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A VERSION OF HILBERT'S 13TH PROBLEM FOR ENTIRE FUNCTIONS

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. It is famous that Hilbert proved that, for any positive integer n, there exists an entire function $f_n(\cdot,\cdot,\cdot)$ of three complex variables which cannot be represented as any n-time nested superposition constructed from several entire fuctions of two complex variables. In this paper, a finer classification of the 13th problem formulated by Hilbert is given. This classification is applied to the theorem showing that there exists an entire function $f(\cdot,\cdot,\cdot)$ of three complex variables which cannot be represented as any finite-time nested superposition constructed from several entire functions of two complex variables. The original result proved by Hilbert can be derived from this theorem.

1. Classification of Hilbert's 13TH Problem

Let $\mathbb{C}$ (resp. $\mathbb{R}$) be the set of all complex numbers (resp. real numbers) and let $f(\cdot,\cdot,\cdot)$ be the function of three variable defined as

$$f(x,y,z) = xy + yz + zx, \quad x,y,z \in \mathbb{C}.$$

Then, we can easily prove that there do not exist any three entire functions of two variables $g(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ satisfying the following equality:

$$f(x,y,z) = g(u(x,y), v(x,z)), \quad x,y,z \in \mathbb{C}.$$

This result shows us that f cannot be represented any one-time nested superposition constructed from several entire functions of two variables. But the following equality:

$$f(x,y,z) = x(y+z) + yz, \quad x,y,z \in \mathbb{C}.$$

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shows that f can be represented as a two-time nested superposition constructed from addition and multiplication.
In 1957, Kolmogorov and Arnold solved Hilbert's 13th problem asking if all continuous functions of several real variables can be represented as appropriate superpositions constructed from several continuous functions of fewer real variables. If their original result is applied to the set of all contiuous functions of three real variables, for any continuous function f of three real variables, we can choose a family of seven continuous functions of one real variable $\{g_i^f; 0 \leq i \leq 6\}$ which is dependent on f, and a family of twenty one continuous functions of one real variable $\{\phi_{ij}; 0 \leq i \leq 6, 1 \leq j \leq 3\}$ which is independent of f, satisfying

$$f(x_1, x_2, x_3) = \sum_{i=0}^{6} g_i^f \left(\sum_{j=1}^{3} \phi_{ij}(x_j) \right), \quad x_1, x_2, x_3 \in \mathbb{R}.$$

This result, which is called Kolmogorov-Arnold representation theorem [**3**], immediately implies that any continuous function of three real variables can be represented as a 7-time nested superposition of continuous functions of two real variables, because the following equality holds:

$$\begin{aligned} f(z_1, z_2, z_3) = \Big(\Big(\Big(\Big(\Big(\Big(& g_0^f\left((\phi_{01}(z_1) + \phi_{02}(z_2)) + \phi_{03}(z_3)\right) \\ & + g_1^f\left((\phi_{11}(z_1) + \phi_{12}(z_2)) + \phi_{13}(z_3)\right)\Big) \\ & + g_2^f\left((\phi_{21}(z_1) + \phi_{22}(z_2)) + \phi_{23}(z_3)\right)\Big) \\ & + g_3^f\left((\phi_{31}(z_1) + \phi_{32}(z_2)) + \phi_{33}(z_3)\right)\Big) \\ & + g_4^f\left((\phi_{41}(z_1) + \phi_{42}(z_2)) + \phi_{43}(z_3)\right)\Big) \\ & + g_5^f\left((\phi_{51}(z_1) + \phi_{52}(z_2)) + \phi_{53}(z_3)\right)\Big) \\ & + g_6^f\left((\phi_{61}(z_1) + \phi_{62}(z_2)) + \phi_{63}(z_3)\right), \quad x_1, x_2, x_3 \in \mathbb{R}. \end{aligned}$$

Let $\mathcal{A}_3$ (resp. $\mathcal{A}_2$) be a set of functions of three variables (resp. two variables) such as the set of all continuous functions of three variables (resp. two variables) or the set of all analytic functions of three variables (resp. two variables). Then, the superposition representation proposition can be classified into the following two propositions:

Proposition I. *There exists a certain positive integer k such that, for any element f of $\mathcal{A}_3$, f can be represented as a certain k-time nested superposition constructed from several elements of $\mathcal{A}_2$.*

