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GENERALIZED JORDAN TRIPLE (θ, ϕ)-DERIVATIONS ON SEMIPRIME RINGS

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Abstract. Let R be a 2-torsion free semiprime ring. In this paper we will show that every Jordan triple (θ, ϕ)-derivation on R is a (θ, ϕ)-derivation. Also every Jordan triple left centralizer on R is a left centralizer. As a consequence, every generalized Jordan triple (θ, ϕ)-derivation on R is a generalized (θ, ϕ)-derivation. This result gives an affirmative answer to the question posed by Wu and Lu in [14].

0. Introduction and Results

Throughout this paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, we denote the commutator $[x, y] = xy - yx$. A ring R is said to be 2-torsion free whenever $2x = 0$, with $x \in R$, implies $x = 0$. Recall that R is said to be semiprime if $xRx = 0$ implies $x = 0$ and R is said to be prime if $xRy = 0$ implies that $x = 0$ or $y = 0$. A mapping $\delta : R \to R$ is called additive if $\delta(x + y) = \delta(x) + \delta(y)$ for all $x, y \in R$. Let θ, ϕ be automorphisms of R and let 1 denote the identity mapping of R. An additive mapping $\delta : R \to R$ is called a (θ, ϕ)-derivation of R if $\delta(xy) = \delta(x)\theta(y) + \phi(x)\delta(y)$ for all $x, y \in R$. An additive mapping $\delta : R \to R$ is called a Jordan (θ, ϕ)-derivation of R if $\delta(x^2) = \delta(x)\theta(x) + \phi(x)\delta(x)$ for all $x \in R$. An additive mapping $\delta : R \to R$ is called a Jordan triple (θ, ϕ)-derivation of R if

$$(\dagger) \qquad \delta(xyx) = \delta(x)\theta(y)\theta(x) + \phi(x)\delta(y)\theta(x) + \phi(x)\phi(y)\delta(x)$$

for all $x, y \in R$. Obviously, every (θ, ϕ)-derivation is a Jordan (θ, ϕ)-derivation. In view of [7, Proposition 3] every Jordan (θ, ϕ)-derivation is a Jordan triple (θ, ϕ)-derivation. For brevity, $(1, 1)$-derivations are simply called derivations. A famous

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result of Herstein [10] states that every Jordan derivation on a 2-torsion free prime ring is a derivation. Later Bresar [4] showed that the same result is true in semiprime rings. Since every Jordan derivation is also a Jordan triple derivation, furthermore Bresar [5] proved that every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Recently, the above results have been extended to Jordan (θ, ϕ)-derivations on prime rings by Bresar and Vukman [7]. In the present paper we will generalize these results to semiprime rings and prove the following:

Theorem 1. *Let R be a 2-torsion free semiprime ring and let θ, ϕ be automorphisms of R. If $\delta : R \to R$ is a Jordan triple (θ, ϕ)-derivation, then δ is a (θ, ϕ)-derivation.*

Corollary 1. *Let R be a 2-torsion free semiprime ring and let θ, ϕ be automorphisms of R. Then every Jordan (θ, ϕ)-derivation of R is a (θ, ϕ)-derivation.*

An additive mapping $T : R \to R$ is called a left (right) centralizer of R if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping $T : R \to R$ is called a Jordan left (right) centralizer of R if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) for all $x \in R$. An additive mapping $T : R \to R$ is called a Jordan triple left (right) centralizer of R if $T(xyx) = T(x)yx$ ($T(xyx) = xyT(x)$) for all $x, y \in R$. In [15], Zalar proved that every Jordan left (right) centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. It is easy to see that every Jordan left (right) centralizer is also a Jordan triple left (right) centralizer. We now generalize Zalar's result as follows.

