TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 4, pp. 1177-1188, September 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

CONTINUITY WITH RESPECT TO SYMBOLS OF COMPOSITION OPERATORS ON THE WEIGHTED BERGMAN SPACE

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Abstract. Let $\alpha > -1$, U be the open unit disk in $\mathbb{C}$ and denote by $H(U)$ the set of all holomorphic functions on U. Let C_φ be a composition operator induced by an analytic self-map φ of U. Composition operators C_φ on the weighted Hilbert Bergman space $\mathcal{A}^2_\alpha(U) = \left\{f \in H(U) \mid \int_U |f(z)|^2(1-|z|^2)^\alpha dm(z) < \infty\right\}$ are considered. We investigate when convergence of sequences (φ_n) of symbols to a given symbol φ, implies the convergence of the induced composition operators. We give a necessary and sufficient condition for a sequence of Hilbert-Schmidt composition operators (C_{φ_n}) to converge in Hilbert-Schmidt norm to C_φ, and we obtain a sufficient condition for convergence in operator norm.

1. Introduction

Let U be the open unit disk in the complex plane $\mathbb{C}$. Let normalized area measure on U be given by $dm(z) = \frac{1}{\pi} r dr d\theta$, and denote by $H(U)$ the set of all holomorphic functions on U. The weighted Bergman space $\mathcal{A}^p_\alpha(U) = \mathcal{A}^p_\alpha$, where $p \in (0,\infty)$ and $\alpha > -1$, is the space of all $f \in H(U)$ such that

$$||f||^p_{p,\alpha} = (\alpha+1)\int_U |f(z)|^p(1-|z|^2)^\alpha dm(z) < \infty.$$

It is well known that $\mathcal{A}^2_\alpha$ is a Hilbert space with the norm $||\cdot||_{2,\alpha}$ coming from the inner product given by

$$\langle f, g\rangle = (\alpha+1)\int_U f(z)\overline{g(z)}(1-|z|^2)^\alpha dm(z).$$

Accepted August 24, 2005.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: Primary 47B38, 47B33; Secondary 46E20.
Key words and phrases: Holomorphic function, Composition operator, Weighted Bergman space, Hilbert-Schmidt operator.

If $f \in \mathcal{A}^2_\alpha$ has the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then a calculation in polar coordinates shows that

$$\begin{aligned} ||f||^2_{2,\alpha} &= \frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} d\theta (1-r^2)^\alpha r dr \\ &= 2(\alpha+1) \int_0^1 \sum_{n=0}^{\infty} |a_n|^2 r^{2n} (1-r^2)^\alpha r dr \\ &= \sum_{n=0}^{\infty} |a_n|^2 \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}, \end{aligned}$$

which can be also used for a definition of the norm $||\cdot||_{2,\alpha}$.

Letting $\alpha \to -1+0$ in the last equation, one observes that if $f \in H^2(U)$, i.e.,

$$||f||^2_{H^2} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$

then

$$\lim_{\alpha \to -1+0} ||f||^2_{2,\alpha} = \sum_{n=0}^{\infty} |a_n|^2 = ||f||^2_{H^2}.$$

Therefore, $\mathcal{A}^2_{-1}$ can be viewed as the Hardy space $H^2(U)$.

Remark 1. If $f \in H^p(U)$, $p > 0$, then $\lim_{\alpha \to -1+0} ||f||_{p,\alpha} = ||f||_{H^p}$. Indeed, by polar coordinates we have

$$||f||^p_{p,\alpha} = \frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p d\theta (1-r^2)^\alpha r dr \le ||f||^p_{H^p},$$

for every $\alpha > -1$.