Proposition II. *For any function f of $\mathcal{A}_3$, there exists a positive integer k_f such that f can be represented as a k_f-time nested superposition constructed from several elements of $\mathcal{A}_2$.*

Here, Proposition I is called the strong superposition representation proposition and Proposition II is called the weak superposition representation proposition. It is clear that Proposition II holds necessarily if Proposition I holds. By the same way as above, the superposition irrepresentation propositions can be classified into the following two propositions:

Proposition III. *There exists a certain element f of $\mathcal{A}_3$ which cannot be represented as any finite-time nested superposition constructed from several elements of $\mathcal{A}_2$.*

Proposition IV. *For any positive integer k, there exists an element f_k of $\mathcal{A}_3$ which cannot be represented as any k-time nested superposition constructed from several elements of $\mathcal{A}_2$.*

Here, Proposition III is called the strong superposition irrepresentation proposition and Proposition IV is called the weak superposition irrepresentation proposition. It is clear that Proposition IV holds necessarily if Proposition III holds. Since Proposition IV is the negative proposition of Proposition I and Proposition III is the negative proposition of Proposition II, only one of the following three cases, namely, the case that Proposition I holds, the case that both Proposition II and Proposition IV hold and the case that Proposition III holds, can be proved exclusively. In other words, if one of these three cases can be proved affirmatively, then the other two cases can be proved negatively.

Remark 1. If we take the set of all continuous functions of three real variables (resp. two real variables) as an example of $\mathcal{A}_3$ (resp. $\mathcal{A}_2$), then Kolmogorov-Arnold theorem assures that only Proposition I holds. If we take the set of all polynomials of three complex variables (resp. two complex variables) as an example of $\mathcal{A}_3$ (resp. $\mathcal{A}_2$), then both Proposition II and Proposition IV holds. If we take the set of all finite-time continuously differentiable functions of three real variables (resp. two real variables) as an example of $\mathcal{A}_3$ (resp. $\mathcal{A}_2$), then Vituskin theorem [6] assures that only Proposition IV holds.

2. ε-Entropy of Analytic Function Spaces

Let $\mathbb{N}$ be the set of all positive integers. Let U be the closed unit disc of $\mathbb{C}$ and, for any positive number s that is greater than or equal to one. Then, sU is defined as $\{sz; z \in U\}$. For any positive integer n, $\mathcal{E}_n$ denotes the set of all entire functions of n variables and $\|\cdot\|_{n,s}$ denotes the norm defined as

$$\|f|_{(sU)^n}\|_{n,s} = \sup_{z_1,\cdots,z_n \in sU} |f(z_1,\cdots,z_n)|, \quad f \in \mathcal{E}_n.$$

Especially, for any positive number M, $\mathcal{E}_n(s,M)$ denotes the subset of $\mathcal{E}_n$ defined as

$$\mathcal{E}_n(s,M) = \left\{f \in \mathcal{E}_n; \|f|_{(sU)^n}\|_{n,s} \le M\right\}.$$

Moreover, $d_n(\cdot,\cdot)$ denotes the metric on $\mathcal{E}_n$ defined as

$$d_n(f,g) = \sum_{k=1}^{\infty} \frac{\|f-g\|_{n,k}}{2^k(1+\|f-g\|_{n,k})}, \quad f,g \in \mathcal{E}_n.$$

Then, it is easy to prove that the metric space $(\mathcal{E}_n, d_n)$ is complete and that $\mathcal{E}_n(s,M)$ is a nonempty closed subset of $\mathcal{E}_n$, and moreover, it is also proved that the topology derived from this metric is exactly equal to the compact open topology over $\mathcal{E}_n$.

Let $\mathcal{X}$ be a metric space. Then, for any positive number ϵ and for any relatively compact subset $\mathcal{F}$ of $\mathcal{X}$, the ϵ-entropy of $\mathcal{F}$, which is denoted by $S(\mathcal{F},\epsilon)$, is defined as the base-2 logarithm of the minimum of the cardinal numbers corresponding to all ϵ-nets of $\mathcal{F}$, and the ϵ-capacity of $\mathcal{F}$, which is denoted by $C(\mathcal{F},\epsilon)$, is defined as the base-2 logarithm of the maximum of the cardinal numbers corresponding to all 2ϵ-separated sets of $\mathcal{F}$.