Theorem 2. *Let R be a 2-torsion free semiprime ring. If $T : R \to R$ is a Jordan triple left (right) centralizer, then T is a left (right) centralizer.*

An additive mapping $F : R \to R$ is called a generalized (θ, ϕ)-derivation of R if there exists a (θ, ϕ)-derivation δ of R such that $F(xy) = F(x)\theta(y) + \phi(x)\delta(y)$ for all $x, y \in R$ (see [11,13]). As usual, generalized $(1, 1)$-derivations are called generalized derivations. Motivated by the concept of generalized derivations, Wu and Lu [14] initiated the study of generalized Jordan derivations and generalized Jordan triple derivations. An additive mapping $F : R \to R$ is called a generalized Jordan (θ, ϕ)-derivation of R if there exists a Jordan (θ, ϕ)-derivation δ of R such that $F(x^2) = \delta(x)\theta(x) + \phi(x)\delta(x)$ for all $x \in R$. An additive mapping $F : R \to R$ is called a generalized Jordan triple (θ, ϕ)-derivation of R if there exists a Jordan triple (θ, ϕ)-derivation δ of R such that

$$(\dagger\dagger) \qquad F(xyx) = F(x)\theta(y)\theta(x) + \phi(x)\delta(y)\theta(x) + \phi(x)\phi(y)\delta(x)$$

for all $x, y \in R$. Moreover, δ is called the relating Jordan triple (θ, ϕ)-derivation of F. In [14], Wu and Lu proved that every generalized Jordan derivation on a prime ring is a generalized derivation. Recently Ashraf et al. [1,2] extended this result to generalized Jordan (θ, ϕ)-derivations on Lie ideals of prime rings. Now applying Theorem 1 and 2, we can solve the conjecture raised by Wu and Lu in [14, page 608 and 611].

Theorem 3. *Let R be a 2-torsion free semiprime ring and let θ, ϕ be automorphisms of R. If $F : R \to R$ is a generalized Jordan triple (θ, ϕ)-derivation, then F is a generalized (θ, ϕ)-derivation.*

Proof. Let δ be the relating Jordan triple (θ, ϕ)-derivation of F satisfying $(\dagger)$. By Theorem 1, δ must be a (θ, ϕ)-derivation. Set $G = F - \delta$. Then in view of $(\dagger)$ and $(\dagger\dagger)$, we have $G(xyx) = G(x)\theta(y)\theta(x)$. So $\theta^{-1}G$ becomes a Jordan triple left centralizer. Applying Theorem 2 yields that $\theta^{-1}G(xy) = \theta^{-1}G(x)y$ for all $x, y \in R$. That is, $G(xy) = G(x)\theta(y)$. Then $F(xy) = \delta(xy) + F(x)\theta(y) - \delta(x)\theta(y) = F(x)\theta(y)+\phi(x)\delta(y)$, implying that F is a generalized (θ, ϕ)-derivation.

Since every generalized Jordan (θ, ϕ)-derivation is also a generalized Jordan triple (θ, ϕ)-derivation [1, Lemma 2.1], we immediately obtain

Corollary 2. *Let R be a 2-torsion free semiprime ring and let θ, ϕ be automorphisms of R. Then every generalized Jordan (θ, ϕ)-derivation of R is a generalized (θ, ϕ)-derivation.*

By using the fact that every linear (θ, ϕ)-derivation on a semisimple Banach algebra is continuous, now we can extend [4, Theorem 6] to generalized (θ, ϕ)-derivations.

Theorem 4. *Let A be a complex semisimple Banach algebra and let θ, ϕ be linear automorphisms of A. If $F : A \to A$ is a linear generalized Jordan triple (θ, ϕ)-derivation and δ is the relating linear Jordan triple (θ, ϕ)-derivation of F, then F is continuous.*

Proof. By Theorem 3, F is a generalized (θ, ϕ)-derivation. Since θ and ϕ are continuous [12], it follows from [8, Corollary 4.3] that δ is continuous. Set $G = F - \delta$. Then $\theta^{-1}G$ becomes a left centralizer. Hence $\theta^{-1}G$ is continuous by [15, Corollary 1.5] and so G is continuous as well. This implies that F is continuous, as desired.

