i}\varphi(\underbrace{\frac{x}{n^{i+1}}, \dots, \frac{x}{n^{i+1}}}_{n \text{ times}}, \underbrace{\frac{y}{n^{i+1}}, \dots, \frac{y}{n^{i+1}}}_{n \text{ times}}) \tag{19}$$

for all $x, y \in X$ and all integers $m \geq k \geq 0$. It follows from (11) and (19) that the sequence $\{n^{2k}f(\frac{x}{n^k}, \frac{y}{n^k})\}_k$ is a Cauchy sequence in Y for all $x, y \in X$. Since Y is a Banach space, it follows that the sequence $\{n^{2k}f(\frac{x}{n^k}, \frac{y}{n^k})\}_k$ converges. We define $T : X \times X \to Y$ by

$$T(x, y) = \lim_{k\to\infty} n^{2k}f(\frac{x}{n^k}, \frac{y}{n^k})$$

for all $x, y \in X$. Therefore, it follows from (13) that

$$\begin{aligned} \|DT(x_1, \dots, x_n, y_1, \dots, y_n)\| &= \lim_{k\to\infty} n^{2k}\left\|Df(\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}, \frac{y_1}{n^k}, \dots, \frac{y_n}{n^k})\right\| \\ &\leq \lim_{k\to\infty} n^{2k}\varphi(\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}, \frac{y_1}{n^k}, \dots, \frac{y_n}{n^k}) = 0 \end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Hence by Theorem 2.3, T is a 2-variable quadratic mapping.

Putting $k = 0$ and letting $m \to \infty$ in (19), we get (14).

It remains to show that T is unique. Suppose that there exists another 2-variable quadratic mapping $Q : X \times X \to Y$ which satisfies (14). Since $n^{2k}Q(\frac{x}{n^k}, \frac{y}{n^k}) = Q(x, y)$ for all $x, y \in X$, we conclude that

$$\|Q(x,y) - T(x,y)\| = \lim_{k\to\infty} n^{2k}\Big\|Q(\frac{x}{n^k}, \frac{y}{n^k}) - f(\frac{x}{n^k}, \frac{y}{n^k})\Big\|$$

$$= \lim_{k\to\infty} n^{2k-2}\widetilde{\varphi}(\frac{x}{n^k}, \frac{y}{n^k}) = 0$$

for all $x, y \in X$. Hence we have $Q(x, y) = T(x, y)$ for all $x, y \in X$ which proves the uniqueness of T. ■

Corollary 3.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \times X \to Y$ be mapping such that*

$$\|Df(x_1, \dots, x_n, y_1, \dots, y_n)\| \le \theta \sum_{i=1}^{n} (\|x_i\|^r + \|y_i\|^r) \tag{20}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Then there exists a unique 2-variable quadratic mapping $T : X \times X \to Y$ such that

$$\|T(x,y) - f(x,y)\| \le \frac{n\theta}{n^r - n^2}(\|x\|^r + \|y\|^r) \tag{21}$$

for all $x, y \in X$.

Theorem 3.3. *Let $\varphi : X^{2n} \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x,y) := \sum_{i=0}^{\infty} \frac{1}{n^{2i}} \varphi(\underbrace{n^i x, \dots, n^i x}_{n \text{ times}}, \underbrace{n^i y, \dots, n^i y}_{n \text{ times}}) < \infty \tag{22}$$

and

$$\lim_{n\to\infty} \frac{1}{n^{2i}} \varphi(n^i x_1, \dots, n^i x_n, n^i y_1, \dots, n^i y_n) = 0 \tag{23}$$

for all $x, y, x_1, \dots, x_n, y_1, \dots, y_n \in X$. Suppose that a mapping $f : X \times X \to Y$ satisfies the inequality

$$\|Df(x_1, \dots, x_n, y_1, \dots, y_n)\| \le \varphi(x_1, \dots, x_n, y_1, \dots, y_n) \tag{24}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Then there exists a unique 2-variable quadratic mapping $T : X \times X \to Y$ such that

$$\Big\|T(x,y) - f(x,y) + \frac{n}{2n+2} f(0,0)\Big\| \le \frac{1}{n^2}\widetilde{\varphi}(x,y) \tag{25}$$

for all $x, y \in X$. The mapping T is given by

$$T(x,y) = \lim_{k\to\infty} \frac{1}{n^{2k}} f(n^k x, n^k y) \tag{26}$$

for all $x, y \in X$.

