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EXISTENCE PROBLEMS IN SECOND ORDER EVOLUTION INCLUSIONS: DISCRETIZATION AND VARIATIONAL APPROACH

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. We prove, under appropriate assumptions, the existence of solutions for a second order evolution inclusion with delay in a separable reflexive Banach space. Several applications are investigated via discretization techniques and variational convergence results.

1. Introduction

In the present paper, we prove, under appropriate assumptions, the existence of solutions for a second order evolution inclusion with delay. Several applications are investigated via discretization techniques and variational convergence results. In Section 3 we present a general existence result for a delayed second order evolution inclusion in a separable reflexive Banach space E such that its strong dual is uniformly convex and its variant involving the normal cone of closed convex moving subsets in a separable Hilbert space. Our techniques provide also new results for second order evolution inclusions. Section 4 is devoted to applications involving variational techniques, biting lemma, the characterization of the second dual of $L^1_{\mathbf{R}^d}$ and Young measures. In particular, a measure solution for a second order evolution inclusion in $\mathbf{R}^d$ of the form

$$0 \in \ddot{u}(t) + Mu(t) + \partial\varphi(u(t))$$

is presented, here M is a linear continuous operator in $\mathbf{R}^d$, φ is a convex proper lower semicontinuous function defined on $\mathbf{R}^d$ and $\partial\varphi(u(t))$ is the subdifferential of the function φ at the point $u(t)$.

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2. Preliminaries and Background

We will use the following definitions and notations and summarize some basic results.

- Let E be a separable reflexive Banach space such that its strong dual is uniformly convex, I_E is the identity mapping on E.
- $\overline{B}_E(0,1)$ is the closed unit ball of E.
- $c(E)$ (resp. $cc(E)$) (resp. $cwk(E)$) is the collection of nonempty closed (resp. closed convex) (resp. weakly compact convex) subsets of E.
- If A is a subset of E, $\delta^*(., A)$ is the support function of A.
- $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue measurable subsets of $[0,T]$.
- If X is a topological space, $\mathcal{B}(X)$ is the Borel tribe of X.
- $L^1_E([0,T], dt)$ (shortly $L^1_E([0,T])$) is the Banach space of Lebesgue-Bochner integrable functions $f : [0,T] \to E$.
- A mapping $u : [0,T] \to E$ is *absolutely continuous* if there is a function $\dot{u} \in L^1_E([0,T])$ such that $u(t) = u(0) + \int_0^t \dot{u}(s)\, ds,\ \forall t \in [0,T]$.
- If X is a topological space, $\mathcal{C}_E(X)$ is the space of continuous mappings $u : X \to E$ equipped with the norm of uniform convergence.
- A set-valued mapping $F : [0,T] \rightrightarrows E$ is measurable if its graph belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(E)$. A convex weakly compact valued mapping $F : X \to cwk(E)$ defined on a topological space X is scalarly upper semicontinuous if for every $y \in E'$, the scalar function $\delta^*(y, F(.))$ is upper semicontinuous on X.
- A multivalued operator $A(t) : E \to 2^E, (t \in [0,T])$ is *m-accretive*, if, for each $t \in [0,T]$ and each $\lambda > 0$, $R(I_E + \lambda A(t)) = E$ and for each $x_1 \in D(A(t)), x_2 \in D(A(t)), y_1 \in A(t)x_1, y_2 \in A(t)x_2$, we have
$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$$
where $D(A(t)) := \{x \in E : A(t)x \neq \emptyset\}$ and j is the single-valued duality mapping in E.

If $A(t)$ is m-accretive, then

$$\begin{aligned} \tfrac{1}{\lambda}||J_\lambda A(t)x - x|| &= ||A_\lambda(t)x|| \leq |A(t)x|_0 \\ &= \inf_{y \in A(t)x} ||y|| \ \forall x \in D(A(t)) \end{aligned} \tag{2.1}$$

where $J_\lambda A(t)x = (I_E + \lambda A(t))^{-1}x$. We refer to [5, 9, 40] for the theory of accretive operators and equations of evolution in Banach spaces. We refer to [23] for measurable multifunctions and Convex Analysis.

3. Some Existence Theorems in Delayed Second Order Evolution Inclusions

We will consider an m-accretive operator $A(t) : E \to 2^E (t \in [0, T])$ with domain $D(A(t))$ in a separable reflexive Banach space satisfying the following assumptions:
(H_1) There exists a continuous function $\rho : [0, T] \to E$ and a nondecreasing function $L : [0, \infty[\to [0, \infty[$ such that

$$||J_\lambda A(t)x - J_\lambda A(s)x|| \leq \lambda|\rho(t) - \rho(s)|L(||x||)$$

for all $\lambda \in]0, 1]$, for all $(t, s) \in [0, T] \times [0, T]$, and for all $x \in E$.
(H_2) $0 \in D(A(0))$ and for each $r > 0$, $\sup_{x \in D(A(0)) \cap \overline{B}_E(0,r)} |A(0)x|_0 < +\infty$.
(H_3)

- (a) For every $L^2_E([0, T])$-mapping $u : [0, T] \to E$ satisfying $u(t) \in D(A(t))$ for all $t \in [0, T]$, the set-valued mapping $t \to A(t)u(t)$ is measurable,
- (b) for every $x \in E$ and for every $\lambda > 0$, $t \mapsto (I_E + \lambda A(t))^{-1}x$ is measurable,
- (c) there exists $\overline{g} \in L^2_E([0, T])$ such that $t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0, T])$ for all $\lambda > 0$.

(H_4) $D(A(t))$ is closed and ball-compact, that is, the intersection of $D(A(t))$ with any closed ball in E is compact.

Remarks.

(1) Assumption (H_1) is similar to the one employed by [9], [28] in the study of quasi-autonomous evolution equations. By ([28], Lemma 3.1), (H_1) implies that the sets $D(A(t))$ are constant, i.e $D(A(t)) := D$ for all $t \in [0, T]$.

(2) If $E = \mathbf{R}^d$ and if $A(t)$ does not depend on $t \in [0, T]$ (i.e. $A(t) = A, t \in [0, T]$), it is obvious that A satisfies (H_1) and (H_3). Indeed, since the graph of A is closed, the set-valued mapping A is Borel, in the sense that for any closed subset B of $\mathbf{R}^d$, the set

$$A^-(B) = \{x \in \mathbf{R}^d : A(x) \cap B \neq \emptyset\} = \text{proj}_{\mathbf{R}^d}[\text{Graph}(A) \cap \mathbf{R}^d \times B]$$

is Borel. Consequently, $D(A)$ is Borel and for any measurable mapping $u : D(A) \to \mathbf{R}^d$, the set-valued mapping $Au(.)$ is measurable, proving $H_3(a)$. Further, for any measurable mapping $g : [0, T] \to \mathbf{R}^d$, the mappings $t \mapsto [I_{\mathbf{R}^d} + \lambda A]^{-1}g(t)$ is measurable, meanwhile $H_3(b)$ and $H_3(c)$ are obvious. We recall first a version of a closure type result in the convergence of approximated solutions. See ([12], Lemma 2.3).

Lemma 3.1. *Suppose that E is a separable reflexive Banach space such that its strong dual is uniformly convex, $A(t) : E \to 2^E (t \in [0,T])$, is an m-accretive operator satisfying (H_3), (u_n) and (v_n) are sequences in $L^2_E([0,T])$ with $u_n(t) \in D(A(t))$ for every n and for every $t \in [0,T]$ and (r_n) is a uniformly bounded sequence of positive measurable functions defined on $[0,T]$ such that $r_n(t) \to 0$ pointwisely on $[0,T]$. Assume that the following conditions are satisfied:*

- (i) *(u_n) converges strongly to $u \in L^2_E([0,T])$ and (v_n) converges to $v \in L^2_E([0,T])$ with respect to the topology $\sigma(L^2_E, L^2_{E'})$,*
- (ii) *$v_n(t) \in A(t)u_n(t) + r_n(t)\overline{B}_E$ for all n and all $t \in [0,T]$.*

Then we have

$$v(t) \in A(t)u(t) \ a.e \ t \in [0,T].$$

Proof. Let $I_{L^2_E([0,T])}$ be the identity operator in $L^2_E([0,T])$. Let $\mathcal{A}$ be the operator in $L^2_E([0,T])$ defined by

$$v \in \mathcal{A}u \Longleftrightarrow v(t) \in A(t)u(t) \ a.e \ t \in [0,T].$$

We claim that $\mathcal{A}$ is m-accretive in $L^2_E([0,T])$. Let $\lambda > 0$ and let $g \in L^2_E([0,T])$. By $(H_3)(c)$ there exists $\overline{g} \in L^2_E([0,T])$ such that $\overline{h} : t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0,T])$. Since $(I_E + \lambda A(t))^{-1}$ is nonexpansive [40], we deduce that the function $h : t \mapsto (I_E + \lambda A(t))^{-1}g(t)$ is measurable and belongs to $L^2_E([0,T])$ thanks to $(H_3)(b)$-(c). Furthermore, we have $g \in h + \lambda\mathcal{A}h \Longleftrightarrow h \in (I_{L^2_E([0,T])} + \lambda\mathcal{A})^{-1}g \Longrightarrow R(I_{L^2_E([0,T])} + \lambda\mathcal{A}) = L^2_E([0,T])$. Let $\mathcal{U}$ be the closed unit ball of $L^\infty_E([0,T])$. In view of (ii), $(H_3)(a)$ and measurable selection theorem, we claim that $v_n \in \mathcal{A}u_n + \mathcal{R}_n$ for all n, where

$$\mathcal{R}_n := \{z \in L^\infty_E([0,T]) : z = r_n w, \ w \in \mathcal{U}\}.$$

Firstly, it is easy to see that $\mathcal{R}_n$ is equal to the set of all measurable selections of the measurable set-valued mapping $r_n(.)\overline{B}_E(0,1)$. Secondly, by (ii) and $(H_3)(a)$, the nonempty set-valued mapping $\Psi_n : [0,T] \to E \times E$ defined by

$$\Psi_n(t) := \{(x,y) \in (A(t)u_n(t), r_n(t)\overline{B}_E(0,1)) : x + y = v_n(t)\}, \ \forall t \in [0,T],$$

is measurable. By measurable selection theorem, there is a measurable selection x_n of $A(.)u_n(.)$ and a measurable selection y_n of the measurable set-valued mapping $r_n(.)\overline{B}_E(0,1)$ such that $v_n(t) = x_n(t) + y_n(t)$ for all $t \in [0,T]$. Moreover there is $w_n \in \mathcal{U}$ such that $y_n = r_n w_n$. So, we have that $v_n = x_n + r_n w_n \in \mathcal{A}u_n + \mathcal{R}_n$ for all n. As $A(t)$ is accretive for each $t \in [0,T]$, it is easy to check that $\mathcal{A}$ is accretive in $L^2_E([0,T])$. Since E' is uniformly convex, the dual $L^2_{E'}([0,T])$ of $L^2_E([0,T])$ is uniformly convex, too, see e.g ([41] , Theorem 4.2 and Remark 4.7). Consequently, by ([40], Theor.1.5.2) the graph of $\mathcal{A}$ is strongly-weakly sequentially closed. By (i) u_n strongly converges to $u \in L^2_H([0,T])$, $v_n - r_n w_n \to v$, weakly in $L^2_H([0,T])$, (because $r_n w_n \to 0$ strongly in $L^2_H([0,T])$ and $v_n - r_n w_n \in \mathcal{A}u_n$ by what has been proved, so we conclude that $v \in \mathcal{A}u \Longleftrightarrow v(t) \in A(t)u(t)$ $a.e$ $t \in [0,T]$. ■

We present first a second order problem for a delayed evolution inclusion governed by an m-accretive operator with convex weakly compact valued upper semicontinuous perturbation. See [12, 13, 18] and the references therein for other delayed evolution inclusions. Let $r > 0$ be a finite delay and let $\mathcal{C}_0 := \mathcal{C}_E([-r,0])$. For any $t \in [0,T]$, let $\tau(t) : \mathcal{C}_E([-r,t]) \to \mathcal{C}_0$ defined by $(\tau(t)u)(s) = u(t+s), \forall s \in [-r,0],\ \forall u \in \mathcal{C}_E([-r,t])$.

Theorem 3.1. *Assume that E is a separable reflexive Banach space such that its strong dual is uniformly convex, $A(t) : E \to cc(E) \cup \{\emptyset\}; t \in [0,T]$, is an m-accretive operator satisfying $(H_1) - (H_4)$, $F : [0,T] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0]) \to cwk(E)$ is separately scalarly measurable on $[0,T]$, separately scalarly upper semicontinuous on $\mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ such that $F(t,x,y) \subset K$ for all $(t,x,y) \in [0,T] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$, for some convex weakly compact set K in E and $G : [0,T] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0]) \to cwk(E)$ is a scalarly upper semicontinuous convex weakly compact valued mapping such that $G(t,x,y) \subset (1 + ||x(0)|| + ||y(0)||)Z$ for all $(t,x,y) \in [0,T] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$, for some convex weakly compact set Z in E. Then, for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) = a \in D(A(0))$ and every $\psi \in \mathcal{C}_0$ with $\psi(0) = b \in E$, there exists two continuous mappings $u : [-r,T] \to E$ and $v : [-r,T] \to E$ such that*

$$(\mathcal{P}_1)\begin{cases} v(t) = \psi(t) \quad \text{on} \quad [-r,0]; \quad v(t) = b + \displaystyle\int_0^t u(s)ds, \forall t \in [0,T], \\ u(t) = \varphi(t) \quad \text{on} \quad [-r,0]; \quad u(t) = a + \displaystyle\int_0^t \dot{u}(s)ds, \forall t \in [0,T], \\ \text{with} \quad \dot{u} \in L^\infty_E([0,T]) \quad \text{and} \quad u(t) \in D(A(t)), \forall t \in [0,T], \\ 0 \in \dot{u}(t) + A(t)u(t) + F(t,\tau(t)u,\tau(t)v) + G(t,\tau(t)u,\tau(t)v) \text{ a.e.} \end{cases}$$

Proof. Our proof follows a technique of discretization developed in [12] for a functional type evolution inclusion, taking account into the trick developed in ([12], Step 2 of Theorem 2.4) involving the multivalued Scorza Dragoni theorem [16] and Multivalued Dugundji extension theorem [7].