On the other hand, for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta > ||f||^p_{H^p} - \varepsilon,$$

for every $r \in (\delta, 1)$. Hence

$$\begin{aligned} ||f||^p_{p,\alpha} &\ge \frac{\alpha+1}{\pi} \int_\delta^1 \int_0^{2\pi} |f(re^{i\theta})|^p d\theta (1-r^2)^\alpha r dr \\ &> (||f||^p_{H^p} - \varepsilon)(\alpha+1) \int_\delta^1 (1-r^2)^\alpha 2r dr = (||f||^p_{H^p} - \varepsilon)(1-\delta^2)^{\alpha+1}. \end{aligned}$$

Successively letting $\alpha \to -1+0$, and $\varepsilon \to 0$, we obtain

$$\liminf_{\alpha \to -1+0} ||f||_{p,\alpha} \ge ||f||_{H^p}.$$

From this and the first inequality in the remark, we have that $\lim_{\alpha\to-1+0}||f||_{p,\alpha} = ||f||_{H^p}$.

As usual, $H^\infty(U)$ denotes the space of all bounded holomorphic functions on U with norm given by $||f||_\infty = \sup_{z\in U}|f(z)|$.

The objects of study here are operators induced on $\mathcal{A}^2_\alpha$ by composition with holomorphic self-maps of U, that is, given a holomorphic function $\varphi : U \to U$ we define the *composition operator* C_φ on the space $H(U)$ by:

$$C_\varphi(f) = f \circ \varphi, \qquad f \in H(U).$$

The operator is obviously linear. Using Littlewood's subordination principle and polar coordinates one sees that if φ is not a unimodular constant then C_φ takes $\mathcal{A}^2_\alpha$ into itself; moreover this operator is bounded ([1, 2]). A generalization of this result can be found in [7].

In this paper we investigate when the convergence of symbols φ_n, say to φ (in some sense that will be clarified in the sequel), implies the convergence of the corresponding composition operators C_{φ_n} to C_φ (in the same or some other sense).

Note that, $||\varphi_n-\varphi||_{p,\alpha} \to 0$ for some $p \in [1,\infty)$ if and only if $||\varphi_n-\varphi||_{1,\alpha} \to 0$. Indeed, by Jensen's inequality we have $||\varphi_n - \varphi||^p_{1,\alpha} \le ||\varphi_n - \varphi||^p_{p,\alpha}$. On the other hand, since $|\varphi_n(z)| < 1$, for every $n \in \mathbb{N}$ and $|\varphi(z)| \le 1$, for every $z \in U$, it follows that

$$\int_U \left(\frac{|\varphi_n(z) - \varphi(z)|}{2}\right)^p (1-|z|^2)^\alpha dm(z) \le \int_U \frac{|\varphi_n(z) - \varphi(z)|}{2}(1-|z|^2)^\alpha dm(z).$$

Thus $||\varphi_n-\varphi||^p_{p,\alpha} \le 2^{p-1}||\varphi_n-\varphi||_{1,\alpha}$, from which the claimed equivalence follows.

In Section 2 we investigate when a sequence of Hilbert-Schmidt composition operators $(C_{\varphi_n})_{n\in\mathbb{N}}$ converges in Hilbert-Schmidt norm to some C_φ.

In Section 3 we give a sufficient condition that a sequence of composition operators $(C_{\varphi_n})_{n\in\mathbb{N}}$ converges to C_φ in the operator norm.

Our results are extension of those in [5].

2. Hilbert-schmidt Norm Convergence

If $\mathcal{H}$ is a separable Hilbert space, then the Hilbert-Schmidt norm $||T||_{HS}$ of an operator $T : \mathcal{H} \to \mathcal{H}$ is defined by:

$$||T||_{HS} = \sqrt{\sum_{n=1}^\infty ||Te_n||^2},$$

where $\{e_n\}$ is an orthonormal basis on $\mathcal{H}$. The right side above does not depend on the choice of basis. Consequently it is larger than the operator norm $||T||_{op}$ of T. Therefore, if $||C_{\varphi_n} - C_\varphi||_{HS} \to 0$, then $||C_{\varphi_n} - C_\varphi||_{op} \to 0$ as $n \to \infty$.