For any positive integer n and for any positive number s that is greater than 1, let $\mathcal{A}_n(s)$ be the set of all complex valued functions of n variables which are continuous on $(sU)^n$ and analytic on the interior of $(sU)^n$. It is known that $(\mathcal{A}_n(s), \|\cdot\|_{n,s})$ is a Banach space [5]. Then, K. I. Babenko [1] and V. D. Erohin [2] had proved that, for any positive number M, the following equality:

$$\lim_{\epsilon \to 0} \frac{S(\{f|_{U^2}; f \in \mathcal{A}_2(s); \|f|_{U^2}\|_{2,1} \le M\}, \epsilon)}{\left(\log \frac{1}{\epsilon}\right)^3} = \frac{2}{3!(\log s)^2}$$

holds. After the above equality had been proved, A. G. Vitushkin [6] gave the following generalization:

$$\lim_{\epsilon \to 0} \frac{S(\{f|_{U^n}; f \in \mathcal{A}_n(s); \|f|_{U^n}\|_{n,1} \le M\}, \epsilon)}{\left(\log \frac{1}{\epsilon}\right)^{n+1}} = \frac{2}{(n+1)!(\log s)^n}, \quad n \ge 2.$$

Here we have the following:

Lemma 1. *For any positive integer n that is greater than one and for any two positive numbers r and s, if the inequalities $1 < r < s$ hold, then the following* *equalities:*

$$\lim_{\epsilon\to 0}\frac{S\left(\overline{\{f|_{(rU)^n}; f\in\mathcal{E}_n(s,M)\}}^{\|\cdot\|_{n,r}},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} = \lim_{\epsilon\to 0}\frac{S\left(\{f|_{(rU)^n}; f\in\mathcal{E}_n(s,M)\},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}}$$
$$= \frac{2}{(n+1)!(\log\frac{s}{r})^n}$$

holds.

Proof. Without loss of generality, we can assume that r is equal to 1. Since the following inclusion:

$$\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\}\subset\{f|_{U^n}; f\in\mathcal{A}_n(s), \|f|_{U^n}\|_{n,1}\le M\}$$

and Babenko-Erohin-Vituskin theorem shows that the following inequality holds:

$$\lim_{\epsilon\to 0}\frac{S\left(\overline{\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\}}^{\|\cdot\|_{n,1}},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} \le \frac{2}{(n+1)!(\log s)^n}.$$

Therefore, we have only to prove the reverse inequality. Let $N(\epsilon)$ be the positive integer defined as

$$N(\epsilon) = \left[\log\frac{1}{\epsilon}+1\right],$$

where $[\cdot]$ means Gaussian symbol. Moreover, let $D(\epsilon)$ be the subset of $\mathbb{Z}_+^n$ defined as

$$D(\epsilon) = \left\{(k_1,\cdots,k_n)\in\mathbb{Z}_+^n; \sum_{i=1}^n k_i\le\frac{N(\epsilon)}{\log s}\right\}.$$

Let ϕ be a mapping defined on $D(\epsilon)$ with values in $\mathbb{C}$ satisfying

$$\left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\epsilon}\right|\in\mathbb{Z}_+,\ \left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\epsilon}\right|\le\left[\frac{M/\sqrt{2}}{2^{n+1}\epsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$
$$\left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\epsilon}\right|\in\mathbb{Z}_+,\ \left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\epsilon}\right|\le\left[\frac{M/\sqrt{2}}{2^{n+1}\epsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$

where $(k_1,\cdots,k_n)$ is an element of $D(\epsilon)$, and let $\Phi(\epsilon)$ be the set of all mappings satisfying the above conditions. For any $\phi\in\Phi(\epsilon)$, $g_\phi(\cdot)$ denotes the polynomial of n complex variables which is defined as

$$g_\phi(z_1,\cdots,z_n) = \sum_{(k_1,\cdots,k_n)\in D(\epsilon)}\phi(k_1,\cdots,k_n)\prod_{i=1}^n z_i^{k_i},\quad z_1,\cdots,z_n\in\mathbb{C}.$$

If $(z_1, \cdots, z_n) \in sU$ holds, then we have

$$\begin{aligned}
|g_\phi(z_1, \cdots, z_n)| &\le \sum_{(k_1,\cdots,k_n)\in D(\epsilon)} |\phi(k_1, \cdots, k_n)| \prod_{i=1}^{n} s^{k_i} \\
&\le \sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \frac{M}{2^n \prod_{i=1}^{n}(k_i+1)^2} \\
&\le \frac{M}{2^n}\left(1 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)}\right)^n \\
&\le M.
\end{aligned}$$