Corollary 3. *Let A be a complex semisimple Banach algebra and let θ, ϕ be linear automorphisms of A. Then every linear generalized Jordan (θ, ϕ)-derivation is continuous.*

1. Preliminaries

Throughout this section we shall denote by δ a Jordan triple $(1, \phi)$-derivation of a ring R. Then

$$\delta(aba) = \delta(a)ba + \phi(a)\delta(b)a + \phi(a)\phi(b)\delta(a) \tag{1}$$

for all $a, b \in R$. Replacing a with $a + c$ in (1), we obtain that

$$\begin{aligned} \delta(abc + cba) &= \delta(a)bc + \phi(a)\delta(b)c + \phi(a)\phi(b)\delta(c) \\ &\quad + \delta(c)ba + \phi(c)\delta(b)a + \phi(c)\phi(b)\delta(a), \end{aligned} \tag{2}$$

for all $a, b, c \in R$. A direct expansion by using (1) yields that

$$\begin{aligned} \delta(abcxcba) &= \delta(a(b(cxc)b)a) \\ &= \delta(a)bcxcba + \phi(a)\delta(b(cxc)b)a + \phi(a)\phi(bcxcb)\delta(a) \\ &= \delta(a)bcxcba + \phi(a)\big(\delta(b)cxcb + \phi(b)\delta(cxc)b \\ &\quad + \phi(b)\phi(cxc)\delta(b)\big)a + \phi(a)\phi(bcxcb)\delta(a) \\ &= \delta(a)bcxcba + \phi(a)\delta(b)cxcba + \phi(a)\phi(b)\delta(c)xcba \\ &\quad + \phi(a)\phi(b)\phi(c)\delta(x)cba + \phi(a)\phi(b)\phi(c)\phi(x)\delta(c)ba \\ &\quad + \phi(a)\phi(b)\phi(cxc)\delta(b)a + \phi(a)\phi(bcxcb)\delta(a). \end{aligned} \tag{3}$$

Following Bresar [5], we write $A(a, b, c) = \delta(abc) - \delta(a)bc - \phi(a)\delta(b)c - \phi(a)\phi(b)\delta(c)$ and $B(a, b, c) = abc - cba$. In view of (2) we have $A(a, b, c) + A(c, b, a) = 0$. We begin with some lemmas which will be used in the sequel.

Lemma 1.1. *Let R be a ring and δ a Jordan triple $(1, \phi)$-derivation of R. Then*

$$A(a, b, c)xB(a, b, c) + \phi(B(a, b, c))\phi(x)A(a, b, c) = 0$$

for all $a, b, c, x \in R$.

Proof. Consider $W = \delta(abcxcba + cbaxabc)$. Use (2) to obtain that

$$\begin{aligned} W &= \delta((abc)x(cba) + (cba)x(abc)) \\ &= \delta(abc)xcba + \phi(abc)\delta(x)cba + \phi(abc)\phi(x)\delta(cba) \\ &\quad + \delta(cba)xabc + \phi(cba)\delta(x)abc + \phi(cba)\phi(x)\delta(abc). \end{aligned}$$

On the other hand, in view of (3)

$$\begin{aligned} W &= \delta((a(b(cxc)b)a) + (c(b(axa)b)c)) \\ &= \delta(a)bcxcba + \phi(a)\delta(b)cxcba + \phi(a)\phi(b)\delta(c)xcba \\ &\quad + \phi(a)\phi(b)\phi(c)\delta(x)cba + \phi(a)\phi(b)\phi(c)\phi(x)\delta(c)ba \\ &\quad + \phi(a)\phi(b)\phi(cxc)\delta(b)a + \phi(a)\phi(bcxcb)\delta(a) \\ &\quad + \delta(c)baxabc + \phi(c)\delta(b)axabc + \phi(c)\phi(b)\delta(a)xabc \\ &\quad + \phi(c)\phi(b)\phi(a)\delta(x)abc + \phi(c)\phi(b)\phi(a)\phi(x)\delta(a)bc \\ &\quad + \phi(c)\phi(b)\phi(axa)\delta(b)c + \phi(c)\phi(baxab)\delta(c). \end{aligned}$$

Comparing the above two equations, we see that

$$A(a,b,c)xcba+\phi(abc)\phi(x)A(c,b,a)+A(c,b,a)xabc+\phi(cba)\phi(x)A(a,b,c)=0$$

for all $a,b,c\in R$. Recall that $A(c,b,a)=-A(a,b,c)$. Thus $A(a,b,c)xB(a,b,c)+\phi(B(a,b,c))\phi(x)A(a,b,c)=0$, as asserted.