Step 1. F is scalarly upper semicontinuous. As G is scalarly upper semicontinuous with

$$G(t,x,y) \subset (1 + ||x(0)|| + ||y(0)||)Z$$

for all $(t,x,y) \in [0,T] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$, for some convex weakly compact set Z in E, $[F+G]$ satisfies obviously the inclusion $[F+G](t,x,y) \subset (1+||x(0)||+$

$|y(0)||)X$ for some convex equilibrated weakly compact set X in E. Consequently, in this particular case, we need to prove our theorem when $F = 0$. Let $h : [0,T]\times\mathcal{C}_E([-r,0])\times\mathcal{C}_E([-r,0])$ be a scalarly $\mathcal{B}([0,T])\times\mathcal{C}_E([-r,0])\times\mathcal{C}_E([-r,0]))$-measurable selection of G (see [23]). We will construct two sequences (v_n), (u_n) in $\mathcal{C}_E([-r,T])$ such that, the associated subsequences converge uniformly to the desired functions v, u satisfying the problem $(\mathcal{P}_1)$. For notational convenience, we take $T = 1$. Let $n \geq 1$ be a fixed integer, we put $u_n(s) = \varphi(s), v_n(s) = \psi(s)$ for all $s \in [-r,0]$ and we consider a partition of $[0,1]$ by the points $t_k^n = ke_n$, $e_n = \frac{1}{n}, k = 0,1,2,...,n.$. For each $t \in [t_0^n, t_1^n]$, we define

$$v_n(t) = b + (t - t_0^n)a$$

$$u_n(t) = \frac{t_1^n - t}{e_n}x_0^n + \frac{t - t_0^n}{e_n}x_1^n.$$

where $b = \psi(0)$, $x_0^n = a = \varphi(0)$ and

$$x_1^n = J_{e_n}A(t_1^n)(x_0^n - e_n h(t_0^n, \tau(t_0^n)u_n, \tau(t_0^n)v_n)$$

so that for $t \in [t_0^n, t_1^n[$

$$\dot{v}_n(t) = x_0^n = a \quad \text{and} \quad \dot{u}_n(t) = \frac{x_1^n - x_0^n}{e_n}.$$

By construction we have $x_1^n \in D(A(t_1^n))$ and for $t \in [t_0^n, t_1^n[$,

$$\dot{u}_n(t) = \frac{x_1^n - x_0^n}{e_n} \in -A(t_1^n)x_1^n - h(t_0^n, \tau(t_0^n)u_n, \tau(t_0^n)v_n). \tag{3.1.1}$$

By induction for $0 \leq k \leq n$, we set

$$x_{k+1}^n = J_{e_n}A(t_{k+1}^n)(x_k^n - e_n h(t_k^n, \tau(t_k^n)u_n, \tau(t_k^n)v_n)$$

then $x_{k+1}^n \in D(A(t_{k+1}^n))$ for $k = 0,1,2,...n-1$ and for $t \in [t_k^n, t_{k+1}^n]$, $0 \leq k \leq n-1$, we define

$$v_n(t) = v_n(t_k^n) + (t - t_k^n)u_n(t_k^n)$$

and

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n}x_k^n + \frac{t - t_k^n}{e_n}x_{k+1}^n.$$

Then for $t \in [t_k^n, t_{k+1}^n[$, we have $\dot{v}_n(t) = u_n(t_k^n)$ and

$$\dot{u}_n(t) = \frac{x_{k+1}^n - x_k^n}{e_n} \in -A(t_{k+1}^n)x_{k+1}^n - h(t_k^n, \tau(t_k^n)u_n, \tau(t_k^n)v_n)). \tag{3.1.2}$$

For each $t \in [0,T]$ and each $n \geq 1$, let $\delta_n(t) = t_k^n,\ \theta_n(t) = t_{k+1}^n,$ if $t \in [t_k^n, t_{k+1}^n[$. So by (3.1.2) we get

$$\dot{u}_n(t) \in -A(\theta_n(t))u_n(\theta_n(t)) - h(\delta_n(t), \tau(\delta_n(t))u_n, \tau(\delta_n(t))v_n), \tag{3.1.3}$$

for a.e. $t \in [0,1]$. It is obvious that, for all $n \geq 1$ and for all $t \in [0,1]$ the following holds:

$$h(\delta_n(t), \tau(\delta_n(t))u_n, \tau(\delta_n(t))v_n) \in G(\delta_n(t), \tau(\delta_n(t))u_n, \tau(\delta_n(t))v_n), \tag{3.1.4}$$

$$u_n(\delta_n(t)) \in D(A(\delta_n(t)));\ u_n(\theta_n(t)) \in D(A(\theta_n(t))), \tag{3.1.5}$$

$$v_n(t) = b + \int_0^t u_n(\delta_n(s))\, ds, \quad \forall t \in [0,1], \tag{3.1.6}$$

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \theta_n(t) = t, \quad \forall t \in [0,1]. \tag{3.1.7}$$

Claim. $(\dot{u}_n)$ is uniformly bounded a.e.

By (2.1) and (H_2) we have $||J_{e_n}A(t_0^n)a - a|| \leq e_n M_a$, for all $n \in \mathbf{N}$ for some positive constant M_a. Let ρ as in (H_1), there is $\gamma > 0$ such that $||\rho(t)|| \leq \frac{\gamma}{2}$ for all $t \in [0,1]$. By using (H_1) and the preceding inequality, we obtain the estimate

$$\begin{aligned}
||x_k^n - a|| &\leq ||J_{e_n}A(t_k^n)(x_{k-1}^n - e_n h(t_{k-1}^n, \tau(t_{k-1}^n)u_n, \tau(t_{k-1}^n)v_n)) - J_{e_n}A(t_k^n)a|| \\
&\quad + ||J_{e_n}A(t_k^n)a - J_{e_n}A(t_0^n)a|| + ||J_{e_n}A(t_0^n)a - a|| \\
&\leq ||x_{k-1}^n - a|| + e_n\, ||h(t_{k-1}^n, \tau(t_{k-1}^n)u_n, \tau(t_{k-1}^n)v_n)|| \\
&\quad + e_n\gamma L(||a||) + e_n M_a \\
&\leq ||x_{k-1}^n - a|| + e_n(\gamma L(||a||) + M_a + |Z|) \\
&\quad + e_n|Z|(||u_n(t_{k-1}^n)|| + ||v_n(t_{k-1}^n)||).
\end{aligned}$$

Iterating the preceding inequality gives

$$\begin{aligned}
&||x_k^n - a|| \\
&\leq k e_n(|Z| + \gamma L(||a||) + M_a) \\
&\quad + e_n|Z|(||u_n(t_0^n)|| + ||u_n(t_1^n)|| + ... + ||u_n(t_{k-1}^n)||) \\
&\quad + e_n|Z|(||v_n(t_0^n)|| + ||v_n(t_1^n)|| + ... + ||v_n(t_{k-1}^n)||) \\
&\leq \gamma L(||a||) + M_a + |Z| + e_n|Z|(||a|| + ||x_1^n|| + ... + ||x_{k-1}^n||) \\
&\quad + e_n|Z|(||b|| + ||v_n(t_1^n)|| + ... + ||v_n(t_{k-1}^n)||)
\end{aligned} \tag{3.1.8}$$

for all $n \geq 1$ and for all $k = 1, 2, ..., n$.

Now

$$v_n(t_1^n) = b + e_n a, \quad v_n(t_2^n) = b + e_n a + e_n x_1^n, \ \dots ,$$
$$v_n(t_{k-1}^n) = b + e_n a + e_n x_1^n + ... + e_n x_{k-2}^n$$

then

$$\begin{aligned}
&||b|| + ||v_n(t_1^n)|| + ... + ||v_n(t_{k-1}^n)|| \\
&\leq ||b|| + (||b|| + e_n||a||) + (||b|| + e_n||a|| + e_n||x_1^n||) + ... \\
&\quad +(||b|| + e_n||a|| + e_n||x_1^n|| + ... + e_n||x_{k-2}^n||) \\
&= k||b|| + e_n(k-1)||a|| + e_n(k-2)||x_1^n|| + ... + e_n||x_{k-2}^n|| \\
&\leq k||b|| + ||a|| + ||x_1^n|| + ... + ||x_{k-2}^n||.
\end{aligned}$$

By substituting this estimate in (3.1.8) we get

$$\begin{aligned}
&||x_k^n - a|| \leq \gamma L(||a||) + M_a + |Z| + e_n|Z|(||a|| + ||x_1^n|| + ... + ||x_{k-1}^n||) \\
&+e_n|Z|(k||b|| + ||a|| + ||x_1^n|| + ... + ||x_{k-2}^n||) \\
&\leq \gamma L(||a||) + M_a + |Z|(1 + ||b|| + 2||a||) + 2e_n|Z|(||x_1^n|| + ... + ||x_{k-1}^n||)
\end{aligned}$$

for all $n \geq 1$ and for all $k = 1, 2, ..., n$. So we have

$$\begin{aligned}
||x_k^n|| \leq ||a|| + \gamma L(||a||) + M_a + |Z|(1 + ||b| + 2||a||) \\
+2e_n|Z|(||x_1^n|| + ... + ||x_{k-1}^n||)
\end{aligned} \tag{3.1.9}$$

for all $n \geq 1$ and for all $k = 1, 2, .., n$. Set for simplicity

$$\delta := ||a|| + \gamma L(||a||) + M_a + |Z|(1 + ||b| + 2||a||) \quad \text{and} \quad \rho_n = 2e_n|Z|$$

we obtain

$$||x_k^n|| \leq \delta + \rho_n(||x_1^n|| + ... + ||x_{k-1}^n||) \tag{3.1.10}$$

for $k = 1, .., n$. Now, we claim that

$$||x_k^n|| \leq \delta(1 + \rho_n)^{k-1}$$

for all $n \geq 1$ and for all $k = 1, 2, ..., n$. For $k = 1$, we have

$$\begin{aligned}
&||x_1^n - a|| \leq ||J_{e_n}A(t_1^n)(a - e_n h(t_0^n, \tau(t_0^n)u_n, \tau(t_0^n)v_n)) - J_{e_n}A(t_1^n)a|| \\
&+||J_{e_n}A(t_1^n)a - J_{e_n}A(t_0^n)a|| + ||J_{e_n}A(t_0^n)a - a|| \\
&\leq e_n\gamma L(||a||) + e_n M_a + e_n|Z|(1 + ||a|| + ||b||) \leq \delta - ||a||.
\end{aligned}$$

Therefore $||x_1^n|| \leq \delta$. Assume by induction that (3.1.10) is true for $1 \leq k < n$, then

$$\begin{aligned}||x_{k+1}^n|| &\leq \delta + \rho_n(||x_1^n|| + ||x_2^n|| + ... + ||x_{k-1}^n|| + ||x_k^n||)\\ &\leq \delta + \rho_n[\delta + \delta(1+\rho_n) + \delta(1+\rho_n)^2 + ... + \delta(1+\rho_n)^{(k-1)}]\\ &= \delta + \delta\rho_n[1 + (1+\rho_n) + (1+\rho_n)^2 + ... + (1+\rho_n)^{(k-1)}]\\ &= \delta + \delta\rho_n[\frac{(1+\rho_n)^k - 1}{(1+\rho_n) - 1}] = \delta + \delta[(1+\rho_n)^k - 1] = \delta(1+\rho_n)^k.\end{aligned}$$

Then for all $n \geq 1$ and for all $k = 1, 2, ...n$, we have the estimate

$$\begin{aligned}||x_k^n|| &\leq \delta(1+\rho_n)^{(k-1)}\\ &\leq \delta(1+\frac{2|Z|}{n})^{(k-1)}\\ &\leq \delta(1+\frac{2|Z|}{n})^n\\ &\leq \delta\exp(2|Z|) = \beta\end{aligned} \tag{3.1.11}$$

thereby proving the required estimate. Consequently, for all $n \geq 1$ and all $k = 0, 1, 2, ..., n$ we have

$$\begin{aligned}||v_n(t_k^n)|| &\leq ||b|| + e_n||a|| + e_n||x_1^n|| + ... + e_n||x_{k-1}^n||\\ &\leq ||b|| + ||a|| + ke_n\beta \leq ||a|| + ||b|| + \beta.\end{aligned} \tag{3.1.12}$$

So by (3.1.12), (H_1) and (H_2) we get the estimate

$$\begin{aligned}||x_{k+1}^n - x_k^n|| &\leq ||J_{e_n}A(t_{k+1}^n)(x_k^n - e_nh(t_k^n, \tau(t_k^n)u_n, \tau(t_k^n)v_n)) - J_{e_n}A(t_{k+1}^n)x_k^n||\\ &\quad + ||J_{e_n}A(t_{k+1}^n)x_k^n - J_{e_n}A(t_k^n)x_k^n|| + ||J_{e_n}A(t_k^n)x_k^n - x_k^n||\\ &\leq e_n|Z|(1 + ||a|| + ||b|| + 2\beta) + e_n(\gamma L(\beta) + M_\beta)\end{aligned}$$

for all $n \geq 1$ and for all $k = 0, 1, 2, .., n$, here $L(\beta)$ and M_β are positive constant, independent of n occurring in (H_1) and (H_2). By (3.1.2) and the preceding estimate, we get

$$||\dot{u}_n(t)|| \leq N := |Z|(1 + ||a|| + ||b|| + 2\beta) + \gamma L(\beta) + M_\beta \tag{3.1.13}$$

for all $n \geq 1$ and a.e. $t \in [0, 1]$. That proves the claim. By (3.1.7) and (3.1.13), we have

$$\lim_{n\to\infty} ||u_n(\delta_n(t)) - u_n(t)|| = \lim_{n\to\infty} ||u_n(\theta_n(t)) - u_n(t)|| = 0 \tag{3.1.14}$$

for all $t \in [0,1]$. By (3.1.5) and (3.1.11) $u_n(\theta_n(t)) \in D(A(\theta_n(t))) \cap \overline{B}_E(0,\beta)$ for all $t \in [0,1]$ and for all $n \in \mathbf{N}$. As $D(A(\theta_n(t))) \cap \overline{B}_E(0,\beta)$ is compact, because $D(A(t))$ is constant, say $D(A(t)) = D$ for all $t \in [0,T]$, $(u_n(\theta_n(t)))$ is relatively compact in E for every $t \in [0,T]$ and, on account of (3.1.14) $(u_n(t))$ is relatively compact too. Thus $(u_n(.))$ is relatively compact in $\mathcal{C}_E([0,1])$. Hence we may suppose that $(\dot{u}_n)$ $\sigma(L^\infty_E, L^1_{E'})$-converges in $L^\infty_E([0,1])$ to a function w with $||w(t)|| \le N$ for a.e $t \in [0,1]$, and (u_n) converges in $\mathcal{C}_E([0,1])$ to an absolutely continuous function u

$$u(t) = a + \int_0^t \dot{u}(s)\, ds,\ \forall t \in [0,1]$$

with $\dot{u} = w$. From (3.1.6), (3.1.13), (3.1.14), we deduce that (v_n) converges uniformly to an absolutely continuous function v with $v(t) = b + \int_0^t u(s)\, ds$, $\forall t \in [0,1]$. As

$$||h(\delta_n(t), \tau(\delta_n(t))u_n, \tau(\delta_n(t))v_n)|| \le (1 + ||u_n(\delta_n(t))|| + ||v_n(\delta_n(t))||)|Z|$$
$$\le (1 + ||a|| + ||b|| + 2\beta)|Z|$$

for all $n \ge 1$ and for all $t \in [0,1]$, we may suppose that

$$(g_n(.)) := (h(\delta_n(.), \tau(\delta_n(.))u_n, \tau(\delta_n(.))v_n))$$

$\sigma(L^\infty_E, L^1_{E'})$-converges in $L^\infty_E([0,1])$ to a function g with $||g(t)|| \le (1 + ||a|| + ||b|| + 2\beta)|Z|$ for a.e $t \in [0,1]$.