Let

$$e_n(z) = \left(\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}\right)^{1/2} z^n = c_{n,\alpha} z^n, \qquad n = 0, 1, 2, \ldots.$$

Since

$$\begin{aligned} ||e_n||_{2,\alpha}^2 &= (\alpha+1)c_{n,\alpha}^2 \int_U |z|^{2n}(1-|z|^2)^\alpha dm(z) \\ &= (\alpha+1)c_{n,\alpha}^2 \int_0^1 \rho^n (1-\rho)^\alpha d\rho = 1 \end{aligned}$$

and $\langle e_n, e_m \rangle = 0$ when $m \neq n$, it follows that $(e_n)_{n\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis for $\mathcal{A}_\alpha^2$.

The Hilbert-Schmidt norm of the composition operator C_φ on $\mathcal{A}_\alpha^2$ is

$$(1) \qquad ||C_\varphi||_{HS} = \left((\alpha+1)\int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}} dm(z)\right)^{1/2}.$$

Indeed,

$$\begin{aligned} ||C_\varphi||_{HS}^2 &= \sum_{n=0}^\infty ||C_\varphi(e_n)||_{2,\alpha}^2 = \sum_{n=0}^\infty c_{n,\alpha}^2 ||\varphi^n||_{2,\alpha}^2 \\ &= (\alpha+1)\sum_{n=0}^\infty c_{n,\alpha}^2 \int_U |\varphi(z)|^{2n}(1-|z|^2)^\alpha dm(z) \\ &= (\alpha+1)\int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}} dm(z), \end{aligned}$$

where in the last equality we have used the formula

$$\frac{1}{(1-x)^{\alpha+2}} = \sum_{n=0}^\infty \frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)} x^n, \qquad |x| < 1.$$

Now we formulate and prove the main results of this section.

Theorem 1. *Let* $\alpha > -1$, φ *be an analytic self-map of* U *and suppose that* $(\varphi_n)_{n\in\mathbb{N}}$ *is a sequence of analytic self-maps of* U *such that* $\varphi_n \to \varphi$ *a.e. on* U,

$$(2) \qquad \int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}} dm(z) < \infty,$$

and

$$\lim_{n\to\infty}\int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}}dm(z) = \int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}}dm(z). \tag{3}$$

Then the sequence $(C_{\varphi_n})_{n\in\mathbb{N}}$ *of Hilbert-Schmidt composition operators converges in Hilbert-Schmidt norm to the composition operator* C_φ.

Proof. We show that the family of functions

$$\mathcal{F} = \left\{ \frac{(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}}\ ,\ n\in\mathbb{N}\right\} \cup \left\{ \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}}\right\}$$

is uniformly integrable. Indeed, from (2) and (3) and by a well-known result, see [3, Lemma 3.17], it follows that

$$\lim_{n\to\infty}\left|\left|\frac{(1-|z|^2)^\alpha}{(1-|\varphi_n|^2)^{\alpha+2}} - \frac{(1-|z|^2)^\alpha}{(1-|\varphi|^2)^{\alpha+2}}\right|\right|_{1,0} = 0,$$

from which the uniform integrability of $\mathcal{F}$ follows. Therefore, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every measurable set $E \subset U$,

$$m(E) < \delta \quad \Rightarrow \quad \int_S \frac{(1-|z|^2)^\alpha}{(1-|g(z)|^2)^{\alpha+2}}dm(z) < \varepsilon, \tag{4}$$

for every $g \in \mathcal{F}$.

Let $E_n = \{z \in U \mid |\varphi(z)| \le 1 - 1/n\}$. It is clear that $\cup_{n=1}^{\infty} E_n = U$, and that $\lim_{n\to\infty} m(E_n) = m(U) = 1$. Choose n_0 large enough such that $m(E_{n_0}^c) < \delta/2$ ($E_{n_0}^c$ denotes the complement of the set E_{n_0} with respect to U). By Egoroff's theorem, there is a $G \subset E_{n_0}$, so that $\varphi_n \to \varphi$ uniformly on G and $m(G^c \cap E_{n_0}) < \delta/2$; that is, for every $\varepsilon > 0$ there is an $n_1 \in \mathbb{N}$ such that

$$\sup_{z\in G}|\varphi_n(z) - \varphi(z)| < \sqrt{\varepsilon} \quad \text{for} \quad n \ge n_1. \tag{5}$$

Now we estimate $||C_{\varphi_n} - C_\varphi||_{HS}$.