Lemma 1.2. *Let R be a semiprime ring and let R_i be additive subgroups of R for $i=1,\cdots,n$, where n is a positive integer. If $H,K:R^n=R\times\cdots\times R\to R$ are n-additive mappings such that $H(a_1,\cdots,a_n)xK(a_1,\cdots,a_n)=0$ for all $a_i\in R_i$ and $x\in R$, then $H(a_1,\cdots,a_n)xK(b_1,\cdots,b_n)=0$ for all $a_i,b_i\in R_i$ and $x\in R$.*

Proof. Replacing a_1 with a_1+b_1 and using the additivity of H and K, we have

$$H(a_1,a_2,\cdots,a_n)xK(b_1,a_2,\cdots,a_n)+H(b_1,a_2,\cdots,a_n)xK(a_1,a_2,\cdots,a_n)=0,$$

for all $x\in R$. Next replacing x with $xK(b_1,a_2,\cdots,a_n)yH(a_1,a_2,\cdots,a_n)x$, it follows $H(a_1,a_2,\cdots,a_n)xK(b_1,a_2,\cdots,a_n)yH(a_1,a_2,\cdots,a_n)\,xK(b_1,a_2,\cdots,a_n)=0$ for all $x,y\in R$. By semiprimeness of R, $H(a_1,a_2,\cdots,a_n)xK(b_1,a_2,\cdots,a_n)=0$ for all $x\in R$. Replacing a_i with a_i+b_i for $i\geq 2$ and continuing the same process as above, we will obtain the assertion of this lemma.

For an arbitrary ring R, we set $S=\{\alpha\in Z(R)\mid \alpha R\subseteq Z(R)\}$. Obviously, S is an ideal of R and $\alpha bc=cb\alpha$ for all $\alpha\in S$ and $b,c\in R$.

Lemma 1.3. *Let R be a semiprime ring and $a\in R$. If $axy=yxa$ for all $x,y\in R$, then $a\in S$.*

Proof. Let $x,y,z,w\in R$. Then $a(wz)yx=yx(wz)a=ya(wz)x=y(zwa)x=(yzwa)x=awyzx$. By semiprimeness of R, $awzy=awyz$. Thus $aw[z,y]=0$ for all $w,z,y\in R$. Hence $ayw[a,y]=yaw[a,y]=0$. In particular, $[a,y]w[a,y]=0$ for all $y,w\in R$. Since R is semiprime, $[a,y]=0$ for all $y\in R$. This implies that $a\in Z(R)$. So now $axy=yxa=yax$ for all $x,y\in R$. Thus $ax\in Z(R)$ for all $x\in R$, as asserted.

We let $Q=Q_s(R)$ be the symmetric Martindale ring of quotients of a semiprime ring R. The center of Q denoted by C is called the extended centroid of R (see [3, chapter 2]). An element $\varepsilon\in C$ is called a central idempotent if $\varepsilon^2=\varepsilon$. The following lemma is a special case of [9, Theorem 3.1] and we state its form needed here.

Lemma 1.4. *Let R be a semiprime ring and let ϕ be an automorphism of R. If $a,b,c,d\in R$ and $axb=c\phi(x)d$ for all $x\in R$, then there exist central idempotents*

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in C$ *and an invertible element* $q \in Q$ *such that* $\varepsilon_i \varepsilon_j = 0$ *for* $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 1_Q$ *and*

$$\varepsilon_1 \phi(x) = \varepsilon_1 q x q^{-1}, \varepsilon_1 a = \varepsilon_1 c q, \varepsilon_1 b = \varepsilon_1 q^{-1} d$$
$$\varepsilon_2 b = \varepsilon_2 d = \varepsilon_3 b = \varepsilon_3 c = \varepsilon_4 a = \varepsilon_4 d = \varepsilon_5 a = \varepsilon_5 c = 0$$

for all $x \in R$.