Claim. $\dot{u}(t) \in -A(t)u(t) - g(t)$ $a.e$ $t \in [0,1]$.

Let us set $w_n(t) := J_{e_n}A(t)(u_n(\delta_n(t)) - e_n h(\delta_n(t), \delta_n(t)u_n, \delta_n(t)v_n))$ for all $n \ge 1$ and for all $t \in [0,T]$. Then $w_n(t) \in D(A(t))$ for all $t \in [0,T]$ and we have

$$u_n(\delta_n(t)) - e_n h(\delta_n(t), \delta_n(t)u_n, \delta_n(t)v_n) \in w_n(t) + e_n A(t)w_n(t). \tag{3.1.15}$$

In view of (H_1) and (3.1.11) and (3.1.12) we have the estimate

$$\begin{aligned} &||u_n(\theta_n(t)) - w_n(t)|| \\ &\le e_n|\rho(\theta_n(t)) - \rho(t)|L(||u_n(\delta_n(t)) - e_n g_n(t)||) \\ &\le e_n|\rho(\theta_n(t)) - \rho(t)|L(\beta + (1 + ||a|| + ||b|| + 2\beta|)|Z|). \end{aligned} \tag{3.1.16}$$

It follows that

$$\lim_{n\to\infty} ||u_n(\theta_n(t)) - w_n(t)|| = 0,\ \forall t \in [0,T]. \tag{3.1.17}$$

Consequently we get

$$\lim_{n\to\infty} w_n(t) = u(t),\ \forall t \in [0,1]. \tag{3.1.18}$$

Let $n \geq 1$ and let $t \in]0,1[$. Then $t \in]t_k^n, t_{k+1}^n[$ for some $0 \leq k < n$. So, taking (3.1.15) (3.1.16) and the preceding estimate into account, we get

$$\begin{aligned}
& d(-\dot{u}_n(t) - g_n(t), A(t)w_n(t)) \\
&= d(\frac{u_n(\delta_n(t)) - u_n(\theta_n(t))}{e_n} - g_n(t), A(t)w_n(t)) \\
&\leq \frac{1}{e_n}||u_n(\theta_n(t)) - w_n(t)|| \\
&\leq |\rho(\theta_n(t)) - \rho(t)|L(\beta + (1 + ||a|| + ||b| + 2\beta)|Z|).
\end{aligned}$$

Since $A(t)w_n(t)$ is closed and convex, the preceding inequality implies that

$$-\dot{u}_n(t) - g_n(t) \in A(t)w_n(t) + r_n(t)\overline{B}_E(0,1)$$

a.e with $r_n(t) := |\rho(\theta_n(t)) - \rho(t)|L(\beta + (1 + ||a|| + ||b| + 2\beta)|Z|) \to 0$ for all $t \in [0,1]$. As $\dot{u}_n + g_n \to \dot{u} + g$ weakly in $L^2_E([0,T])$ and $w_n \to u$ strongly in $L^2_E([0,1])$ by (3.1.11), (3.1.16), (3.1.18), and from Lemma 3.1, we deduce that

$$-\dot{u}(t) \in A(t)u(t) + g(t)\ a.e\ t \in [0,T].$$

Claim. $g(t) \in G(t, \tau(t)u, \tau(t)v)\ a.e\ t \in [0,1]:$

Let $t \in [0,1]$, we have

$$\begin{aligned}
& ||\tau(\delta_n(t))u_n - \tau(t)u||_{\mathcal{C}_E([-r,0])} \leq \ ||\tau(\delta_n(t))u_n - \tau(t)u_n||_{\mathcal{C}_E([-r,0])} \\
& + ||\tau(t)u_n - \tau(t)u||_{\mathcal{C}_E([-r,0])} \\
& \leq \sup_{\{s_1,s_2\in[-r,1],|s_1-s_2|<e_n\}} ||u_n(s_1) - u_n(s_2)|| + ||\tau(t)u_n - \tau(t)u|| \\
& \leq \sup_{\{s_1,s_2\in[-r,0],|s_1-s_2|<e_n\}} ||u_n(s_1) - u_n(s_2)|| \\
& \quad \sup_{\{s_1,s_2\in[0,1],|s_1-s_2|<e_n\}} ||u_n(s_1) - u_n(s_2)|| \\
& + ||\tau(t)u_n - \tau(t)u||_{\mathcal{C}_E([-r,0])} \\
& \leq \sup_{\{s_1,s_2\in[-r,0],|s_1-s_2|<e_n\}} ||\varphi(s_1) - \varphi(s_2)|| \\
& + Ne_n + ||\tau(t)u_n - \tau(t)u||_{\mathcal{C}_E([-r,0])}.
\end{aligned}$$

Using the continuity of φ, the uniform convergence of u_n towards u, and the preceding estimate, we see that

$$\lim_{n\to\infty} ||\tau(\delta_n(t))u_n - \tau(t)u||_{\mathcal{C}_E([-r,0])} = 0. \tag{3.1.19}$$

Similarly we have

$$\lim_{n\to\infty} ||\tau(\delta_n(t))v_n - \tau(t)v||_{\mathcal{C}_E([-r,0])} = 0. \tag{3.1.20}$$

From (3.1.4), (3.1.19), (3.1.20), the scalarly upper semicontinuity of G and a closure type result ([23], Theorem VI-14) we get the required claim. So existence for $(\mathcal{P}_1)$ is completely demonstrated in the case when F is globally scalarly upper semicontinuous on $[0,1] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$.

Step 2. For simplicity assume that $K = \overline{B}_E(0,1)$ and $F : [0,1] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ is separately measurable on $[0,1]$ and separately upper semicontinuous on $\mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$. If F is globally scalarly upper semicontinous, then the multifunction $F(t,x,y)+G(t,x,y)$ is scalarly upper semicontinuous on $[0,1] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ and satisfies the inclusion $F(t,x,y)+G(t,x,y) \subset 2(1+||x(0)||+||y(0)||)\overline{B}_E(0,1)$, so that we can apply the existence result in first step when G is substituted to $F+G$. Then, for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) = a \in D(A(0))$ and every $\psi \in \mathcal{C}_0$ with $\psi(0) = b \in E$, there exists two continuous mappings $u : [-r,T] \to E$ and $v : [-r,T] \to E$ such that

$$\begin{cases} v(t) = \psi(t) \quad \text{on} \quad [-r,0]; \quad v(t) = \psi(0) + \displaystyle\int_0^t u(s)ds, \forall t \in [0,1] \\ u(t) = \varphi(t) \quad \text{on} \quad [-r,0]; \quad u(t) = a + \displaystyle\int_0^t \dot{u}(s)ds, \forall t \in [0,1] \\ u(t) \in D(A(t)), \forall t \in [0,1] \\ 0 \in \dot{u}(t) + A(t)u(t) + F(t,\tau(t)u,\tau(t)v) + G(t,\tau(t)u,\tau(t)v) \text{ a.e.} \end{cases}$$

with $\dot{u} \in L^\infty_E([0,T]$. Now we pass to the general case. Here we imitate an argument developed in [12] in this particular situation. By Scorza-Dragoni theorem ([6], [16]) there exists an increasing sequence of compact sets (J_n) in $[0,1]$ such that the Lebesgue measure of $[0,1] \setminus J_n$ tends to 0 when $n \to \infty$ and that the restriction of F on $J_n \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ is scalarly upper semicontinuous. Let $\widetilde{F}_n$ be the upper semicontinuous Dugundji extension (e.g. [7]) of $F|J_n \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ to $[0,1] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$ with $\widetilde{F}_n(t,x,y) \subset K$, for all $(t,x,y) \in [0,1] \times \mathcal{C}_E([-r,0]) \times \mathcal{C}_E([-r,0])$. We now apply step 1 with $\widetilde{F}_n + G$. Thus, using the above remark, for every n, for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) = a \in D(A(0))$ and every $\psi \in \mathcal{C}_0$ with $\psi(0) = b \in E$, there exist two continuous mappings $u_n : [-r, 1] \to E$ and $v_n : [-r, 1] \to E$ such that

$$(\mathcal{P}_1^n) \begin{cases} v_n(t) = \psi(t) \quad \text{on} \quad [-r, 0]; \quad v_n(t) = \psi(0) + \displaystyle\int_0^t u_n(s) ds, \forall t \in [0, 1] \\ u_n(t) = \varphi(t) \quad \text{on} \quad [-r, 0]; \quad u_n(t) = a + \displaystyle\int_0^t \dot{u}_n(s) ds, \forall t \in [0, 1] \\ u_n(t) \in D(A(t)), \forall t \in [0, 1] \\ 0 \in \dot{u}_n(t) + A(t)u_n(t) + \widetilde{F}_n(t, \tau(t)u_n, \tau(t)v_n) + G(t, \tau(t)u_n, \tau(t)v_n) \end{cases}$$

where $(\dot{u}_n)$ is bounded in $L^\infty_E([0, T]$. As D is ball-compact, we may suppose that (u_n) converges uniformly to $u \in \mathcal{C}_E([0, 1]), (\dot{u}_n)$ converges $\sigma(L^\infty_E, L^1_{E'})$ to $\dot{u} \in L^\infty_E([0, T]$ and $u(t) = a + \int_0^t \dot{u}(s)\, ds,\ \forall t \in [0, 1]$, and (v_n) converges uniformly to $v(t) = b + \int_0^t u(s) ds$. We now finish the proof as follows. There is a measurable mapping $z_n : [0, 1] \to K$ with $z_n(t) \in \widetilde{F}_n(t, \tau(t)u_n, \tau(t)v_n) + G(t, \tau(t)u_n, \tau(t)v_n)$ for a.e. $t \in [0, 1]$ and

$$-\dot{u}_n(t) \in A(t)u_n(t) + z_n(t),$$

a.e. $t \in [0, 1]$. It is obvious that (z_n) is bounded in $L^\infty_E([0, 1])$. We may suppose that (z_n) converges $\sigma(L^\infty_E, L^1_{E'})$ in $L^\infty_E([0, 1])$ to $z \in L^\infty_E([0, 1])$ so that, in view of Lemma 3.1, we get

$$-\dot{u}(t) \in A(t)u(t) + z(t)$$

a.e. $t \in [0, 1]$. By construction, there is a Lebesgue null set N_n such that $z_n(t) \in F_n(t, \tau(t)u_n, \tau(t)v_n) + G(t, \tau(t)u_n, \tau(t)v_n)$ for all $t \in J_n \setminus N_n$. Let $N_0 := ([0, 1] \setminus \cup_n J_n) \cup (\cup_n N_n)$ which is Lebesgue-negligible. If $t \in [0, 1] \setminus N_0$, there is an integer $p := p(t)$ such that $z_n(t) \in F_n(t, \tau(t)u_n, \tau(t)v_n) + G(t, \tau(t)u_n, \tau(t)v_n)$ for all $n \geq p(t)$. As $F + G$ is scalarly upper semicontinuous on $\mathcal{C}_E([-r, 0]) \times \mathcal{C}_E([-r, 0])$ and $u_n(t) \to u(t)$ in E, we have

$$\begin{aligned} \limsup_n \langle x', z_n(t) \rangle &\leq \limsup_n [\delta^*(x', F(t, \tau(t)u_n, \tau(t)v_n)) \\ &\quad + \delta^*(x', G(t, \tau(t)u_n, \tau(t)v_n))] \\ &\leq [\delta^*(x', F(t, \tau(t)u, \tau(t)v)) + \delta^*(x', G(t, \tau(t)u, \tau(t)v))] \end{aligned}$$

for all $x' \in E'$ and for all $n \geq p$. Thus

$$\begin{aligned} \limsup_n \langle x', z_n(t) \rangle &\leq \delta^*(x', F(t, \tau(t)u(t), \tau(t)v(t))) \\ &\quad + \delta^*(x', G(t, \tau(t)u(t), \tau(t)v(t))) \end{aligned}$$

for a.e $t \in [0, 1]$. By Fatou's lemma, it follows that, for every measurable set

$A \subset [0,1]$ and for every $x' \in E'$,

$$\int_A \langle x', z(t)\rangle dt = \lim_n \int_A \langle x', z_n(t)\rangle dt$$
$$= \limsup_n \int_A \langle x', z_n(t)\rangle dt \leq \int_A \limsup_n \langle x', z_n(t)\rangle dt$$
$$\leq \int_A [\delta^*(x', F(t, \tau(t)u, \tau(t)v)) + \delta^*(x', G(t, \tau(t)u, \tau(t)v))]\, dt.$$

Consequently

$$z(t) \in F(t, \tau(t)u, \tau(t)v)) + G(t, \tau(t)u, \tau(t)v)$$

a.e., thereby proving the inclusion

$$-\dot{u}(t) \in A(t)u(t) + F(t, \tau(t)u, \tau(t)v) + G(t, \tau(t)u, \tau(t)v) \quad \text{a.e.}$$

and completing the proof. ■

Now we proceed to some variants of Theorem 3.1 involving the normal cone of closed convex moving sets in Hilbert space. It is well known that the normal cone of a closed convex set is the subdifferential of its indicator function which is a maximal monotone operator. This problem, so-called ”sweeping process”, was introduced and solved by Moreau [30]. A study of second order evolution problem involving the normal cone of closed convex moving sets in Hilbert spaces was initiated by the first author [10], subsequently there has been a deal of research on this problem [1, 26]. We present here an existence result for the second order sweeping process with undelayed perturbations. For more information on sweeping process and, convex and nonconvex evolution problems, we refer to [14, 21, 22, 24-27, 32, 33, 37-39].