By Stirling's formula ([9, p. 77]), we have

$$c_{n,\alpha}^2 \asymp \frac{n^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{6}$$

By equations (4)-(6), the following estimates hold:

(7)

$$
\begin{aligned}
&||C_{\varphi_n}-C_\varphi||_{HS}^2=\sum_{k=0}^{\infty}c_{k,\alpha}^2||\varphi_n^k-\varphi^k||_{2,\alpha}^2\\
=\ &(\alpha+1)\sum_{k=0}^{\infty}c_{k,\alpha}^2\int_G|\varphi_n^k(z)-\varphi^k(z)|^2(1-|z|^2)^\alpha dm(z)\\
&+(\alpha+1)\sum_{k=0}^{\infty}c_{k,\alpha}^2\int_{G^c}|\varphi_n^k(z)-\varphi^k(z)|^2(1-|z|^2)^\alpha dm(z)\\
\le\ &(\alpha+1)\sup_{z\in G}|\varphi_n(z)-\varphi(z)|^2\\
&\times\sum_{k=0}^{\infty}c_{k,\alpha}^2\int_G|\varphi_n^{k-1}(z)+\varphi_n^{k-2}(z)\varphi(z)+\cdots+\varphi^{k-1}(z)|^2(1-|z|^2)^\alpha dm(z)\\
&+2(\alpha+1)\sum_{k=0}^{\infty}\int_{G^c}c_{k,\alpha}^2(|\varphi_n(z)|^{2k}+|\varphi(z)|^{2k})(1-|z|^2)^\alpha dm(z)\\
\le\ &\varepsilon\sum_{k=0}^{\infty}c_{k,\alpha}^2k^2\left(1-\frac{1}{n_0}\right)^{2k-2}\\
&+2(\alpha+1)\int_{G^c}\left(\frac{(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}}+\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+2}}\right)dm(z)\\
<\ &\varepsilon\left(\sum_{k=0}^{\infty}c_{k,\alpha}^2k^2\left(1-\frac{1}{n_0}\right)^{2k-2}+4(\alpha+1)\right),
\end{aligned}
$$

for sufficiently large n. In view of (6) the series

$$\sum_{k=0}^{\infty}c_{k,\alpha}^2k^2\left(1-\frac{1}{n_0}\right)^{2k-2}$$

is equiconvergent to $\sum_{k=0}^{\infty}k^{\alpha+3}(1-1/n_0)^{2k-2}$, which converges, for example, by Dalambert's criterion. From this and (7) the result follows.

Remark 2. Note that in view of equation (1) Theorem 1 can be written in the following form:

Let φ be an analytic self-map of U and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of U such that C_φ and $(C_{\varphi_n})_{n\in\mathbb{N}}$ are Hilbert-Schmidt operators, $\varphi_n\to\varphi$ a.e. on U, and $\lim_{n\to\infty}||C_{\varphi_n}||_{HS}=||C_\varphi||_{HS}$. Then $\lim_{n\to\infty}||C_{\varphi_n}-C_\varphi||_{HS}=0$.

Theorem 2. *If C_φ is a Hilbert-Schmidt operator, then $||C_{\varphi_n}-C_\varphi||_{HS}\to 0$ if and only if $||C_{\varphi_n}||_{HS}\to||C_\varphi||_{HS}$ and $||\varphi_n-\varphi||_{2,\alpha}\to 0$.*

Proof. If $||C_{\varphi_n} - C_\varphi||_{HS} \to 0$ then clearly, $||C_{\varphi_n}||_{HS} \to ||C_\varphi||_{HS}$. On the other hand,

$$\begin{aligned} ||\varphi_n - \varphi||_{2,\alpha} &= ||C_{\varphi_n}(z) - C_\varphi(z)||_{2,\alpha} \leq ||C_{\varphi_n} - C_\varphi||_{op}||z||_{2,\alpha} \\ &\leq \frac{1}{\sqrt{\alpha+2}}||C_{\varphi_n} - C_\varphi||_{HS}, \end{aligned}$$

from which it follows that $||\varphi_n - \varphi||_{2,\alpha} \to 0$.