Corollary 1.5. *Let* R *be a 2-torsion free semiprime ring and* ϕ *an automorphism of* R. *If* $a, b \in R$ *and* $axb + \phi(b)\phi(x)a = 0$ *for all* $x \in R$, *then* $axb = 0$ *for all* $x \in R$.

Proof. In view of Lemma 1.4, there exist central idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in C$ and an invertible element $q \in Q$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 1_Q$ and $\varepsilon_1 \phi(x) = \varepsilon_1 q x q^{-1}, \varepsilon_1 a = \varepsilon_1 \phi(b) q, \varepsilon_1 b = -\varepsilon_1 q^{-1} a, \varepsilon_2 b = \varepsilon_3 b = \varepsilon_4 a = \varepsilon_5 a = 0$. So $\varepsilon_1 a = -q(-\varepsilon_1 q^{-1} a) = -q \varepsilon_1 b = -\varepsilon_1 q b$ and $\varepsilon_1 a = \varepsilon_1 \phi(b) q = \varepsilon_1 q b q^{-1} q = \varepsilon_1 q b$. Hence $2\varepsilon_1 a = 0$. Since R is 2-torsion free, $\varepsilon_1 a = 0$ and then $\varepsilon_1 axb = 0$. So it is easy to see that $axb = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)axb = 0$, as desired.

Corollary 1.6. *Let* R *be a semiprime ring and* ϕ *an automorphism of* R. *If* $\alpha \in Z(R)$, $b \in R$ *and* $(\phi(\alpha x) - \alpha x)b = 0$ *for all* $x \in R$, *then* $(\phi(x) - x)\alpha b = 0$ *for all* $x \in R$.

Proof. By assumption, $-\alpha x b + \phi(\alpha)\phi(x) b = 0$. In view of Lemma 1.4, there exist central idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in C$ and an invertible element $q \in Q$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 1_Q$ and $\varepsilon_1 \phi(x) = \varepsilon_1 q x q^{-1}, \varepsilon_2 b = \varepsilon_3 b = \varepsilon_4 b = \varepsilon_5 \alpha = 0$. In particular, $\varepsilon_1 \phi(\alpha) = \varepsilon_1 q \alpha q^{-1} = \varepsilon_1 \alpha$. Thus $0 = \varepsilon_1(-\alpha x b + \phi(\alpha)\phi(x) b) = \varepsilon_1(-x + \phi(x))\alpha b$. So $(\phi(x) - x)\alpha b = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(\phi(x) - x)\alpha b = 0$, as desired.

2. Proof of Theorem 1

Proof. Since $\theta^{-1}\delta$ is a Jordan triple $(1, \theta^{-1}\phi)$-derivation, replacing δ by $\theta^{-1}\delta$ we may assume that δ is a Jordan triple $(1, \phi)$-derivation. Then we have $A(a,b,c)xB(a,b,c) + \phi(B(a,b,c))\phi(x)A(a,b,c) = 0$ for all $a, b, c, x \in R$ by Lemma 1.1. It follows from Corollary 1.5 that $A(a,b,c)xB(a,b,c) = 0$ for all $a, b, c, x \in R$. Thus by Lemma 1.2 $A(a,b,c)xB(r,s,t) = 0$ for all $a, b, c, r, s, t, x \in R$. For $a, b, c, x, r, s \in R$, we have

$$\begin{aligned} & B(A(a,b,c), r, s)xB(A(a,b,c), r, s) \\ &= \big(A(a,b,c)rs - srA(a,b,c)\big)xB(A(a,b,c), r, s) \\ &= A(a,b,c)rsxB(A(a,b,c), r, s) - srA(a,b,c)xB(A(a,b,c), r, s) = 0 \end{aligned}$$