Therefore, g_ϕ is an element of $\mathcal{E}_n(s, M)$. Let ϕ_1 and ϕ_2 be two elements belonging to $\Phi(\epsilon)$. Then, there exists an element $(k_1, \cdots, k_n) \in D(\epsilon)$ satisfying

$$|\phi_1(k_1, \cdots, k_n) - \phi_2(k_1, \cdots, k_n)| \ \ge\ 2\epsilon.$$

This inequality implies that

$$\|g_{\phi_1}|_{U^n} - g_{\phi_2}|_{U^n}\|_{n,1} \ge 2\epsilon$$

holds, and $\{g_\phi|_{U^n}; \phi \in \Phi(\epsilon)\}$ is a 2ϵ-separated set of $\{f|_{U^n}; f \in \mathcal{E}_n(s, M)\}$. Let $\Delta(\epsilon)$ be the subset of $\mathbb{R}_+^n$ defined as

$$\Delta(\epsilon) \ =\ \left\{(x_1, \cdots, x_n) \in \mathbb{R}_+^n; \sum_{i=1}^{n} x_i \le \frac{N(\epsilon)}{\log s}\right\}.$$

If $(k_1, \cdots, k_n)$ is an element belonging to $D(\epsilon)$, then, for any sufficiently small ϵ, we obtain the following inequalities:

$$\prod_{i=1}^{n}(k_i+1)^2 \le \left(\frac{2}{\log s}\right)^{2n} N(\epsilon)^{2n},$$

$$\begin{aligned}
card(D(\epsilon)) &\ge \int \cdots \int_{\Delta(\epsilon)} dx_1 \cdots dx_n \\
&= \frac{N(\epsilon)^n}{n!(\log s)^n}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \log\left(\prod_{i=1}^{n} \frac{1}{s^{k_i}}\right) &= -\sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \log s \sum_{i=1}^{n} k_i \\
&\ge -\int \cdots \int_{\Delta(\frac{\epsilon}{2})} \log s \sum_{i=1}^{n}(x_i+1)\, dx_1 \cdots dx_n
\end{aligned}$$

$$\geq \frac{-nN\left(\frac{\epsilon}{2}\right)^{n+1}}{(n+1)!(\log s)^n} + \mathcal{O}\left(N(\epsilon)^n\right).$$

Therefore, a lower bound of $C(\{f|_{U^n}; f \in \mathcal{E}_n(s, M)\}, \epsilon)$ can be estimated as follows:

$$\begin{aligned}
& C(\{f|_{U^n}; f \in \mathcal{E}_n(s, M)\}, \epsilon) \\
\geq\ & \log card(\Phi(\epsilon)) \\
\geq\ & \log \prod_{(k_1,\cdots,k_n)\in D(\epsilon)} \left(\frac{\sqrt{2}M}{2^{n+1}\epsilon\prod_{i=1}^n (k_i+1)^2}\prod_{i=1}^n \frac{1}{s^{k_i}} + 1\right)^2 \\
\geq\ & \frac{2N(\epsilon)^n}{n!(\log s)^n}\log\frac{1}{\epsilon} - \frac{2nN(\epsilon)^{n+1}}{(n+1)!(\log s)^n} + \mathcal{O}\left(\left(\log\frac{1}{\epsilon}\right)^n \log\log\frac{1}{\epsilon}\right).
\end{aligned}$$

These results imply that the following inequality:

$$\liminf_{\epsilon\to 0} \frac{C(\{f|_{U^n}; f \in \mathcal{E}_n(s, M)\}, \epsilon)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} \geq \frac{2}{(n+1)!(\log s)^n}$$