Proof. Letting $x_1 = \cdots = x_n = x$ and $y_1 = \cdots = y_n = y$ in (24), we have

$$\left\| \frac{1}{n^2} f(nx, ny) - f(x,y) + \frac{\alpha}{n^2} f(0,0) \right\| \le \frac{1}{n^2}\varphi(\underbrace{x, \ldots, x}_{n \text{ times}}, \underbrace{y, \ldots, y}_{n \text{ times}}) \tag{27}$$

for all $x, y \in X$ where $\alpha = \frac{n^2-n}{2}$. Replacing x by nx and y by ny in (27), respectively, and dividing by n^2, we get

$$\begin{aligned} &\left\| \frac{1}{n^4} f(n^2x, n^2y) - \frac{1}{n^2} f(nx, ny) + \frac{\alpha}{n^4} f(0,0) \right\| \\ &\le \frac{1}{n^4}\varphi(\underbrace{nx, \ldots, nx}_{n \text{ times}}, \underbrace{ny, \ldots, ny}_{n \text{ times}}) \end{aligned} \tag{28}$$

for all $x, y \in X$. Using the induction on $m \ge 0$, we obtain that

$$\begin{aligned} &\left\| \frac{1}{n^{2m+2}} f(n^{m+1}x, n^{m+1}y) - \frac{1}{n^{2m}} f(n^m x, n^m y) + \frac{\alpha}{n^{2m+2}} f(0,0) \right\| \\ &\le \frac{1}{n^{2m+2}}\varphi(\underbrace{n^m x, \ldots, n^m x}_{n \text{ times}}, \underbrace{n^m y, \ldots, n^m y}_{n \text{ times}}) \end{aligned} \tag{29}$$

for all $x, y \in X$. Therefore, we have

$$\begin{aligned} &\left\| \sum_{i=k}^{m} \left[\frac{1}{n^{2i+2}} f(n^{i+1}x, n^{i+1}y) - \frac{1}{n^{2i}} f(n^i x, n^i y) + \frac{\alpha}{n^{2i+2}} f(0,0) \right] \right\| \\ &\le \sum_{i=k}^{m} \left\| \frac{1}{n^{2i+2}} f(n^{i+1}x, n^{i+1}y) - \frac{1}{n^{2i}} f(n^i x, n^i y) + \frac{\alpha}{n^{2i+2}} f(0,0) \right\| \\ &\le \sum_{i=k}^{m} \frac{1}{n^{2i+2}}\varphi(\underbrace{n^i x, \ldots, n^i x}_{n \text{ times}}, \underbrace{n^i y, \ldots, n^i y}_{n \text{ times}}) \end{aligned}$$

for all $x, y \in X$ and all integers $m \ge k \ge 0$. Hence

$$\begin{aligned} &\left\| \frac{1}{n^{2m+2}} f(n^{m+1}x, n^{m+1}y) - \frac{1}{n^{2k}} f(n^k x, n^k y) + \sum_{i=k}^{m} \frac{\alpha}{n^{2i+2}} f(0,0) \right\| \\ &\le \sum_{i=k}^{m} \frac{1}{n^{2i+2}}\varphi(\underbrace{n^i x, \ldots, n^i x}_{n \text{ times}}, \underbrace{n^i y, \ldots, n^i y}_{n \text{ times}}) \end{aligned} \tag{30}$$

for all $x, y \in X$ and all integers $m \geq k \geq 0$. It follows from (22) and (30) that the sequence $\{\frac{1}{n^{2k}} f(n^k x, n^k y)\}_k$ is a Cauchy sequence in Y for all $x, y \in X$. Since Y is a Banach space, it follows that the sequence $\{\frac{1}{n^{2k}} f(n^k x, n^k y)\}_k$ converges. We define $T : X \times X \to Y$ by

$$T(x, y) = \lim_{k\to\infty} \frac{1}{n^{2k}} f(n^k x, n^k y)$$

for all $x, y \in X$. The rest of the proof is similar to the proof of Theorem 3.1. ■

Corollary 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \times X \to Y$ be mapping satisfying* (20). *Then there exists a unique* 2*-variable quadratic mapping $T : X \times X \to Y$ such that*

$$\left\|T(x, y) - f(x, y) + \frac{n}{2n+2} f(0, 0)\right\| \leq \frac{n\theta}{n^2 - n^r}(\|x\|^r + \|y\|^r) \tag{31}$$

for all $x, y \in X$. If $0 < r < 2$ then $f(0, 0) = 0$ and if $r < 0$ the inequality (20) *holds for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X \setminus \{0\}$ and* (31) *for all $x, y \in X \setminus \{0\}$.*

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