By semiprimeness of R, $B(A(a,b,c),r,s) = A(a,b,c)rs - srA(a,b,c) = 0$ for all $a,b,c,r,s \in R$. In light of Lemma 1.3, we see that $A(a,b,c) \in S$ for all $a,b,c \in R$. Let $\alpha \in S$ and $b,c \in R$. Then $\alpha, \alpha b, \alpha c \in Z(R)$ and $cb\alpha = c(\alpha b) = \alpha bc$. Similarly, $\delta(c)b\alpha = \alpha b\delta(c)$ and $\alpha\delta(b)c = c\delta(b)\alpha$ $(\dagger\dagger\dagger)$. Consider $W = \delta(\alpha bcxcb\alpha)$. Use (3) to obtain

$$\begin{aligned} W = \delta(\alpha(b(cxc)b)\alpha) = \delta(\alpha)bcxcb\alpha + \phi(\alpha)\delta(b)cxcb\alpha + \phi(\alpha)\phi(b)\delta(c)xcb\alpha \\ +\phi(\alpha)\phi(b)\phi(c)\delta(x)cb\alpha + \phi(\alpha)\phi(b)\phi(c)\phi(x)\delta(c)b\alpha \\ +\phi(\alpha)\phi(b)\phi(cxc)\delta(b)\alpha + \phi(\alpha)\phi(bcxcb)\delta(\alpha). \end{aligned}$$

On the other hand, using (1) we have

$$W = \delta((\alpha bc)x(\alpha bc)) = \delta(\alpha bc)x\alpha bc + \phi(\alpha bc)\delta(x)\alpha bc + \phi(\alpha bc)\phi(x)\delta(\alpha bc).$$

Comparing the above two equations and noticing that $cb\alpha = \alpha bc$, we see that

$$\phi(\alpha bc)\phi(x)A(c,b,\alpha) + A(\alpha,b,c)x\alpha bc = 0.$$

Recall that $A(c,b,\alpha) = -A(\alpha,b,c)$ and $\alpha bc \in Z(R)$. So

$$\phi(\alpha bc)\phi(x)A(\alpha,b,c) - A(\alpha,b,c)x\alpha bc = 0,$$

for all $\alpha \in S$ and $b,c,x \in R$. In view of Corollary 1.6, $(\phi(x)-x)\alpha bcA(\alpha,b,c) = 0$. Multiplying y form the right hand side, we have $(\phi(x)-x)\alpha bcyA(\alpha,b,c) = 0$ since $A(\alpha,b,c) \in Z(R)$. Thus it follows from Lemma 1.2 that $(\phi(x)-x)\beta styA(\alpha,b,c) = 0$ for all $\alpha,\beta \in S$ and $b,c,s,t,y \in R$. Replace β by $A(\alpha,b,c)$ to yield that $(\phi(x)-x)A(\alpha,b,c)^2 = 0$ for all $x \in R$. Note that $A(\alpha,b,c)[x,y] = [A(\alpha,b,c)x,y] = 0$. This implies that $A(\alpha,b,c)[R,R] = 0$. Thus