holds, and therefore, we can conclude the proof. ■

2. A Revised Version of the Proof Presented by Hilbert

For any two positive numbers s and L that are greater than one, $\mathcal{E}'_2(s, L)$ denotes the intersection of $\mathcal{E}_2(s, L)$ and $\mathcal{E}_2(1, 1)$. Let $\mathcal{N}_\epsilon(\mathcal{E}'_2(s, L))$ be a subset of $\mathcal{E}'_2(s, L)$ satisfying the condition that $\{f|_{U^2}; f \in \mathcal{N}_\epsilon(\mathcal{E}'_2(s, L))\}$ is an ϵ-net of $\{f|_{U^2}; f \in \mathcal{E}'_2(s, L)\}$. For any positive integer n, $\mathcal{I}_n(\mathcal{E}'_2(s, L))$ denotes the set of all entire functions of three variables which can be represented as n-time nested superposition representations of elements of $\mathcal{E}'_2(s, L)$. Especially, $\mathcal{I}_0(\mathcal{E}'_2(s, L))$ is defined as $\mathcal{E}'_2(s, L)$. $\mathcal{I}_n(\mathcal{N}_\epsilon(\mathcal{E}'_2(s, L)))$ can be defined by the same way as stated above. Then, we can prove the following:

Lemma 2. *For any fixed positive integer n, there exists a positive constant c_n, which is independent of ϵ, satisfying the condition that $\{f|_{U^3}; f \in \mathcal{I}_n(\mathcal{N}_\epsilon(\mathcal{E}'_2(s, L)))\}$ is a $c_n\epsilon$-net of $\overline{\{f|_{U^3}; f \in \mathcal{I}_n(\mathcal{E}'_2(s, L))\}}^{\|\cdot\|_{3,1}}$.*

Proof. We prove this lemma inductively. It is easy to prove the case of $n = 0$, because $\{f|_{U^2}; f \in \mathcal{E}'_2(s, L)\}$ is a relatively compact subset of $(\mathcal{A}_2(1), \|\cdot\|_{2,1})$. Assume that $\{f|_{U^3}; f \in \mathcal{I}_n(\mathcal{N}_\epsilon(\mathcal{E}'_2(s, L)))\}$ is a $c_n\epsilon$-net and let g be an element of $\overline{\{f|_{U^3}; f \in \mathcal{I}_n(\mathcal{E}'_2(s,L))\}\})}^{\|\cdot\|_{3,1}}$. Then, there exist three functions $f \in \mathcal{E}'_2(s,L)$, $u \in \mathcal{I}_n(\mathcal{E}'_2(s,L))$ and $v \in \mathcal{I}_n(\mathcal{E}'_2(s,L))$ satisfying

$$\|g - f(u,v)|_{U^3}\|_{3,1} < \epsilon.$$

Moreover, there exist three functions $f' \in \mathcal{N}_\epsilon(\mathcal{E}'_2(s,L))$, $u' \in \mathcal{I}_n(\mathcal{N}_\epsilon(\mathcal{E}'_2(s,L)))$ and $v' \in \mathcal{I}_n(\mathcal{N}_\epsilon(\mathcal{E}'_2(s,L)))$ satisfying

$$\begin{aligned} \|f|_{U^2} - f'|_{U^2}\|_{2,1} &< \epsilon, \\ \|u|_{U^3} - u'|_{U^3}\|_{3,1} &< c_n\epsilon, \\ \|v|_{U^3} - v'|_{U^3}\|_{3,1} &< c_n\epsilon. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|g - f'(u',v')|_{U^3}\|_{3,1} &\le \|g - f(u,v)|_{U^3}\|_{3,1} + \|f(u,v)|_{U^3} - f'(u',v')|_{U^3}\|_{3,1} \\ &\le \|g - f(u,v)|_{U^3}\|_{3,1} + \|f(u,v)|_{U^3} - f(u',v)|_{U^3}\|_{3,1} \\ &\quad + \|f(u',v)|_{U^3} - f(u',v')|_{U^3}\|_{3,1} \\ &\quad + \|f(u',v')|_{U^3} - f'(u',v')|_{U^3}\|_{3,1} \\ &< \left(\frac{2c_nL}{s-1} + 2\right)\epsilon, \end{aligned}$$

because the following inequalities:

$$\sup\left\{\left|\frac{\partial f}{\partial z_1}(z_1,z_2)\right|; (z_1,z_2) \in U^2\right\} \le \frac{L}{s-1}$$

and

$$\sup\left\{\left|\frac{\partial f}{\partial z_2}(z_1,z_2)\right|; (z_1,z_2) \in U^2\right\} \le \frac{L}{s-1}$$

hold. Therefore, $\{f|_{U^3}; f \in \mathcal{I}_{n+1}(\mathcal{N}_\epsilon(\mathcal{E}'_2(s,L)))\}$ is a $((2c_nL)/(s-1)+2)\epsilon$-net of $\overline{\{f|_{U^3}; f \in \mathcal{I}_n(\mathcal{E}'_2(s,L))\}}^{\|\cdot\|_{3,1}}$. ■