Theorem 3.2. *Let $H = \mathbf{R}^d$ and $T > 0$. Assume that $C : H \to cc(H)$ is a closed convex valued Λ-Lipschitzean mapping: that is*

$$\mathcal{H}(C(x), C(y)) \leq \Lambda||x - y||, \quad \forall (x, y) \in H \times H$$

here $\mathcal{H}$ denotes the Hausdorff distance on $cc(H)$. Assume that

$$G : [0, T] \times H \times H \to ck(H)$$

is convex compact valued scalarly upper semicontinuous mapping such that

$$G(t, x, y) \subset (1 + ||x|| + ||y||)Z$$

for all $(t,x,y) \in [0,T]\times H\times H$, *for some convex compact set* Z *in* H. *Then, for every every* $b \in H$, *and for every* $a \in C(b)$ *there exist two absolutely continuous mappings* $u : [0,T] \to H$ *and* $v : [0,T] \to H$ *such that*

$$(\mathcal{P}_2)\begin{cases} v(t) = b + \displaystyle\int_0^t u(s)ds, \forall t \in [0,T], \\ u(t) = a + \displaystyle\int_0^t \dot{u}(s)ds, \forall t \in [0,T], \\ \text{with} \quad \dot{u} \in L^\infty_H([0,T]) \quad \text{and} \quad u(t) \in C(v(t)), \forall t \in [0,T], \\ 0 \in \dot{u}(t) + N(C(v(t)); u(t)) + G(t, u(t), v(t)) \text{ a.e.} \end{cases}$$

here $N(C(v(t)); u(t))$ *denotes the normal cone of* $C(v(t))$ *at the point* $u(t)$.

Proof. Our proof follows some techniques of discretization developed in [12], and some arguments of the proof of Theorem 3.1. Nevertheless, this need a careful look. Let $h : [0,T]\times H\times H \to H$ be a scalarly $\mathcal{B}([0,T]\times H\times H)$-measurable selection of G (see [23]). For notational convenience we will take $T = 1$ and $Z = \overline{B}_H(0,1)$ so that $|Z| \le 1$. We consider a partition of $[0,1]$ by the points $t^n_k = ke_n$, $e_n = \frac{1}{n}, k = 0,1,2,...,n$. For each $t \in [t^n_0, t^n_1]$, we define

$$v_n(t) = b + (t - t^n_0)a$$

$$u_n(t) = \frac{t^n_1 - t}{e_n} x^n_0 + \frac{t - t^n_0}{e_n} x^n_1.$$

where $x^n_0 = a \in C(b)$ and

$$x^n_1 = \text{proj}_{C(v_n(t^n_1))}(x^n_0 - e_n h(t^n_0, a, b))$$

so that $v_n(t^n_0) = b$ and $u_n(t^n_0) = a$. Then we have the estimate

$$\begin{aligned} &d(x^n_0 - e_n h(t^n_0, u_n(t^n_0), v_n(t^n_0)); C(v(t^n_1))) \\ &\le \mathcal{H}(C(v_n(t^n_0)), C(v_n(t^n_1))) + e_n||h(t^n_0, u_n(t^n_0), v_n(t^n_0))|| \\ &\le (\Lambda||u_n(t^n_0)|| + 1 + ||u_n(t^n_0)|| + ||v_n(t^n_0)||)e_n. \end{aligned}$$

Hence for $t \in [t^n_0, t^n_1[$, we have $\dot{v}_n(t) = a$ and

$$\dot{u}_n(t) = \frac{x^n_1 - x^n_0}{e_n} \in -N(C(v_n(t^n_1)); x^n_1) - h(t^n_0, u_n(t^n_0), v_n(t^n_0)). \tag{3.2.1}$$

with

$$||\frac{x^n_1 - x^n_0}{e_n}|| \le \Lambda||u_n(t^n_0)|| + 2(1 + ||u_n(t^n_0)|| + ||v_n(t^n_0)||).$$

For $t \in [t_1^n, t_2^n[$, we define $v_n(t) = v_n(t_1^n) + (t - t_1^n)u_n(t_1^n)$ and

$$u_n(t) = \frac{t_2^n - t}{e_n} x_1^n + \frac{t - t_1^n}{e_n} x_2^n$$

where

$$x_2^n = \mathrm{proj}_{C(v_n(t_2^n))}(x_1^n - e_n h(t_1^n, u_n(t_1^n), v_n(t_1^n)))$$

Then for $t \in [t_1^n, t_2^n[$, we have $\dot{v}_n(t) = x_1^n = u_n(t_1^n)$ and

$$\text{(3.2.2)} \qquad \dot{u}_n(t) = \frac{x_2^n - x_1^n}{e_n} \in -N(C(v_n(t_2^n)); x_2^n) - h(t_1^n, u_n(t_1^n), v_n(t_1^n)).$$

As $v_n(t_2^n) = v_n(t_1^n) + e_n u_n(t_1^n)$ we have the estimate

$$\begin{aligned} & d(x_1^n - e_n h(t_1^n, u_n(t_1^n), v_n(t_1^n)); C(v_n(t_2^n))) \\ & \leq \mathcal{H}(C(v_n(t_1^n)), C(v_n(t_2^n))) + e_n \|h(t_1^n, u_n(t_1^n), v_n(t_1^n)))\| \\ & \leq (\Lambda||u_n(t_1^n)|| + 1 + ||u_n(t_1^n)|| + ||v_n(t_1^n)||)e_n \end{aligned}$$

so that

$$||\frac{x_2^n - x_1^n}{e_n}|| \leq \Lambda||u_n(t_1^n)|| + 2(1 + ||u_n(t_1^n)|| + ||v_n(t_1^n)||).$$

Suppose that $(v_n), (u_n)$ are well defined on $[t_0^n, t_k^n]$ and recall that

$$\begin{aligned} & v_n(t_1^n) = b + e_n a, \quad v_n(t_2^n) = b + e_n a + e_n x_1^n, ..., \\ & v_n(t_k^n) = b + e_n a + e_n x_1^n + ... + e_n x_{k-1}^n \end{aligned}$$

with $u_n(t_k^n) = x_k^n$ and

$$||\frac{x_k^n - x_{k-1}^n}{e_n}|| \leq \Lambda||u_n(t_{k-1}^n)|| + 2(1 + ||u_n(t_{k-1}^n)|| + ||v_n(t_{k-1}^n)||).$$

For each $t \in [t_k^n, t_{k+1}^n]$, we define

$$v_n(t) = v_n(t_k^n) + (t - t_k^n)u_n(t_k^n)$$

and

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n} x_k^n + \frac{t - t_k^n}{e_n} x_{k+1}^n$$

where

$$x_{k+1}^n = \mathrm{proj}_{C(v_n(t_{k+1}^n)}(x_k^n - e_n h(t_k^n, u_n(t_k^n), v_n(t_k^n)).$$

Then for $t \in [t_k^n, t_{k+1}^n[$ we have $\dot{v}_n(t) = u_n(t_k^n)$ and

$$\text{(3.2.3)} \quad \dot{u}_n(t) = \frac{x_{k+1}^n - x_k^n}{e_n} \in -N(C(v_n(t_{k+1}^n)); x_{k+1}^n) - h(t_k^n, u_n(t_k^n), v_n(t_k^n))$$

with the estimate

$$(3.2.4)\qquad ||\frac{x_{k+1}^n - x_k^n}{e_n}|| \leq \Lambda||u_n(t_k^n)|| + 2(1 + ||u_n(t_k^n)|| + ||v_n(t_k^n)||).$$

For each $t \in [0, 1]$ and each $n \geq 1$, let $\delta_n(t) = t_k^n,\ \theta_n(t) = t_{k+1}^n,$ if $t \in [t_k^n, t_{k+1}^n[$. So by (3.2.3) we get

$$(3.2.5)\quad \dot{u}_n(t) \in -N(C(v_n(\theta_n(t)); u_n(\theta_n(t)) - h(\delta_n(t), u_n(\delta_n(t)), v_n(\delta_n(t)))$$

for a.e. $t \in [0, 1]$. It is obvious that, for all $n \geq 1$ and for all $t \in [0, 1]$, the following hold:

$$(3.2.6)\qquad h(\delta_n(t), u_n(\delta_n(t)), v_n(\delta_n(t))) \in G(\delta_n(t), u_n(\delta_n(t)), v_n(\delta_n(t))).$$

$$(3.2.7)\qquad u_n(\delta_n(t)) \in C(v_n(\delta_n(t))); u_n(\theta_n(t)) \in C(v_n(\theta_n(t))).$$

$$(3.2.8)\qquad v_n(t) = b + \int_0^t u_n(\delta_n(s))ds, \quad \forall t \in [0, 1].$$

$$(3.2.9)\qquad \lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \theta_n(t) = t, \quad \forall t \in [0, 1].$$

Claim. $(\dot{u}_n)$ *is uniformly bounded a.e.*

By iterating the estimate (3.2.4) we get

$$\begin{aligned}
||x_k^n - x_0^n|| &\leq e_n\Lambda(||u_n(t_0^n)|| + ... + ||u_n(t_{k-1}^n)|| \\
&+2ke_n + 2e_n(||u_n(t_0^n)|| + ... + ||u_n(t_{k-1}^n)||) \\
&+2e_n(||v_n(t_0^n)|| + ... + ||v_n(t_{k-1}^n)||)
\end{aligned}$$

for $k = 1, .., n$. On account of the expression of $v_n(t_j^n)$, $j = 0, .., k-1$ and the preceding estimate we get

$$\begin{aligned}
||x_k^n - x_0^n|| &\leq e_n\Lambda(||a|| + ||x_1^n|| + ... + ||x_{k-1}^n|| \\
&+2ke_n + 2e_n(||a|| + ||x_1^n|| + ... + ||x_{k-1}^n||) \\
&+2e_n(k||b|| + ||a|| + ||x_1^n|| + ... + ||x_{k-2}^n||).
\end{aligned}$$

So we have

$$(3.2.10)\qquad \begin{aligned}
||x_k^n|| &\leq (\Lambda + 5)||a|| + 2 + 2||b|| \\
&+(\Lambda + 4)e_n(||x_1^n|| + ... + ||x_{k-1}^n)||).
\end{aligned}$$

Set for simplicity $\delta = (\Lambda + 5)||a|| + 2 + 2||b||$ and $\rho_n = (\Lambda + 4)e_n$ yields

$$(3.2.11a) \qquad ||x_k^n|| \le \delta + \rho_n(||x_1^n|| + ... + ||x_{k-1}^n)||).$$

for all $n \in \mathbf{N}$ and for all $k = 1, 2, ...n$. Using this estimate and arguing as in the proof of Theorem 3.1, we see that

$$(3.2.11b) \qquad ||x_k^n|| \le \delta \exp(2) := \beta$$

for all $n \in \mathbf{N}$ and for all $k = 0, 1, 2, ...n$. Coming back to the estimate

$$||\frac{x_{k+1}^n - x_k^n}{e_n}|| \le \Lambda||u_n(t_k^n)|| + 2(1 + ||u_n(t_k^n)|| + ||v_n(t_k^n)||)$$

we get by (3.2.11)

$$\begin{aligned}||\frac{x_{k+1}^n - x_k^n}{e_n}|| &\le \Lambda||u_n(t_k^n)|| + 2(1 + ||u_n(t_k^n)|| + ||v_n(t_k^n)||)\\ &\le \Lambda||u_n(t_k^n)|| + 2(1 + ||u_n(t_k^n)|| + ||b||\\ &\quad + e_n||a|| + e_n||x_1^n|| + ... + e_n||x_{k-1}^n||)\\ &\le \Lambda\beta + 2(1 + \beta + ||b|| + ||a|| + ke_n\beta)\\ &\le \Lambda\beta + 2(1 + 2\beta + ||b|| + ||a||)(= N)\end{aligned}$$

we finally conclude that

$$(3.2.12) \qquad ||\dot{u}_n(t)|| \le N$$

for all $n \in \mathbf{N}$ and for a.e $t \in [0, 1]$. As

$$||u_n(\theta_n(t)) - u_n(t)|| \le N(\theta_n(t) - t)$$

we have

$$(3.2.13) \qquad \lim_{n\to\infty} ||u_n(\theta_n(t)) - u_n(t)|| = 0$$

By construction $(u_n(\theta_n(t))$ is relatively compact in H, for every $t \in [0, 1]$, so is $(u_n(t))$. Thus $(u_n(.))$ is relatively compact in $\mathcal{C}_H([0, 1])$. Hence we may suppose that $(\dot{u}_n)$ $\sigma(L_H^\infty, L_H^1)$ converges in $L_H^\infty([0, 1])$ to a function z with $||z(t)|| \le N$ for a.e $t \in [0, 1]$, and (u_n) converges in $\mathcal{C}_H([0, 1])$ to an absolutely continuous function u

$$u(t) = a + \int_0^t \dot{u}(s)\, ds, \;\; \forall t \in [0, 1]$$

with $\dot{u} = z$. From (3.2.8), (3.2.13) and the convergence of (u_n) we deduce that (v_n) converges uniformly to an absolutely continuous function v with $v(t) = b + \int_0^t u(s)\, ds$. As

$$\begin{aligned}&||h(\delta_n(t), u_n(\delta_n(t), v_n(\delta_n(t)))\\ &\leq 1 + ||u_n(\delta_n(t))|| + ||v_n(\delta_n(t))||\\ &\leq 1 + ||b|| + ||a|| + 2\beta(= L)\end{aligned}$$

for all $n \in \mathbf{N}$ and for all $t \in [0,1]$, from the above estimate, we may suppose that

$$(g_n(.)) := (h(\delta_n(.), u_n(\delta_n(.)), v_n(\delta_n(.)))$$

$\sigma(L^\infty_H, L^1_H)$ converges in $L^\infty_H([0,1])$ to a function g

Claim: $g(t) \in G(t, u(t), v(t))$ *a.e* $t \in [0,1]$.