Now assume that $||C_{\varphi_n}||_{HS} \to ||C_\varphi||_{HS}$ and $||\varphi_n - \varphi||_{2,\alpha} \to 0$, but $||C_{\varphi_n} - C_\varphi||_{HS} \not\to 0$. Then there is a subsequence $(C_{\varphi_{n_k}})_{k\in\mathbb{N}}$ and an $\varepsilon_0 > 0$ such that

$$||C_{\varphi_{n_k}} - C_\varphi||_{HS} \geq \varepsilon_0 > 0, \qquad k \in \mathbb{N}.$$

Since $||\varphi_{n_k} - \varphi||_{2,\alpha} \to 0$, it follows that there is a subsequence $(\varphi_{n_{k_l}})_{l\in\mathbb{N}}$ that converges to φ a.e. By Theorem 1 we have that $||C_{\varphi_{n_{k_l}}} - C_\varphi||_{HS} \to 0$, which is a contradiction.

Theorem 3. *Let φ be an analytic self-map of U and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of U such that $||\varphi_n - \varphi||_{2,\alpha} \to 0$ as $n \to \infty$. If there is a measurable function $\chi : U \to [0,1]$ such that for every $n \in \mathbb{N}$, $|\varphi_n| \leq |\chi|$ a.e. on U and*

$$\int_U \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+2}} dm(z) < \infty,$$

then the sequence $(C_{\varphi_n})_{n\in\mathbb{N}}$ of composition operators converges in Hilbert-Schmidt norm to C_φ.

Proof. Note that the condition $|\varphi_n| \leq |\chi|$ a.e. on U implies that

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}} \leq \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+2}}, \quad \text{a.e. on } U.$$

Assume now that $||C_{\varphi_n} - C_\varphi||_{HS} \not\to 0$. Then there is a subsequence $(C_{\varphi_{n_k}})_{k\in\mathbb{N}}$ and an $\varepsilon_0 > 0$ such that

$$||C_{\varphi_{n_k}} - C_\varphi||_{HS} \geq \varepsilon_0 > 0, \qquad k \in \mathbb{N}.$$

Since $||\varphi_{n_k} - \varphi||_{2,\alpha} \to 0$, it follows that there is a subsequence $(\varphi_{n_{k_l}})_{l\in\mathbb{N}}$ that converges to φ a.e. By Lebesgue's dominated convergence theorem we have that

$$\lim_{k\to\infty} ||C_{\varphi_{n_{k_l}}}||^2_{HS} = ||C_\varphi||^2_{HS} \leq (\alpha+1)\int_U \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+2}} dm(z).$$

On the other hand, by Theorem 2 it follows that $||C_{\varphi_{n_k}} - C_\varphi||_{HS} \to 0$ as $k \to \infty$, which is a contradiction.

Remark 3. If we apply Theorem 1 then the condition $||\varphi_n - \varphi||_{2,\alpha} \to 0$ as $n \to \infty$, can be replaced by $\varphi_n \to \varphi$ a.e. on U.

Remark 4. Note that from the inequality

$$\int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}} dm(z) \le \int_U \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+2}} dm(z) < \infty, \quad n \in \mathbb{N},$$

it follows that $(C_{\varphi_n})_{n\in\mathbb{N}}$ is a sequence of Hilbert-Schmidt composition operators. Clearly C_φ is also a Hilbert-Schmidt composition operator.