Let $\mathcal{I}_n(\mathcal{E}_2(1,1))$ be the set of all entire functions of three variables that are represented as n-time nested superpositions of elements of $\mathcal{E}_2(1,1)$. Then, we can prove the following:

Theorem 3. *There exists an element of* $\mathcal{E}_3(1,1)$ *which does not belong to* $\{mf; m \in \mathbb{N}, f \in \cup_{n=1}^{\infty}\mathcal{I}_n(\mathcal{E}_2(1,1))\}$.

Proof. Since $\mathcal{E}_3(1,1)$ is a closed subset of the complete metric space $(\mathcal{E}_3, d_3)$ and $\cup_{n=1}^{\infty}\mathcal{I}_n(\mathcal{E}_2(1,1))$ is also a subset of $\mathcal{E}_3(1,1)$, Baire's category theorem assures that it is sufficient to prove that $\cup_{n=1}^{\infty}\mathcal{I}_n(\mathcal{E}_2(1,1))$ is a subset of the first category. Since we have

$$\mathcal{E}_2(1,1) = \cup_{k=1}^{\infty}\mathcal{E}_2'\left(1+\frac{1}{k},2\right)$$

and

$$\mathcal{I}_n\left(\mathcal{E}_2(1,1)\right) = \cup_{k=1}^{\infty}\mathcal{I}_n\left(\mathcal{E}_2'\left(1+\frac{1}{k},2\right)\right), \quad n\in\mathbb{N},$$

we have only to prove that $\mathcal{I}_n(\mathcal{E}_2'(1+(1/k),2))$ is nowhere dense. Assume that, for two positive integers n and k, $\mathcal{I}_n(\mathcal{E}_2'(1+(1/k),2))$ is not nowhere dense. Then, there exists a certain positive number δ which satisfies the following inclusion:

$$\overline{\mathcal{I}_n\left(\mathcal{E}_2'\left(1+\frac{1}{k},2\right)\right)}^{d_3} \supset \{g\in\mathcal{E}_3; d_3(0,g)<\delta\}+g_\delta,$$

where g_δ is a certain element of $\mathcal{E}_3$. Since there exist a sufficiently large positive number s and a sufficiently small positive number L satisfying the following inclusion:

$$\{g\in\mathcal{E}_3; d_3(0,g)<\delta\}\supset\left\{g\in\mathcal{E}_3; \|g|_{(sU)^3}\|_{3,s}<L\right\},$$

we have

$$S\left(\left\{f|_{U^3}; f\in\overline{\mathcal{I}_n\left(\mathcal{E}_2'\left(1+\frac{1}{k},2\right)\right)}^{d_3}\right\},\epsilon\right)\geq S\left(\{g|_{U^3}; g\in\mathcal{E}_3(s,L)\},\epsilon\right),\ \epsilon>0.$$

It is clear that $\{f|_{U^3}; f\in\overline{\mathcal{I}_n\left(\mathcal{E}_2'\left(1+1/k,2\right)\right)}^{d_3}\}$ is a subset of $\overline{\{f|_{U^3}; f\in\mathcal{I}_n\left(\mathcal{E}_2'\left(1+1/k,2\right)\right)\}}^{\|\cdot\|_{3,1}}$. Therefore, Lemma 1 implies that, for any sufficiently small number α, there exists a certain positive number ε_α satisfying the following two inequalities:

$$\begin{aligned}&S\left(\left\{f|_{U^3}; f\in\overline{\mathcal{I}_n\left(\mathcal{E}_2'\left(1+\tfrac{1}{k},2\right)\right)}^{d_3}\right\},\varepsilon\right)\\ &\leq\left(\frac{2}{3!(\log s)^2}+\alpha\right)\left(\log\frac{1}{c_n\varepsilon}\right)^3, \quad 0<\varepsilon<\varepsilon_\alpha\end{aligned}$$

and

$$S\left(\{g|_{U^3}; g\in\mathcal{E}_3(s,L)\},\varepsilon\right)\geq\left(\frac{2}{4!(\log s)^3}-\alpha\right)\left(\log\frac{1}{\varepsilon}\right)^4, \quad 0<\varepsilon<\varepsilon_\alpha.$$