Using the convergence of u_n towards u in $\mathcal{C}_H([0,T]$, and the preceding estimate, we see that

$$\lim_{n\to\infty} u_n(\delta_n(t)) = u(t). \tag{3.2.14}$$

Similarly we have

$$\lim_{n\to\infty} v_n(\delta_n(t)) = v(t). \tag{3.2.15}$$

Using (3.2.6), (3.2.14), (3.2.15) and by invoking the scalarly upper semicontinuity of G on $[0,1] \times H \times H$ and a closure type result in ([23], Theorem VI-14) we get the desired claim.

Claim. $\dot{u}(t) \in -N(C(v(t)); u(t)) - G(t, u(t), v(t))a.e\, t \in [0,1]$ *and conclusion*

First we show that $u(t) \in C(v(t)), \forall t \in [0,1]$. Indeed, for every $t \in [0,1]$ and for every $n \in \mathbf{N}$, by (3.2.7) we have

$$\begin{aligned}d(u_n(t), C(v(t)) &\leq ||u_n(t) - u_n(\theta_n(t))|| + d(u_n(\theta_n(t)), C(v(t)))\\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + \mathcal{H}(C(v_n(\theta_n(t))), C(v(t)))\\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + \Lambda||v_n(\theta_n(t)) - v(t)||.\end{aligned}$$

Since $\lim_{n\to\infty} ||u_n(\theta_n(t)) - u_n(t)|| = 0$ and $\lim_{n\to\infty} ||v_n(\theta_n(t)) - v(t)|| = 0$ and $C(v(t))$ is closed, by passing to the limit when $n \to \infty$, in the preceding inequality, we get $u(t) \in C(v(t))$.

In order to prove the required claim we will use some limiting arguments developed in ([23], Theorems VII-18-19) involving lower semicontinuity of convex integral functionals. Indeed the inclusion

$$-\dot{u}_n(t) - g_n(t) \in N(C(v_n(\theta_n(t))); u_n(\theta_n(t)))$$

is equivalent to

$$\delta^*(-\dot{u}_n(t) - g_n(t); C(v_n(\theta_n(t)))) + \langle \dot{u}_n(t) + g_n(t), u_n(\theta_n(t))\rangle \le 0$$

by recalling that $u_n(\theta_n(t)) \in C(v_n(\theta_n(t)))$. By integrating on $[0, T]$ we have

$$0 \ge \int_0^T \delta^*(-\dot{u}_n(t) - g_n(t); C(v_n(\theta_n(t))))dt$$
$$+ \int_0^T \langle \dot{u}_n(t) + g_n(t), u_n(\theta_n(t))\rangle dt.$$

It is easy to see that

$$\lim_n \int_0^T \langle \dot{u}_n(t) + g_n(t), u_n(\theta_n(t))\rangle dt = \int_0^T \langle \dot{u}(t) + g(t), u(t)\rangle dt.$$

Furthermore as C is Λ-Lipschitzean with respect to the Hausdorff distance, we have the estimate

$$\int_0^T |\delta^*(-\dot{u}_n(t) - g_n(t); C(v_n(\theta_n(t)))) - \delta^*(-\dot{u}_n(t) - g_n(t); C(v(t)))|dt$$
$$\le \int_0^T \Lambda||\dot{u}_n(t) - g_n(t)||\, ||v_n(\theta_n(t)) - v(t)||dt$$

As $||v_n(\theta_n(t)) - v(t)|| \to 0$ these integrals go to 0 when $n \to \infty$ while by using the lower semicontinuity for convex integral functional [23]

$$\liminf_n \int_0^T \delta^*(-\dot{u}_n(t) - g_n(t); C(v(t)))dt$$
$$\ge \int_0^T \delta^*(-\dot{u}(t) - g(t); C(v(t)))dt.$$

Hence we deduce that

$$\liminf_n \int_0^T \delta^*(-\dot{u}_n(t) - g_n(t); C(v_n(\theta_n(t))))dt$$
$$\ge \int_0^T \delta^*(-\dot{u}(t) - g(t); C(v(t)))dt.$$

Whence it follows that

$$0 \geq [\liminf_n \int_0^T \delta^*(-\dot{u}_n(t) - g_n(t); C(v_n(\theta_n(t))))dt$$
$$+ \int_0^T \langle \dot{u}_n(t) + g_n(t), u_n(\theta_n(t))\rangle dt]$$
$$\geq \int_0^T \delta^*(-\dot{u}(t) - g(t); C(v(t))dt + \int_0^T \langle \dot{u}(t) + g(t), u(t)\rangle dt$$

that is

$$\int_0^T \delta^*(-\dot{u}(t) - g(t); C(v(t)))dt + \int_0^T \langle \dot{u}(t) + g(t), u(t)\rangle dt \leq 0$$

As $u(t) \in C(v(t))$, the last inequality is classically equivalent to

$$-\dot{u}(t) - g(t) \in N(C(v(t)); u(t)) \quad a.e.$$

Since $g(t) \in G(t, u(t), v(t)$ a.e, the proof is therefore complete. ■

Remarks.

(1) Theorem 3.2 is valid when the inclusion has the form

$$0 \in \dot{u}(t) + N(C(v(t)); u(t)) + F(t, u(t), v(t)) + G(t, u(t), v(t))$$

by repeating the arguments developed in the proof of Theorem 3.1 involving the multivalued Scorza-Dragoni theorem and Dungundji theorem. We refer to [26] for other variants of this second order evolution inclusion.

(2) In first order case, a different approach can be used to obtain a delayed version of theorem 3.2 (see [8], [18]); it consists to consider a partition of the interval $[0, T]$ and reduce the problem with delay to a problem without delay in each subinterval and apply the known results for this case.

4. Applications, Towards Variational Convergence

The techniques developed in section 3 can be used to prove the existence of absolutely continuous solution for second order evolution inclusions in Mechanics, Mathematical Economics and Control theory. For that purpose let us mention a useful corollary of Theorem 3.1. For simplicity we consider only the undelayed evolution case, i.e. $r = 0$.

Proposition 4.1. *Assume that $E = \mathbf{R}^d$, $A : E \to cc(E)$ is a maximal monotone operator with closed domain $D(A)$ satisfying $0 \in A(0)$ and the condition: for* *every* $l > 0$*,* $\sup\{|Ax|_0 : x \in D(A); ||x|| \leq l\} < \infty$*,* $G : [0,T] \times E \times E \to ck(E)$ *is upper semicontinuous such that* $|G(t,x,y)| \leq \alpha + \beta(||x|| + ||y||)$ *for all* $(t,x,y) \in [0,T] \times E \times E$*, where* α *and* β *are positive constant. Then, for every* $a \in D(A), b \in E$*, there exists an absolutely continuous mapping* $q : [0,T] \to E$ *such that*

$$(\mathcal{P}_4)\begin{cases} 0 \in \ddot{q}(t) + A\dot{q}(t) + G(t, \dot{q}(t), q(t)) \quad \text{a.e.} \quad t \in [0,T] \\ q(t) = b + \displaystyle\int_0^t \dot{q}(s)ds, \forall t \in [0,T] \\ \dot{q}(t) = a + \displaystyle\int_0^t \ddot{q}(s)ds, \forall t \in [0,T] \\ \dot{q}(t) \in D(A), \forall t \in [0,T] \\ \ddot{q} \in L^\infty_{\mathbf{R}^d}([0,T]) \end{cases}$$

The following is a corollary of Theorem 3.2

Proposition 4.2. *Let* H *be a separable Hilbert space and* $T > 0$*. Assume that* $C : H \to cc(H)$ *is a closed convex valued* Λ*-Lipschitzean mapping satisfying: there is a closed, ball-compact subset* D *of* H *such that,* $C(x) \subset D$ *for all* $x \in H$*. Assume that*

$$G : [0,T] \times H \times H \to cwk(H)$$

is convex weakly compact valued scalarly upper semicontinuous mapping such that

$$G(t,x,y) \subset (1 + ||x|| + ||y||)Z$$

for all $(t,x,y) \in [0,T] \times H \times H$*, for some convex weakly compact set* Z *in* H*. Then, for every every* $b \in H$*, and for every* $a \in C(b)$ *there exists an absolutely continuous mapping* $q : [0,T] \to H$ *such that*

$$(\mathcal{P}_2)\begin{cases} q(t) = b + \displaystyle\int_0^t \dot{q}(s)ds, \forall t \in [0,T], \\ \dot{q}(t) = a + \displaystyle\int_0^t \ddot{q}(s)ds, \forall t \in [0,T], \\ \text{with} \quad \ddot{q} \in L^\infty_H([0,T]) \quad \text{and} \ \dot{q}(t) \in C(q(t)), \forall t \in [0,T], \\ 0 \in \ddot{q}(t) + N(C(q(t)); \dot{q}(t)) + G(t, \dot{q}(t), q(t)) \ \textit{a.e.} \end{cases}$$

here $N(C(q(t)); \dot{q}(t))$ *denotes the normal cone of* $C(q(t))$ *at the point* $\dot{q}(t)$*.*

Proof. It is an easy reformulation of Theorem 3.2 in a separable Hilbert space. When $H = \mathbf{R}^d$, one can take $D = \mathbf{R}^d$, the ball-compactness assumption combined with the estimations of the derivatives of the approximants obtained in the proof of Theorem 3.2 ensures by Ascoli theorem the uniform convergence of theses approximants in the same vein as in the proof Theorem 3.2. So we omit the details. ■

Remarks. Other variants are available by introducing some anti-monotone conditions for the mapping C (see [10, 26]) for details.

In the present framework we don't expect to have uniqueness of solution by constrast to the first order case (5, 7, 9, 12, 18, 24, 29, 26). In the litterature, there are some existence and uniqueness results for $W^{2,2}_H([0,T])$-solution in the evolution inclusion in a Hilbert space H of the form

$$(\mathcal{P}_6)\begin{cases} \ddot{v}(t)+\gamma\dot{v}(t)\in\partial\varphi(v(t))+f(t), \quad a.e. \quad t\in[0,T], \\ v(T)=-v(0), \quad \dot{v}(T)=-\dot{v}(0). \end{cases}$$

here $\gamma\in\mathbf{R}^+$, f is antiperiodic and belongs to $L^2_H([0,T]$, and φ is proper, convex lower semicontinuous function with $\varphi(x)=\varphi(-x)$. See [2], [3] and the references therein. A recent result dealing with upper semicontinuous convex weakly compact perturbation F of $(\mathcal{P}_6)$

$$(\mathcal{P}_7)\begin{cases} \ddot{v}(t)+\gamma\dot{v}(t)\in\partial\varphi(v(t))+F(t,v(t)), \quad a.e. \quad t\in[0,T], \\ v(T)=-v(0), \quad \dot{v}(T)=-\dot{v}(0). \end{cases}$$

has been demonstrated in [11]. For notational convenience, we denote by $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$ the set of all absolutely continuous mappings $v:[0,T]\to\mathbf{R}^d$ such that $v(t)=v(0)+\int_0^t\dot{v}(s)ds$, and $\dot{v}(t)=\dot{v}(0)+\int_0^t\ddot{v}(s)ds$, $\forall t\in[0,T]$ where $\ddot{v}\in L^\infty_H([0,T])$. Before going further let us mention some particular cases of Proposition 4.1 and 4.2. Let $\varphi:[0,T]\times\mathbf{R}^d\to\mathbf{R}$ is continuous and satisfies a Lipschitz type condition

$$|\varphi(t,x)-\varphi(t,y)|\le k||x-y||$$

for all $(t,x,y)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}^d$ for some $k>0$, $\partial^c\varphi_t$ is the Clarke subdifferential of φ_t, then $F(t,x):=\partial^c\varphi_t(x)$ is scalarly upper semicontinuous with convex compact values see e.g. [36], [37]. Consequently the problems

$$\begin{cases} 0\in\ddot{v}(t)+A(t)\dot{v}(t)+G(t,\dot{v}(t),v(t))+\partial^c\varphi_t(v(t)) \\ v(0)=v_0;\dot{v}(0)=a\in D(A);\dot{v}(t)\in D(A),\forall t\in[0,T], \\ \ddot{v}\in L^\infty_H([0,T]), \end{cases}$$

and

$$\begin{cases} 0\in\ddot{v}(t)+N(C(v(t)),\dot{v}(t))+G(t,\dot{v}(t),v(t))+\partial^c\varphi_t(v(t)), \\ v(0)=v_0;\dot{v}(0)=a\in C(v_0);\dot{v}(t)\in C(v(t)),\forall t\in[0,T], \\ \ddot{v}\in L^\infty_H([0,T]), \end{cases}$$

admit at least a $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution. Theses inclusions shed a new light in second order evolution inclusions that we develop below using variational techniques essentially in the case when $\varphi_t(.)$ is convex lower semicontinuous. Let us mention first some new applications illustrating these considerations.

Proposition 4.3. *Assume that* $f : \mathbf{R}^d \to \mathbf{R}$ *is convex continuous so that* $x \to \partial f(x)$ *upper semicontinuous with convex compact values in* $\mathbf{R}^d$ *and* $g : \mathbf{R}^d \to \mathbf{R}^d$ *is a* k*-Lipschitzean mapping:* $|g(x) - g(y)| \leq k||x - y||$ *for all* $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ *for some* $k > 0$ *and is Gateaux-differentiable, so that* $\partial^c g(x) = \nabla g(x)$*,* $h : [0, T] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ *is continuous such that* $||h(t, x, y)|| \leq \alpha + \beta(||x|| + ||y||)$ *for all* $(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^d$*, where* α *and* β *are positive constant. Then any* $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$*-solution* u *of the inclusion*

$$\begin{cases} 0 \in \ddot{u}(t) + \partial f(\dot{u}(t))) + h(t, \dot{u}(t), u(t)) + \nabla g(u(t)) \quad a.e. \quad t \in [0, T] \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in \mathbf{R}^d \end{cases}$$

satisfies the variational equality

$$0 = \frac{d}{dt}[f(\dot{u}(t))] + \langle \ddot{u}(t), \ddot{u}(t) + h(t, \dot{u}(t), u(t)) + \nabla g(u(t)) \rangle \quad a.e.$$

In particular, any $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$*-solution* u *of the inclusion*

$$\begin{cases} 0 \in \ddot{u}(t) + \gamma \dot{u}(t) + \partial f(\dot{u}(t)) \quad a.e. \quad t \in [0, T] \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in \mathbf{R}^d \end{cases}$$

satisfies the variational equality

$$0 = \frac{d}{dt}[f(\dot{u}(t))] + \frac{\gamma}{2}||\dot{u}(t)||^2] + ||\ddot{u}(t)||^2 \quad a.e.$$

Proof. By virtue of Theorem 3.1, the problem

$$\begin{cases} 0 \in \ddot{u}(t) + \partial f(\dot{u}(t)) + \nabla g(u(t)) + h(t, \dot{u}(t), u(t)) \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in \mathbf{R}^d \end{cases}$$

admits at least a $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u since $A(x) := \partial f(x)$ is maximal monotone, upper semicontinuous with nonempty convex compact values. Multiplying scalarly the inclusion

$$-\ddot{u}(t) - h(t, \dot{u}(t), u(t)) \in \nabla g(u(t)) + \partial f(\dot{u}(t)) \quad a.e.$$

by $\ddot{u}$ and applying the chain rule theorem ([31], Theorem 1) for the convex continuous function f and the absolutely continuous function $\dot{u}$, gives

$$0 = \frac{d}{dt}[f(\dot{u}(t))] + \langle \ddot{u}(t), \ddot{u}(t) + h(t, \dot{u}(t), u(t)) + \nabla g(u(t))\rangle \quad a.e.$$

In particular, any $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u of the inclusion

$$\begin{cases} 0 \in \ddot{u}(t) + \gamma\dot{u}(t)) + \partial f(\dot{u}(t)) \quad \text{a.e.} \quad t \in [0,T] \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in \mathbf{R}^d \end{cases}$$

satisfies the variational equality

$$0 = \frac{d}{dt}[f(\dot{u}(t)) + \frac{\gamma}{2}||\dot{u}(t)||^2] + ||\ddot{u}(t)||^2 \quad a.e. \qquad \blacksquare$$

Here is a variant of Proposition 4.1.