$$\begin{aligned} & 2A(\alpha,b,c)^3 \\ &= A(\alpha,b,c)^2\big(A(\alpha,b,c) - A(c,b,\alpha)\big) \\ &= A(\alpha,b,c)^2\big(\delta(\alpha bc) - \delta(\alpha)bc - \phi(\alpha)\delta(b)c - \phi(\alpha)\phi(b)\delta(c) \\ &\quad -\delta(cb\alpha) + \delta(c)b\alpha + \phi(c)\delta(b)\alpha + \phi(c)\phi(b)\delta(\alpha)\big) \\ &= A(\alpha,b,c)^2\big(- \delta(\alpha)bc - \phi(\alpha)\delta(b)c - \phi(\alpha)\phi(b)\delta(c) \\ &\quad +\delta(c)b\alpha + \phi(c)\delta(b)\alpha + \phi(c)\phi(b)\delta(\alpha)\big) \\ &= A(\alpha,b,c)^2\Big(\delta(\alpha)(\phi(bc) - bc) - \delta(\alpha)(\phi(b)\phi(c) - \phi(c)\phi(b)) \\ &\quad +(\phi(cb)\delta(\alpha) - \delta(\alpha)\phi(cb)) - (\phi(\alpha) - \alpha)\delta(b)c \\ &\quad +(\phi(c) - c)\delta(b)\alpha + (\alpha b - \phi(\alpha b))\delta(c)\Big) \text{ (by } (\dagger\dagger\dagger)) \\ &= A(\alpha,b,c)^2\Big(\delta(\alpha)(\phi(bc) - bc) - \delta(\alpha)[\phi(b),\phi(c)] + [\phi(cb),\delta(\alpha)] \\ &\quad -(\phi(\alpha) - \alpha)\delta(b)c + (\phi(c) - c)\delta(b)\alpha + (\alpha b - \phi(\alpha b))\delta(c)\Big) = 0. \end{aligned}$$

It is easy to see that $Z(R)$ does not contain nonzero nilpotent elements. So it follows that $A(\alpha, b, c) = 0$ for all $\alpha \in S$ and $b, c \in R$. That is,

$$\delta(\alpha bc) - \delta(\alpha)bc - \phi(\alpha)\delta(b)c - \phi(\alpha)\phi(b)\delta(c) = 0. \tag{4}$$

Note that if $\alpha \in S$, then $\alpha R \subseteq S$ and $\phi(\alpha), \phi^{-1}(\alpha) \in S$. Let $\alpha \in S$ and $x, b, c \in R$. Applying (4) we obtain that

$$\delta(\alpha xabc) = \delta(\alpha)xabc + \phi(\alpha)\delta(x)abc + \phi(\alpha)\phi(x)\delta(abc)$$

and

$$\begin{aligned}
&\delta((\alpha xa)bc) \\
&= \delta(\alpha xa)bc + \phi(\alpha xa)\delta(b)c + \phi(\alpha xa)\phi(b)\delta(c) \\
&= \big(\delta(\alpha)xa + \phi(\alpha)\delta(x)a + \phi(\alpha)\phi(x)\delta(a)\big)bc + \phi(\alpha xa)\delta(b)c + \phi(\alpha xa)\phi(b)\delta(c).
\end{aligned}$$

Comparing the above two equations, $\phi(\alpha)\phi(x)A(a, b, c) = 0$ for all $\alpha \in S$ and $a, b, c \in R$. Replacing α by $\phi^{-1}(A(a, b, c))$, we see that $A(a, b, c)\phi(x)A(a, b, c) = 0$. By semiprimeness of R, $A(a, b, c) = 0$ for all $a, b, c \in R$. That is $\delta(abc) = \delta(a)bc + \phi(a)\delta(b)c + \phi(a)\phi(b)\delta(c)$ for all $a, b, c \in R$. Consider $W = \delta(abxab)$. Then

$$\begin{aligned}
W &= \delta(a(bxa)b) \\
&= \delta(a)bxab + \phi(a)\delta(bxa)b + \phi(a)\phi(bxa)\delta(b) \\
&= \delta(a)bxab + \phi(a)\big(\delta(b)xa + \phi(b)\delta(x)a + \phi(b)\phi(x)\delta(a)\big)b + \phi(a)\phi(bxa)\delta(b).
\end{aligned}$$

On the other hand,

$$W = \delta((ab)x(ab)) = \delta(ab)xab + \phi(ab)\delta(x)ab + \phi(ab)\phi(x)\delta(ab).$$

Comparing the above two equations, we have

$$(\delta(ab) - \phi(a)\delta(b) - \delta(a)b)xab + \phi(ab)\phi(x)(\delta(ab) - \phi(a)\delta(b) - \delta(a)b) = 0.$$

By Corollary 1.5, $(\delta(ab) - \phi(a)\delta(b) - \delta(a)b)xab = 0$. Thus it follows from Lemma 1.2 that $(\delta(ab) - \phi(a)\delta(b) - \delta(a)b)xcd = 0$ for all $a, b, c, d, x \in R$. By semiprimeness of R, $\delta(ab) - \phi(a)\delta(b) - \delta(a)b = 0$, as desired.