Since these two inequalities contradict each other, $\mathcal{I}_n\left(\mathcal{E}_n\left(1+1/k, 2\right)\right)$ is nowhere dense. Therefore, we can conclude the proof. ■

Here, Theorem 3 can lead us to the following:

Theorem 4. *There exists an entire function of three variables which cannot be represented as any finite-time nested superposition constructed from several entire functions of two variables.*

Proof. Let f^* be a function which does not belong to $\{mf; m \in \mathbb{N}, f \in \cup_{n=1}^{\infty}\mathcal{I}_n(\mathcal{E}_2(1,1))\}$. Then, it is sufficient to prove that f^* cannot be represented as any finite-time nested superposition constructed from several entire functions of two variables. For a certain positive integer n, assume that f^* can be represented as a certain n-time nested superposition constructed from several entire functions of two variables. Then, we have only to prove the fact that there exists a positive integer M satisfying

$$\frac{f^*(\cdot,\cdot,\cdot)}{M} \in \bigcup_{k=1}^{n} \mathcal{I}_k(\mathcal{E}_2(1,1)).$$

It is easy to prove the case of $n = 1$. If $n \geq 2$ holds, mathematical induction assures that there exists a positive integer M satisfying the inclusion which is stated above. Since f is assumed to be represented as n-time nested superposition constructed from several entire functions of two variables, there exist one entire function of two variables $g(\cdot,\cdot)$ and two entire functions of three variables $u(\cdot,\cdot,\cdot)$ and $v(\cdot,\cdot,\cdot)$, which are $n-1$-time nested superpositions satisfying the following equality:

$$f^*(x,y,z) = g(u(x,y,z), v(x,y,z)), \quad x,y,z \in \mathbb{C}.$$

Since the assumption, which is based on the mathematical induction, assures that there exists a sufficiently large constant L satisfying

$$\frac{u(\cdot,\cdot,\cdot)}{L} \in \bigcup_{k=1}^{n-1} \mathcal{I}_k(\mathcal{E}_2(1,1))$$

and

$$\frac{v(\cdot,\cdot,\cdot)}{L} \in \bigcup_{k=1}^{n-1} \mathcal{I}_k(\mathcal{E}_2(1,1)),$$

Therefore, M can be defined as the following:

$$M = \sup_{z_1,z_2 \in U} |g(Lz_1, Lz_2)|,$$

and we can obtain

$$\frac{g\left(L\left(\frac{u(\cdot,\cdot,\cdot)}{L}\right), L\left(\frac{v(\cdot,\cdot,\cdot)}{L}\right)\right)}{M} \in \bigcup_{k=1}^{n} \mathcal{I}_k(\mathcal{E}_2(1,1)).$$

Actually, Theorem 3 assures that $f^*(\cdot,\cdot,\cdot)/M$ does not belong to $\cup_{k=1}^{\infty}\mathcal{I}_k(\mathcal{E}_2(1,1))$, because f does not belong to $\{mf; m \in \mathbb{N}, f \in \cup_{k=1}^{\infty}\mathcal{I}_k(\mathcal{E}_2(1,1))\}$ either. Therefore, we have a contradiction. ■

Remark 2. By the same way as above, it can be proved that, for any positive integer n, there exists an element of $\mathcal{E}_{n+1}(1,1)$ which cannot be represented as any finite-time nested superposition constructed from several elements of $\mathcal{E}_n(1,1)$.

References

1. S. Akashi, A version of Hilbert's 13th problem for analytic functions, *The Bulletinof the London Mathematical Society*, **35** (2003), 8-14.
2. K. I. Babenko, On the best approximation of a class of analytic functions, Izv. **22** (1958), 631-640.
3. V. D. Erohin, On the asymptotic behavior of the ϵ-entropy of analytic functions, *Dokl.*, **120** (1958), 949-952.
4. A. N. Kolmogorov, On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, *Dokl.*, **114** (1957), 679-681.
5. G. G. Lorentz, *Approximation of Functions, Holt*, Rinehart and Winston Inc., New York, 1966.
6. W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, Third Edition, New York, 1987.
7. A. G. Vitushkin, Some properties of linear superpositions of smooth functions, *Dokl.*, **156** (1964), 1003-1006.

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