Proposition 4.4. *Assume that $A : R^d \to \mathbf{R}^d$ is a single-valued maximal monotone operator with closed domain $D(A)$ satisfying: $0 = A(0)$ and for each $r > 0$, $\sup_{x\in D(A)\cap \overline{B}_{\mathbf{R}^d}(0,r)} ||Ax|| < \infty$, $g : \mathbf{R}^d \to \mathbf{R}^d$ is a k-Lipschitzean mapping: $|g(x) - g(y)| \leq k||x-y||$ for all $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$ for some $k > 0$ and $h : [0,T]\times\mathbf{R}^d\times\mathbf{R}^d \to \mathbf{R}^d$ is continuous such that $||h(t,x,y)|| \leq \alpha + \beta(||x|| + ||y||)$ for all $(t,x,y) \in [0,T]\times\mathbf{R}^d\times\mathbf{R}^d$, where α and β are positive constant. Then any $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u of the inclusion*

$$\begin{cases} 0 \in \ddot{u}(t) + A\dot{u}(t) + h(t, \dot{u}(t), u(t)) + \partial^c g(u(t)) \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in D(A), \dot{u}(t) \in D(A) \end{cases}$$

satisfies the variational equality

$$0 = \langle \ddot{u}(t) + A\dot{u}(t) + h(t, \dot{u}(t), u(t)), \dot{u}(t)\rangle + \frac{d}{dt}[g(u(t))] \quad a.e.$$

In particular, if A verifies $\langle x, Ax\rangle \geq \omega||x||^2$ for all $x \in D(A)$, for some positive constant $\omega > 0$, and γ is a positive constant, then any $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u of the inclusion

$$\begin{cases} 0 \in \ddot{u}(t) + A\dot{u}(t) + \gamma\dot{u}(t) + \partial^c g(u(t)) \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in D(A), \dot{u}(t) \in D(A) \end{cases}$$

satisfies the variational inequality

$$\frac{d}{dt}[g(u(t)) + \frac{1}{2}||\dot{u}(t)||^2]] \leq -(\omega + \gamma)||\dot{u}(t)||^2 \quad a.e.$$

Proof. Taking account into the remarks of Theorem 3.1 and the above considerations, the existence of at least a $W^{1,\infty}_{\mathbf{R}^d}([0,T])$ solution of the inclusion under consideration follows. As

$$-\ddot{u}(t) - A\dot{u}(t) - h(t, \dot{u}(t), u(t)) \in \partial^c g(u(t)) \quad a.e.$$

and u is absolutely continuous, by virtue of Theorem 2 of Chain rule formula in [31], we have,

$$0 = \langle \ddot{u}(t) + A\dot{u}(t) + h(t, \dot{u}(t), u(t)), \dot{u}(t)\rangle + \frac{d}{dt}[g(u(t))] \quad a.e.$$

If A verifies $\langle x, Ax\rangle \geq \omega||x||^2$ for all $x \in D(A)$, for some positive constant $\omega > 0$ and γ is > 0, then any $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u of the inclusion

$$\begin{cases} 0 \in \ddot{u}(t) + A\dot{u}(t) + \gamma\dot{u}(t) + \partial^c g(u(t)) \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = a \in D(A), \dot{u}(t) \in D(A) \end{cases}$$

satisfies

$$\begin{aligned} 0 &= \frac{d}{dt}[g(u(t))] + \langle \ddot{u}(t) + A\dot{u}(t) + \gamma\dot{u}(t), \dot{u}(t)\rangle \\ &\geq \frac{d}{dt}[g(u(t))] + \frac{1}{2}\frac{d}{dt}||\dot{u}(t)||^2] + \omega||\dot{u}(t)||^2 + \gamma||\dot{u}(t)||^2 \quad a.e. \end{aligned}$$

that is

$$\frac{d}{dt}[g(u(t)) + \frac{1}{2}||\dot{u}(t)||^2] \leq -(\omega+\gamma)||\dot{u}(t)||^2 \quad \text{a.e.} \qquad \blacksquare$$

There is a useful corollary of Proposition 4.4 that corresponds to the case when $A(t) = 0$ for all $t \in [0,T]$ and $E = R^d$.

Corollary 4.1. *Assume that $E = \mathbf{R}^d$, $M : \mathbf{R}^d \to R^d$ is a continuous linear mapping, and let $g : \mathbf{R}^d \to \mathbf{R}^d$ is a k-Lipschitzean mapping. Then the problem*

$$\begin{cases} 0 \in \ddot{u}(t) + M\dot{u}(t) + \partial^c g(u(t)) \quad a.e. \quad t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \end{cases}$$

admits at least a $\mathcal{W}^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution u, and any solution u of the preceding problem satisfies the variational equality

$$-\langle M\dot{u}(t), \dot{u}(t)\rangle = \frac{d}{dt}[g(u(t)) + \frac{1}{2}||\dot{u}(t)||^2].$$

In the light of the preceding applications, several open problems appear here. For instance, it is interesting to study the inclusion

$$\begin{cases} 0 \in \ddot{u}(t) + A\dot{u}(t) + Mu(t) + \partial^c g(u(t)) & \text{a.e.} \quad t \in [0,T] \\ u(0) = u_0 \in \mathbf{R}^d; \dot{u}(0) = \dot{u}_0 \in \mathbf{R}^d \end{cases}$$

when M is a linear continuous operator in $\mathbf{R}^d$, and g is a proper, lower semicontinuous convex function. For this purpose we will provide some variational versions of the preceding problem. Similar results in this direction are obtained by ([2-4, 11, 15, 20, 35).

Let us recall a useful Gronwall type lemma [18].

Lemma 4.1. (A Gronwall-like inequality). *Let* $p, q, r : [0,T] \to [0,\infty[$ *be three nonnegative Lebesgue integrable functions such that for almost all* $t \in [0,T]$

$$r(t) \leq p(t) + q(t) \int_0^t r(s)\, ds.$$

Then

$$r(t) \leq p(t) + q(t) \int_0^t [p(s) \exp(\int_s^t q(\tau)\, d\tau)]\, ds$$

for all $t \in [0,T]$.

We recall below some notations and summarize some results which describe the limiting behavior of a bounded sequence in $L^1_H([0,T])$. See ([19], Proposition 6.5.17).

Proposition 4.5. *Let* H *be a separable Hilbert space. Let* (ζ_n) *be a bounded sequence in* $L^1_H([0,T])$*. Then the following hold:*

(1) (ζ_n) *(up to an extracted subsequence) stably converges to a Young measure* ν *that is, there exist a subsequence* (ζ'_n) *of* (ζ_n) *and a Young measure* ν *belonging to the space of Young measure* $\mathcal{Y}([0,T]; H_\sigma)$ *with* $t \mapsto bar(\nu_t) \in L^1_H([0,T])$ *(here* $bar(\nu_t)$ *denotes the barycenter of* ν_t*) such that*

$$\lim_{n\to\infty} \int_0^T h(t, \zeta'_n(t)))\, dt) = \int_0^T [\int_H h(t,x)\, \nu_t(dx)]\, dt$$

for all bounded Carathéodory integrands $h : [0,T] \times H_\sigma \to \mathbf{R}$,

(2) (ζ_n) *(up to an extracted subsequence) weakly biting converges to an integrable function* $f \in L^1_H([0,T])$*, which means that, there is a subsequence*

(ζ'_m) of (ζ_n) and an increasing sequence of Lebesgue-measurable sets (A_p) with $\lim_p \lambda(A_p) = 1$ and $f \in L^1_H([0,T])$ such that, for each p,

$$\lim_{m\to\infty} \int_{A_p} \langle h(t), \zeta'_m(t)\rangle\, dt = \int_{A_p} \langle h(t), f(t)\rangle\, dt$$

for all $h \in L^\infty_H([0,T])$,

(3) *(ζ_n) (up to an extracted subsequence) Komlós converges to an integrable function $g \in L^1_H([0,T])$, which means that, there is a subsequence $(\zeta_{\beta(m)})$ and an integrable function $g \in L^1_H([0,T])$, such that*

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{j=1}^n \zeta_{\gamma(j)}(t) = g(t),\ a.e. \in [0,T],$$

for every subsequence $(f_{\gamma(n)})$ of $(f_{\beta(n)})$.

(4) *There is a filter $\mathcal{U}$ finer than the Fréchet filter such that $\mathcal{U} - \lim_n \zeta_n = l \in (L^\infty_H)'_{weak}$ where $(L^\infty_H)'_{weak}$ is the second dual of $L^1_H([0,T])$.*

Let $w_{l_a} \in L^1_H([0,T])$ be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s.

If we have considered the same extracted subsequence in $(1), (2), (3), (4)$, then one has

$$f(t) = g(t) = bar(\nu_t) = w_{l_a}(t)\ a.e.\ t \in [0,T]$$

Combining the techniques in Proposition 4.4, we are able to provide a variational convergence problem for the inclusion of the form

$$\begin{cases} 0 \in \ddot{u}(t) + M\dot{u}(t) + \partial\varphi(u(t)), \\ u(0) = u_0; \dot{u}(0) = \dot{u}_0. \end{cases}$$

here M is a linear continuous operator in $\mathbf{R}^d$, and φ is convex lower semicontinuous. To illustrate these facts and to end this paper, let us summarize them in the results below.

Proposition 4.6. *Assume that $M : R^d \to \mathbf{R}^d$ is linear continuous operator. Let $n \in \mathbf{N}$ and $\varphi_n : \mathbf{R}^d \to \mathbf{R}^+$ be a convex, Lipchitzean function and let φ_∞ be a nonnegative l.s.c proper function defined on $\mathbf{R}^d$. For each $n \in \mathbf{N}$, let u^n be a $W^{1,\infty}_{\mathbf{R}^d}([0,T])$-solution to the problem*

$$(\mathcal{Q}_n) \begin{cases} 0 \in \ddot{u}^n(t) + M\dot{u}^n(t) + \partial\varphi_n(u^n(t)), \\ u^n(0) = u^n_0; \dot{u}^n(0) = \dot{u}^n_0. \end{cases}$$

Assume that

(i) *φ_n epi-converges to φ_∞.*

(ii) $\sup_{n\geq 1} ||u_0^n|| < +\infty$, $\sup_{n\geq 1} ||\dot{u}_0^n|| < +\infty$ *and* $\sup_{n\geq 1} \varphi_n(u_0^n) < +\infty$.

(iii) *There exist $r_0 > 0$ and $x_0 \in \mathbf{R}^d$ such that*

$$\sup_{n\in\mathbf{N}} \sup_{v\in\overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}} \int_0^T \varphi_n(x_0 + r_0 v(t))) < +\infty$$

here $\overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$ is the closed unit ball in $L^\infty_{\mathbf{R}^d}([0,T])$.

(a) *Then up to extracted subsequences, (u^n) converges uniformly to an absolutely continuous function u^∞ and $(\dot{u}^n)$ pointwisely converges to a BV function v^∞ with $v^\infty = \dot{u}^\infty$, $(d\dot{u}^n)$ weakly converges to the differential measure dv^∞ of the BV function v^∞, and $(\ddot{u}^n)$ weakly biting converges to a function $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$ which satisfy the variational inclusion*

$$(\mathcal{Q}_\infty) \qquad 0 \in \zeta^\infty + M\dot{u}^\infty + \partial I_{\varphi_\infty}(u^\infty)$$

here $\partial I_{\varphi_\infty}$ denotes the subdifferential of the convex lower semicontinuous integral functional I_{φ_∞} defined on $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt,\ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Furthermore $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt.$

(b) *There are a filter $\mathcal{U}$ finer than the Fréchet filter, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ such that*

$$\mathcal{U} - \lim_n[-\ddot{u}^n - M\dot{u}^n] = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ is the second dual of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ and $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ such that

$$\lim_n[-\ddot{u}^n - M\dot{u}^n] = m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$$

here $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s. Then

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ *so that*

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-\zeta^\infty - M\dot{u}^\infty) + \delta^*(l_s, dom I_{\varphi_\infty})$$

here φ^*_∞ *is the conjugate of* φ_∞, $I_{\varphi^*_\infty}$ *the integral functional defined on* $L^1_{\mathbf{R}^d}([0,T])$ *associated with* φ^*_∞, $I^*_{\varphi_\infty}$ *the conjugate of the integral functional* I_{φ_∞}, $dom I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}$ *and*

$$\langle m, h\rangle = \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt + \langle m_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

with $\langle m_s, h\rangle = l_s(h)$, $\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$. *Further* m *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex lower semicontinuous integral functional* J_{φ_∞} *defined on* $\mathcal{C}_{\mathbf{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt, \ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

(c) *Consequently the density* $-\zeta^\infty - M\dot{u}^\infty$ *of the absolutely continuous part* m_a

$$m_a(h) := \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

satisfies the inclusion

$$-\zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e..}$$

and for any nonnegative measure θ *on* $[0,T]$ *with respect to which* m_s *is absolutely continuous*

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ *denotes the recession function of* φ^*_∞. *Furthermore, the following equality holds*

$$\langle dy^\infty, h\rangle = \int_0^T \langle \zeta^\infty(t), h(t)\rangle dt - \langle m_s, h\rangle$$

for all $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$.