Corollary 1. *The map $\varphi \to C_\varphi$ is continuous from the open unit ball of $H^\infty(U)$ into the set of Hilbert-Schmidt composition operators.*

Proof. Assume that $||\varphi||_\infty < 1$. Then there is a $\delta \in (0,1)$ such that $||\varphi||_\infty + \delta < 1$. On the other hand, if $||\varphi_n - \varphi||_\infty \to 0$ as $n \to \infty$, then $||\varphi_n - \varphi||_{2,\alpha} \to 0$ as $n \to \infty$, and $||\varphi_n||_\infty \le ||\varphi||_\infty + \delta$ for sufficiently large n. Applying Theorem 3 with $\chi = ||\varphi||_\infty + \delta$, the result follows.

Corollary 2. *Let φ be an analytic self-map of U and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of self-maps of U. If*

$$\lim_{n\to\infty} \int_U \frac{|\varphi_n(z) - \varphi(z)|}{(1-|\varphi_n(z)|^2)^{\alpha+2}(1-|\varphi(z)|^2)^{\alpha+2}} (1-|z|^2)^\alpha dm(z) = 0, \tag{8}$$

then $\lim_{n\to\infty} ||C_{\varphi_n} - C_\varphi||_{HS} = 0$.

Proof. Since

$$||\varphi_n - \varphi||_{1,\alpha} \le (\alpha+1) \int_U \frac{|\varphi_n(z) - \varphi(z)|(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}(1-|\varphi(z)|^2)^{\alpha+2}} dm(z),$$

from (8) and the inequality

$$||\varphi_n - \varphi||_{2,\alpha}^2 \le 2||\varphi_n - \varphi||_{1,\alpha},$$

it follows that $||\varphi_n - \varphi||_{2,\alpha} \to 0$ as $n \to \infty$.

Let $I = |\ ||C_{\varphi_n}||_{HS}^2 - ||C_\varphi||_{HS}^2\ |$. Then by Lagrange's theorem it follows that

$$\begin{aligned} I &= (\alpha+1) \left| \int_U \frac{(1-|\varphi_n(z)|^2)^{\alpha+2} - (1-|\varphi(z)|^2)^{\alpha+2}}{(1-|\varphi_n(z)|^2)^{\alpha+2}(1-|\varphi(z)|^2)^{\alpha+2}} (1-|z|^2)^\alpha dm(z) \right| \\ &\le (\alpha+1)(\alpha+2) \int_U \frac{|\ |\varphi_n(z)|^2 - |\varphi(z)|^2\ |}{(1-|\varphi_n(z)|^2)^{\alpha+2}(1-|\varphi(z)|^2)^{\alpha+2}} (1-|z|^2)^\alpha dm(z) \\ &\le 2(\alpha+1)(\alpha+2) \int_U \frac{|\varphi_n(z) - \varphi(z)|(1-|z|^2)^\alpha}{(1-|\varphi_n(z)|^2)^{\alpha+2}(1-|\varphi(z)|^2)^{\alpha+2}} dm(z) \end{aligned}$$

The statement of the corollary follows from the above estimates and Theorem 2.

Theorem 4. *For every pair of distinct symbols* φ, ψ, *let* $\chi = \max\{|\varphi|, |\psi|\}$. *Then the following upper estimates hold*

$$||C_\varphi - C_\psi||_{HS} \leq M \left(\int_U \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+4}} dm(z) \right)^{1/2} ||\varphi - \psi||_\infty, \tag{9}$$

where the constant M *is independent of* φ *and* ψ. *Therefore, for each* $R > 0$, *the map* $\varphi \to C_\varphi$ *is Lipschitz continuous on the set*

$$\mathcal{S}_R = \left\{ \varphi \;\middle|\; \int_U \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+4}} dm(z) \leq R \right\}.$$