3. Proof of Theorem 2

Proof. Proof. Suppose T is a Jordan triple left centralizer. We write $A(a, b, c) = T(abc) - T(a)bc$ and $B(a, b, c) = abc - bca$. By assumption, $T(aba) = T(a)ba$ for all $a, b \in R$. Replacing a by $a+c$, we see that $T(abc+cba) = T(a)bc+T(c)ba$. Consider $W = T(abcxcba + cbaxabc)$. Then

$$W = T((abc)x(cba) + (cba)x(abc)) = T(abc)xcba + T(cba)xabc.$$

On the other hand,

$$W = T((a(b(cxc)b)a) + (c(b(axa)b)c)) = T(a)bcxcba + T(c)baxabc.$$

So $A(a,b,c)xcba + A(c,b,a)xabc = 0$ for all $a, b, c, x \in R$. Recall that $A(c,b,a) = -A(a,b,c)$. Thus $A(a,b,c)xB(a,b,c) = 0$. By Lemma 1.2, $A(a,b,c)xB(r,s,t) = 0$ for all $a, b, c, r, s, t, x \in R$. For $a, b, c, x, r, s \in R$, we have

$$\begin{aligned} &B(A(a,b,c),r,s)xB(A(a,b,c),r,s) \\ &= \big(A(a,b,c)rs - srA(a,b,c)\big)xB(A(a,b,c),r,s) \\ &= A(a,b,c)rsxB(A(a,b,c),r,s) - srA(a,b,c)xB(A(a,b,c),r,s) = 0 \end{aligned}$$

By semiprimeness of R, $B(A(a,b,c),r,s) = A(a,b,c)rs - srA(a,b,c) = 0$ for all $a, b, c, r, s \in R$. In light of Lemma 1.3, we see that $A(a,b,c) \in S$ for all $a, b, c \in R$. Let $\alpha \in S$ and $b, c \in R$. Consider $W = T(\alpha bcxcb\alpha)$. Then $T(\alpha)bcxcb\alpha = T(\alpha(b(cxc)b)\alpha) = W = T((\alpha bc)x(\alpha bc)) = T(\alpha bc)x\alpha bc$. Thus $A(\alpha,b,c)x\alpha bc = 0$. By Lemma 1.2, $A(\alpha,b,c)x\beta st = 0$ for all $\alpha, \beta \in S$ and $b, c, s, t, x \in R$. Since $A(\alpha,b,c) \in S$, we have $A(\alpha,b,c)^2xst = 0$. By semiprimeness of R, $A(\alpha,b,c)^2 = 0$. Recall that $Z(R)$ contains no nonzero nilpotent elements. Hence $A(\alpha,b,c) = 0$. In particular, $A(c,b,\alpha) = 0$. That is, $T(cb\alpha) = T(c)b\alpha$ for all $b, c \in R$ and $\alpha \in S$.

Let $a, b, c \in R$ and $\alpha = A(a,b,c)$. Then

$$T(abc)\alpha^2 = T((abc)\alpha\alpha) = T(a(bc)(\alpha^2)) = T(a)bc\alpha^2.$$

Hence $\alpha^3 = (T(abc) - T(a)bc)\alpha^2 = 0$. Thus $\alpha = 0$. This means that $T(abc) = T(a)bc$ for all $a, b, c \in R$. Now we have $T(a)bxab = T(abxab) = T(ab)xab$. Then $(T(ab) - T(a)b)xab = 0$. By Lemma 1.2, $(T(ab) - T(a)b)xst = 0$ for all $a, b, x, s, t \in R$. By semiprimeness of R, $T(ab) - T(a)b = 0$, as desired.

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