Proof.

Step 1. $||\dot{u}^n(.)||$ and $\varphi_n(u_n(.))$ are uniformly bounded.

By Theorem 4.1, there is a $W^{1,\infty}_{\mathbf{R}^d}([0,T])$ solution, u^n, of the problem under consideration. Applying the inclusion

$$-\ddot{u}^n(t) - M\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

and the chain rule theorem ([31], Theorem 2) yields

$$-\langle \dot{u}^n(t), \ddot{u}^n(t)\rangle - \langle \dot{u}^n(t), M\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))]$$

that is

$$-\langle M\dot{u}^n(t), \dot{u}^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2].$$

By integrating on $[0,t]$ this equality we get

$$\begin{aligned}
&\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 \\
&= \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 - \int_0^t \langle M\dot{u}^n(s), \dot{u}^n(s)\rangle ds \\
&\leq \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 + \gamma\int_0^t ||\dot{u}^n(s)||^2 ds
\end{aligned}$$

here $\gamma = ||M||$ is the norm of the operator M. We will assume $\gamma > 0$. Then from (ii), the preceding estimate and the Gronwall like inequality (Lemma 4.1), it is immediate that

$$\text{(4.6.2)} \qquad \sup_{n\geq 1}\sup_{t\in[0,T]} ||\dot{u}^n(t)|| < +\infty \quad \text{and} \quad \sup_{n\geq 1}\sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty.$$

Step 2. Estimation of $||\ddot{u}^n(.)||$. As

$$z^n(t) := -\ddot{u}^n(t) - M\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

by the subdifferential inequality for convex lower semi continuous functions we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. Now let $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, the closed unit ball of $L^\infty_{\mathbf{R}^d}[0,T])$. By taking $x = w(t) := x_0 + r_0 v(t)$ in the preceding inequality we get

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle w(t) - u^n(t), z^n(t)\rangle.$$

Integrating the preceding inequality gives

$$\begin{aligned}
&\int_0^T \langle x_0 + r_0 v(t) - u^n(t), z^n(t)\rangle dt \\
&= \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt + r_0\int_0^T \langle v(t), z^n(t)\rangle dt \\
&\leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt - \int_0^T \varphi_n(u^n(t))dt.
\end{aligned}$$

Whence follows

$$
\begin{aligned}
(4.6.3)\qquad & r_0\int_0^T \langle v(t), z^n(t)\rangle dt \le \int_0^T \varphi_n(x_0+r_0v(t))dt\\
&-\int_0^T \varphi_n(u^n(t))dt - \int_0^T \langle x_0-u^n(t), z^n(t)\rangle dt.
\end{aligned}
$$

We compute the last integral in the preceding inequality. For simplicity, let us set $v^n(t) = u^n(t) - x_0$ for all $t \in [0,T]$. By integration by parts and taking account into (4.6.2) we have

$$
\begin{aligned}
& -\int_0^T \langle x_0-u^n(t), z^n(t)\rangle dt = -\int_0^T \langle v^n(t), \ddot{v}^n(t)+M\dot{v}^n(t)\rangle dt\\
(4.6.4)\qquad &= -[\langle v^n(t), \dot{v}^n(t)+Mv^n(t)]_0^T + \int_0^T \langle \dot{v}^n(t), \dot{v}^n(t)+Mv^n(t)\rangle dt\\
&\le -\langle v^n(T), \dot{v}^n(T)\rangle + \langle v^n(0), \dot{v}^n(0)\rangle - \langle Mv^n(T), v^n(T)\rangle\\
&\quad +\langle Mv^n(0), v^n(0)\rangle + \int_0^T ||\dot{v}^n(t)||^2dt + \int_0^T \langle \dot{v}^n(t), Mv^n(t)\rangle dt.
\end{aligned}
$$

By $(4.6.2) - (4.6.4)$, we get

$$
(4.6.5)\qquad r_0\textstyle\int_0^T \langle v(t), z^n(t)\rangle dt \le \int_0^T \varphi_n(x_0+r_0v(t))dt + L
$$

for all $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, here L is a suitable positive constant independent of $n \in \mathbf{N}$. By (iii) we conclude that $(\ddot{u}^n + M\dot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, then so is $(\ddot{u}^n)$. It turns out that the sequence $(\dot{u}^n)$ of absolutely continuous functions is bounded in variation and by Helly theorem, we may assume that $(\dot{u}^n)$ pointwisely converges to a BV function $v^\infty : [0,T] \to \mathbb{R}^d$ and the sequence (u^n) converges uniformly to an absolutely continuous function u^∞ with $\dot{u}^\infty = v^\infty$ a.e. At this point, it is clear that $(\dot{u}_n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to v^∞, using (4.6.2) and the dominated convergence theorem. Hence $(M\dot{u}^n(.))$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to $Mv^\infty(.)$.

Step 3. Weak biting limit of $\ddot{u}_n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ weakly biting converges to a function $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$, that is, there exists a decreasing sequence of Lebesgue-measurable sets (B_p) with $\lim_p \lambda(B_p) = 0$ such that the restriction of $(\ddot{u}_n)$ on each B_p^c converges weakly in $L^1_{\mathbf{R}^d}([0,T])$ to ζ^∞. Noting that $(M\dot{u}^n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to Mv^∞. It follows that the restriction of $(z^n = -\ddot{u}^n - M\dot{u}^n)$ to each B_p^c weakly converges in $L^1_{\mathbf{R}^d}([0,T])$ to $z^\infty := -\zeta^\infty - Mv^\infty$, because

$$
\lim_n \int_B \langle \ddot{u}_n + M\dot{u}_n, h\rangle\, dt = \int_B \langle \zeta^\infty + Mv^\infty, h\rangle\, dt
$$

for every $B \in B_p^c \cap \mathcal{L}([0,T])$ and for every function $h \in L^\infty_{\mathbf{R}^d}([0,T])$.

Step 4. Localization of the limits: $z^\infty = -\zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty)$.

We will adapt the techniques developed in ([20], Lemma 3.7, Proposition 4.2). As (φ_n) epiconverges to φ_∞, by virtue of Lemma 3.4 in [20] we have

$$\liminf_n \int_B \varphi_n(u^n(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt,$$

for every $B \in \mathcal{L}([0,T])$. Let $h \in L^\infty_{\mathbf{R}^d}([0,T])$. Using (4.6.2) and applying Lemma 3.7 in [20] provides a bounded sequence (h^n) in $L^\infty_H([0,T])$, such that (h^n) pointwisely converges to h and such that

$$\limsup_n \int_B \varphi_n(h^n(t))\,dt \leq \int_B \varphi_\infty(h(t))\,dt$$

for every $B \in \mathcal{L}([0,T])$. Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. By substituting x by $h^n(t)$ in this inequality and by integrating on each $B \in B_p^c \cap \mathcal{L}([0,T])$,

$$\int_B \varphi_n(h^n(t))\,dt \geq \int_B \varphi_n(u^n(t))\,dt + \int_B \langle h^n(t) - u^n(t), z^n(t)\rangle\,dt$$

and passing to the limit in the preceding inequality when n goes to $+\infty$, we get

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle h(t) - u^\infty(t), z^\infty(t)\rangle\,dt$$

As this inequality is true on each $B \cap B_p^c$

$$\begin{aligned}\int_{B\cap B_p^c} \varphi_\infty(h(t))\,dt \geq & \int_{B\cap B_p^c} \varphi_\infty(u^\infty(t))\,dt \\ & + \int_{B\cap B_p^c} \langle h(t) - u^\infty(t), z^\infty(t)\rangle\,dt\end{aligned}$$

and $B_p^c \uparrow [0,T]$, by passing to the limit when p goes to ∞ in the last inequality, we get

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle z^\infty(t), h(t) - u^\infty(t)\rangle\,dt$$

for all $B \in \mathcal{L}([0,T])$ and for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$. In other words,

$$z^\infty = -\zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty).$$

Step 5. $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt$.

From the chain rule theorem given in Step 1, recall that

$$-\langle \dot{u}^n(t), \ddot{u}^n(t) + M\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))]$$

that is

$$\langle \dot{u}^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))].$$

By the estimate (4.6.2) and the boundedness in $L^1_{\mathbf{R}^d}([0,T])$ of (z^n), it is immediate that $(\frac{d}{dt}[\varphi_n(u_n(t))])$ is bounded in $L^1_{\mathbf{R}}([0,T])$ so that $(\varphi_n(u_n(.))$ is bounded in variation. By Helly theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function ψ. By (4.6.2), $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbf{R}}([0,T])$ to ψ. In particular, for every $k \in L^\infty_{\mathbf{R}^+}([0,T])$ we have

$$\lim_{n\to\infty} \int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\psi(t)dt.$$

Using this fact and repeating the biting arguments via the epi-limit results given in Step 4, it is easy to see that

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \psi(t)\,dt + \int_B \langle z^\infty(t), h(t) - u^\infty(t)\rangle\,dt$$

for all $B \in \mathcal{L}([0,T])$ and for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$. In particular, we get the estimate

$$\int_B \varphi_\infty(u^\infty(t))\,dt \geq \int_B \psi(t)\,dt$$

for all $B \in \mathcal{L}([0,T])$. Again by the epi-lower convergence result in Step 4, we have

$$\int_B \psi(t)\,dt = \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt = \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt$$

for all $B \in \mathcal{L}([0,T])$. It turns out that $\varphi_\infty(u^\infty(t)) = \psi(t)$ a.e.

Step 6. Localization of further limits and final step.

As $(z^n = -\ddot{u}^n - M\dot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$ in view of Step 3, it is relatively compact in the second dual $L^\infty_{\mathbf{R}^d}([0,T])'$ of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the weak topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$. Furthermore, (z^n) can be viewed as a bounded sequence in $\mathcal{C}_{\mathbf{R}^d}([0,T])'$. Hence there are a filter $\mathcal{U}$ finer than the Fréchet filter, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ and $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$ such that

$$\mathcal{U} - \lim_n z^n = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak} \tag{4.6.6}$$

and

$$\lim_n z^n = m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak} \tag{4.6.7}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ is the second dual of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ and $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$, because $\mathcal{C}_{\mathbf{R}^d}([0,T])$ is a separable Banach space for the norm sup, so that we may assume by extracting subsequences that (z^n) weakly converges to $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s, in the sense there is an decreasing sequence (A_n) of Lebesgue measurable sets in $[0,T]$ with $A_n \downarrow \emptyset$ such that $l_s(h) = l_s(1_{A_n} h)$ for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ and for all $n \geq 1$. As $(z^n = -\ddot{u}^n - M\dot{u}^n)$ weakly biting converges to $z^\infty = -\zeta^\infty(t) - M\dot{u}^\infty$ in Step 4, it is already seen (cf. Proposition 4.5) that

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$, shortly $z^\infty = -\zeta^\infty(t) - M\dot{u}^\infty$ coincides a.e. with the density of the absolutely continuous part l_a. By ([23], [34]) we have

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-\zeta^\infty - M\dot{u}^\infty) + \delta^*(l_s, domI_{\varphi_\infty})$$

here φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ is the integral functional defined on $L^1_{\mathbf{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ is the conjugate of the integral functional I_{φ_∞} and

$$domI_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}.$$

Using the inclusion

$$z^\infty = -\zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty).$$

that is

$$I_{\varphi^*_\infty}(-\zeta^\infty - M\dot{u}^\infty) = \langle -\zeta^\infty - M\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty)$$

we see that

$$I^*_{\varphi_\infty}(l) = \langle -\zeta^\infty - M\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty) + \delta^*(l_s, domI_{\varphi_\infty}).$$

Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. By substituting x by $h(t)$ in this inequality, here $h \in C_{\mathbb{R}^d}([0,T])$, and by integrating

$$\int_0^T \varphi_n(h(t))\,dt \geq \int_0^T \varphi_n(u^n(t))\,dt + \int_0^T \langle h(t) - u^n(t), z^n(t)\rangle\,dt.$$

Arguing as in Step 4 by passing to the limit in the preceding inequality, involving the epilimsup property for integral functionals $\int_0^T \varphi_n(h(t))dt$ defined on $L^\infty_{\mathbf{R}^d}([0,T])$, it is easy to see that

$$\int_0^T \varphi_\infty(h(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \langle h - u^\infty, m\rangle.$$

Since this holds, in particular, when $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$, we conclude that m belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

As $(z^n = -\ddot{u}^n - M\dot{u}^n)$ weakly biting converges to $z^\infty = -\zeta^\infty(t) + M\dot{u}^\infty$ in Step 4, we see that

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ (see Proposition 4.5) so that

$$l(h) = \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt + l_s(h), \quad \forall h \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Now let $B : \mathcal{C}_{\mathbf{R}^d}([0,T]) \to L^\infty_{\mathbf{R}^d}([0,T])$ be the continuous injection and let $B^* : L^\infty_{\mathbf{R}^d}([0,T])' \to \mathcal{C}_{\mathbf{R}^d}([0,T])'$ be the adjoint of B given by

$$\langle B^* l, h\rangle = \langle l, Bh\rangle = \langle l, h\rangle, \quad \forall l \in L^\infty_{\mathbf{R}^d}([0,T])', \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

Then we have $B^* l = B^* l_a + B^* l_s$, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ being the limit of $(z_n = -\ddot{u}^n - M\dot{u}^n)$ under the filter $\mathcal{U}$ given in section 4 and $l = l_a + l_s$ being the decomposition of l in absolutely continuous part l_a and singular part l_s. It follows that

$$\langle B^* l, h\rangle = \langle B^* l_a, h\rangle + \langle B^* l_s, h\rangle = \langle l_a, h\rangle + \langle l_s, h\rangle$$

for all $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$. But it is already seen that

$$\begin{aligned}\langle l_a, h\rangle &= \langle -\zeta^\infty - M\dot{u}^\infty, h\rangle \\ &= \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in L^\infty_{\mathbf{R}^d}([0,T])\end{aligned}$$

so that the measure B^*l_a is absolutely continuous

$$\langle B^*l_a, h\rangle = \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

and its density $-\zeta^\infty - M\dot{u}^\infty$ satisfies the inclusion

$$-\zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e..}$$

and the singular part B^*l_s satisfies the equation

$$\langle B^*l_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

As $B^*l = m$, using (4.6.6)-(4.6.7), it turns out that m is the sum of the absolutely continuous measure m_a with

$$\langle m_a, h\rangle = \int_0^T \langle -\zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

and the singular part m_s given by

$$\langle m_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

which satisfies the property: for any nonnegative measure θ on $[0,T]$ with respect to which m_s is absolutely continuous

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ denotes the recession function of φ^*_∞. Indeed, as m belongs to $\partial J_{\varphi_\infty}(u^\infty)$ by applying Theorem 5 in [34] we have

$$J^*_{\varphi_\infty}(m) = I_{\varphi^*_\infty}(\frac{dm_a}{dt}) + \int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) \tag{4.6.8}$$

with

$$I_{\varphi^*_\infty}(v) := \int_0^T \varphi^*_\infty(v(t))dt, \forall v \in L^1_{\mathbf{R}^d}([0,T]).$$

Recall that

$$\frac{dm_a}{dt} = -\zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty)$$

that is

$$I_{\varphi^*_\infty}(\frac{dm_a}{dt}) = \langle -\zeta^\infty - M\dot{u}^\infty, u^\infty\rangle_{\langle L^1_{\mathbf{R}^d}([0,T]), L^\infty_{\mathbf{R}^d}([0,T])\rangle} - I_{\varphi_\infty}(u^\infty) \tag{4.6.9}$$

From (4.6.9) we deduce

$$\begin{aligned}
J^*_{\varphi_\infty}(m) &= \langle u^\infty, m\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]),\mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - J_{\varphi_\infty}(u^\infty)\\
&= \langle u^\infty, m\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]),\mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - I_{\varphi_\infty}(u^\infty)\\
&= \int_0^T \langle u^\infty(t), -\zeta^\infty(t) - M\dot{u}^\infty(t)\rangle\, dt\\
&\quad + \int_0^T \left\langle u^\infty(t), \frac{dm_s}{d\theta}(t)\right\rangle d\theta(t)) - I_{\varphi_\infty}(u^\infty)\\
&= I_{\varphi^*_\infty}\left(\frac{dm_a}{dt}\right) + \int_0^T \left\langle u^\infty(t), \frac{dm_s}{d\theta}(t)\right\rangle d\theta(t)).
\end{aligned}$$

Coming back to (4.6.8) we get the equality

$$\int_0^T h_{\varphi^*_\infty}\left(\frac{dm_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \left\langle u^\infty(t), \frac{dm_s}{d\theta}(t)\right\rangle d\theta(t)).$$

It remain to check the last equality

$$dv^\infty = \zeta^\infty dt - m_s.$$

Indeed, since $(z^n = -\ddot{u}^n - M\dot{u}^n)$ converges in $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ to m with

$$m = [-\zeta^\infty - M\dot{u}^\infty]dt + m_s$$

and $(d\dot{u}^n)$ converges in $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ to dv^∞ by virtue of Helly theorem, it follows that $(z^n = -\ddot{u}^n - M\dot{u}^n)$ converges in $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ to $-dv^\infty - M\dot{u}^\infty dt$. Hence we get

$$m = [-\zeta^\infty - M\dot{u}^\infty]dt + m_s = -dv^\infty - M\dot{u}^\infty dt$$

thereby proving the required equality. ■

Remark. Combining biting argument with the characterization of the decomposition formula in the dual of $L^\infty_{\mathbf{R}^d}([0,T])$ allows to localize the limits under consideration and their relationships via Proposition 4.5 and the continuous injection $B : \mathcal{C}_{\mathbf{R}^d}([0,T]) \to L^\infty_{\mathbf{R}^d}([0,T])$, namely the absolute continuous part m_a of the measure limit m and its singular part m_s. At this point, it is easy to see that, up to extracted subsequences, $(z_n = -\ddot{u}^n - M\dot{u}^n)$ stably converges to a Young measure $\nu^\infty \in \mathcal{Y}([0,T], \mathcal{M}^1_+(\mathbf{R}^d))$ with

$$bar(\nu_t) = \int_{\mathbf{R}^d} x\, \nu_t(dx) = -\zeta^\infty(t) - M\dot{u}^\infty(t) \quad a.e. \quad t \in [0,T]).$$

Taking account into the above remark and the results given in Proposition 4.6 and its proofs, we obtain

Corollary 4.2. *Under the hypotheses and notations of Proposition 4.6, assume that φ_n^* is non negative for all $n \in \mathbf{N} \cup \{\infty\}$ and $(\varphi^*_n)_{n\geq 1}$ epilower converges to φ^*_∞, then the following hold:*

$$(^*) \quad \liminf_n \int_0^T \varphi_n^*(-\ddot{u}^n(t) - M\dot{u}^n(t))\, dt \geq \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt.$$

Consequently the limits under consideration satisfy

$$(^{**}) \quad \begin{aligned} 0 \geq & \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt - \int_0^T \langle bar(\nu_t^\infty), u^\infty(t)\rangle\, dt \\ & + \int_0^T \varphi_\infty(u^\infty(t))\, dt - \int_0^T h_{\varphi^*_\infty}\left(\frac{dm_s}{d\theta}(t)\right) d\theta(t). \end{aligned}$$

Proof. As (φ_n^*) epilower converges to φ^*_∞ and $(z_n = -\ddot{u}^n - M\dot{u}^n)$ stably converges to $\nu^\infty \in \mathcal{Y}([0,T], \mathcal{M}^1_+(\mathbf{R}^d))$, by virtue of Lemma 3.4 in [20], we have

$$(^*) \quad \liminf_n \int_0^T \varphi_n^*(-\ddot{u}^n(t) - M\dot{u}^n(t))\, dt \geq \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt.$$

Using the results obtained in the proof of Proposition 4.6 and $(^*)$, it is not difficult to check that

$$\begin{aligned} 0 \geq & \liminf_n \left[\int_0^T \varphi_n^*(-\ddot{u}^n(t) - M\dot{u}^n(t))\ dt \right. \\ & \left. + \int_0^T \langle \ddot{u}^n(t) + M\dot{u}^n(t), u^n(t)\rangle dt \ + \int_0^T \varphi_n(u^n(t)\, dt\right] \\ \geq & \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt - \langle u^\infty, m\rangle + \int_0^T \varphi_\infty(u^\infty(t))\, dt \\ = & \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt + \int_0^T \langle \zeta^\infty(t) + M\dot{u}^\infty(t), u^\infty(t)\rangle\, dt \\ & - \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)) + \int_0^T \varphi_\infty(u^\infty(t))\, dt \\ = & \int_0^T \left[\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)\right] dt + \int_0^T \langle \zeta^\infty(t) + M\dot{u}^\infty(t), u^\infty(t)\rangle\, dt \\ & - \int_0^T h_{\varphi^*_\infty}\left(\frac{dm_s}{d\theta}(t)\right) d\theta(t) + \int_0^T \varphi_\infty(u^\infty(t))\, dt \end{aligned}$$

thus proving (**). ■

Remark. The techniques of the proof in Proposition 4.6 permit to treat also similar variational problems involving second order evolution, in particular, the following one involving anti-periodic boundary conditions

$$\begin{cases} 0 = \ddot{u}^n(t) + \gamma \dot{u}^n(t) - \nabla\psi(u^n) + \nabla\varphi_n(u^n(t)) + f_n(t), \\ u^n(T) = -u^n(0); \dot{u}^n(T) = \dot{u}^n(0). \end{cases}$$

where γ is a positive constant, ψ and φ_n are convex lipschitzian, Gateaux differentiable, even, functions and φ_n epiconverges to a convex lower semicontinuous even function φ_∞, (f_n) is a sequence in $L^2_H([0,T])$ weakly converges to a function $f_\infty \in L^2_H([0,T])$. For shortness we omit the details.

References

1. D. Azzam and M. Bounkhel, Existence results on the second-order nonconvex sweeping processes with perturbationss, *Set-Valued Analysis*, **12** (2004), 291-318.

2. N. Apreutesei, Continuous dependence on data for quasiautonomous nonlinear boundary value problems, *Abstract and Applied Analysis*, **1** (2005), 67-86.

3. A. R. Aftabizadeh and N. H. Pavel, Boundary value problems for second order problems for second order differential equations and a convex problem of Bolza, *Differential Integral Equations*, **2(4)** (1989) 495-509.

4. H. Attouch, A. Cabot and P. Redont, The Dynamics of Elastic Shocks via Epigraphical Regularization of a Differential Inclusion. Barrier and Penalty Approximations, Advances in Mathematical Sciences ans Applications, *Gakkotosho Tokyo*, **12(1)** (2002), 273-306.

5. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Int. Publ., Leyden, 1976.

6. H. Benabdellah, Differential inclusions on closed sets in Banach spaces with application to sweeping process, *TMNA*, **23(1)** (2004), 115-148.

7. H. Benabdellah and A. Faik, Perturbations convexes et nonconvexes des équations d'évolution, *Portugaliae Matematica*, **53(2)** (1996), 187-208.

8. M. Bounkhel and M. Yarou, Existence results for first and second order nonconvex sweeping process with delay, *Portugaliae Matematica*, **61(2)** (2004), 2007-2030.

9. H. Brezis, *Opérateurs maximaux monotones et Semi-groupes de contractions dans un espace de Hilbert*, North Holland 1979.

10. C. Castaing, Quelques problèmes d'évolution du second ordre, *Sem. Anal. Convexe, Montpellier*, 1988, Exposé No 5.

11. C. Castaing and T. Haddad, *Relaxation problem for second order evolution inclusion and a Bolza type problem*, Working paper, June 2007.

12. C. Castaing and A. G. Ibrahim, Functional evolution equations governed by m-accretive operators, *Adv. Math. Econ.*, **5** (2003), 23-54.

13. C. Castaing and M. D. P. Monteiro Marques, Topological properties of solutions sets for sweeping processes with delay, *Portugaliae Mathematica*, **54(4)** (1997), 485-507

14. C. Castaing and S. Marcellin, Evolution Inclusions with PLN Functions and Applications to Viscosity and Control, *JNCA*, **8(2)** (2007), 227-255.

15. C. Castaing, A. G. Ibrahim and S. Marcellin, *Variational inequalities involving pln functions with application to evolution inclusions*, Working paper, November 2006.

16. C. Castaing and M. D. P. Monteiro Marques, A multivalued version of Scorza-Dragoni's theorem with an application to normal integrands, *Bull. Pol. Acad. Sci. Mathematics*, **42** (1994) 133-140.

17. C. Castaing and M. D. P. Monteiro Marques, Periodic solutions of evolution problem associated with moving convex sets Discussiones Mathematicae *Differential inclusions*, **15** (1995), 99-127

18. C. Castaing, A. Salvadori and L. Thibault, Functional evolution equations governed by nonconvex sweeeping process, *Journal of Nonlinear and Convex Analysis*, **2(2)** (2001), 217-241.

19. C. Castaing, P. Raynaud de Fitte and M. Valadier, Young measures on Topological Spaces. With Applications in Control Theory and Probability Theory, Kluwer Academic Publishers, Dordrecht, 2004.

20. C. Castaing, P. Raynaud de Fitte and A. Salvadori, Some variational convergence results with application to evolution inclusions, *Adv. Math. Econ.*, **8** (2006), 33-73.

21. J. F. Edmond, *Problèmes d'évolution associés a des ensembles prox-reguliers. Inclusions et intégration des sous-différentiels*, Thèse, Université de Montpellier II, Juin 2004.

22. J. F. Edmond and L. Thibault, Relaxation and optimal control problem involving a perturbed sweeping proocess, *Math. Program, Ser. B*, **104** (2005), 347-373.

23. C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions, Lectures Notes in Mathematics*, Springer-Verlag, Berlin, 1977, p. 580.

24. A. Faik and A. Syam, Differential inclusions governed by a nonconvex sweeping process, *JNCA*, **2(3)** (2001) 381-391.

25. C. Castaing, A. Jofre and A. Salvadori, Control problems governed by functional evolution inclusions with Young measures, *Journal of Nonlinear and Convex Analysis*, **5(1)** 2004, 131-152.

26. S. Marcellin, *Intégration d'epsilon-sous-différentiels et problèmes d 'evolution non convexes, Thèse de Doctororat*, Université de Montpellier 2, Decembre 2004.

27. G. Colombo and V. V. Goncharov, The sweeping process without convexity, *Set-Valued Analysis*, **7** (1999), 357-374.

28. L. C. Evans, Nonlinear evolution equation in an arbitrary Banach space, *Israel J. Math*. **26(1)** (1977), 1-24.

29. M. D. P. Monteiro Marques, Differential inclusions non Nonsmooth Mechanical Problems, Shocks and Dry Friction, Progress in Nonlinear Differential Equations and Their Applications, *Birkhauser*, **9** (1993).

30. J. J. Moreau, Evolution problem asssociated with a moving convex set in a Hilbert space, *J. Diff. Eqs.*, **26** (1977), 347-374.

31. J. J. Moreau and M. Valadier, A chain rule involving vector functions of bounded variations, *Journal of Functional Analysis*, **74(2)** (1987), 333-345.

32. J. C. Péralba, Un problème d'évolution relatif à un opérateur Sous-différentiel dépendant du temps, *C.R. Acad. Sci. Paris*, **275** (1972).

33. J. C. Péralba, *Equations d'évolution dans une espace de Hillbert, associées à des opérateurs sous-differentiels*, Thèse, Université des Sciences et Techniques du Languedoc, 1973.

34. R.T. Rockafellar, Integrals which are convex functionals, II, *Pacific Journal of Mathematics*, **39(2)** (1971), 439-369.

35. D. Tataru, Stabilizing second order differential equations, *Differential and Integral Equations*, **2(2)** (1989), 132-143.

36. L. Thibault, *Propriétés des sous-différentiels de fonctions localement Lipschitziennes définies sur un espace de Banach séparable. Applications*, Thèse, Université Montpellier, 1976.

37. L. Thibault, Sweeping process with regular and nonregular sets, *J. Differential Equations*, **193** (2003), 1-26.

38. M. Valadier, Quelques résultats de base concernant le processus de la rafle, *Sém. Anal. Convexe, Montpellier*, **3** (1988).

39. M. Valadier, Lipschitz approximations of the sweeping process (or Moreau) process, *J. Differential Equations*, **88(2)** (1990), 248-264.

40. I. L. Vrabie, *Compactness methods for Nonlinear evolutions equations, Pitman Monographs and Surveys in Pure and Applied mathematics*, Longman Scientific and Technical, John Wiley and Sons, Inc. New York, 1987, p. 32.

41. C. Zalinescu, On uniformly convex functions, *J. Math. Anal. Appl.*, **95** (1983), 344-374.

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