Proof. Using (6) we get

$$\begin{aligned}
||C_\varphi - C_\psi||_{HS}^2 &= \sum_{n=0}^{\infty} c_{n,\alpha}^2 ||\varphi^n - \psi^n||_{2,\alpha}^2 \\
&\leq ||\varphi - \psi||_\infty^2 \sum_{n=1}^{\infty} c_{n,\alpha}^2 ||\varphi^{n-1} + \varphi^{n-2}\psi + \cdots + \psi^{n-1}||_{2,\alpha}^2 \\
&\leq (\alpha+1)||\varphi - \psi||_\infty^2 \int_U \big(\sum_{n=1}^{\infty} c_{n,\alpha}^2 n^2 \chi^{2(n-1)}(z) \big) (1-|z|^2)^\alpha dm(z) \\
&\leq M\, ||\varphi - \psi||_\infty^2 \int_U \big(\sum_{n=0}^{\infty} c_{n,\alpha+2}^2\, \chi^{2n}(z) \big) (1-|z|^2)^\alpha dm(z) \\
&\leq M\, ||\varphi - \psi||_\infty^2 \int_U \frac{(1-|z|^2)^\alpha}{(1-|\chi(z)|^2)^{\alpha+4}} dm(z),
\end{aligned}$$

for some positive constant M independent of φ and ψ, as desired.

3. Uniform Operator Convergence

An upper norm estimate for the operator norm of a difference of two composition operators is established in this section. Before we formulate the main result of this section, we need an auxiliary result, which is incorporated in the following lemma.

Lemma 1. ([6,9]) *Assume that* $z \in U$, $\alpha \in \mathbb{R}$, $t > -1$, *and*

$$I_{\gamma,t}(z) = \int_U \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{t+\gamma+2}} dm(w).$$

Then we have

(a) *If* $\gamma < 0$, *then* $\sup_{z\in U} I_{\gamma,t}(z)$ *is finite.*

(b) *If* $\gamma = 0$, *then* $I_{\gamma,t}(z) \sim \ln \frac{1}{1-|z|^2}$, *as* $|z| \to 1-0$.

(c) *If* $\gamma > 0$, *then* $I_{\gamma,t}(z) \sim \frac{1}{(1-|z|^2)^\gamma}$, *as* $|z| \to 1-0$.

Theorem 5. *Let* $\alpha > -1$, *and* φ *and* ψ *are arbitrary symbols. Then*

$$\begin{aligned} &||C_\varphi - C_\psi||_{op} \le \\ (10)\qquad & c_\alpha \Big(\int_U \frac{|\varphi(w)-\psi(w)|(1-|w|^2)^\alpha}{(1-|\varphi(w)|)^{\alpha+2}(1-|\psi(w)|)^{\alpha+2}} \ln \frac{e}{(1-|\varphi(w)|)(1-|\psi(w)|)} dm(w) \Big)^{1/2}, \end{aligned}$$

for some c_α *depending only on* α.

Proof. Let $dm_\alpha(w) = (\alpha+1)(1-|w|^2)^\alpha dm(w)$. Note that this equation defines normalized Lebesgue measure on U. Applying the reproduction formula for the weighted Bergman space ([8, p. 53]) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |f(z)-f(\zeta)|^2 &= \left| \int_U \left(\frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{1}{(1-\zeta\bar{w})^{\alpha+2}} \right) f(w) dm_\alpha(w) \right|^2 \\ &\le \int_U \left| \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{1}{(1-\zeta\bar{w})^{\alpha+2}} \right|^2 dm_\alpha(w)\ ||f||^2_{2,\alpha} \\ (11)\qquad &\le \sup_{w\in U} \left| \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{1}{(1-\zeta\bar{w})^{\alpha+2}} \right| \times \\ &\quad \int_U \left| \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{1}{(1-\zeta\bar{w})^{\alpha+2}} \right| dm_\alpha(w)\ ||f||^2_{2,\alpha}. \end{aligned}$$

We have

$$\begin{aligned} &\sup\nolimits_{w\in U} \left| \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{1}{(1-\zeta\bar{w})^{\alpha+2}} \right| \\ \le\ &\sup\nolimits_{w\in U} \frac{|(1-z\bar{w})^{\alpha+2} - (1-\zeta\bar{w})^{\alpha+2}|}{(1-|z|)^{\alpha+2}(1-|\zeta|)^{\alpha+2}} \\ (12)\qquad =\ &\sup_{w\in U} \frac{|\int_\zeta^z \big((1-u\bar{w})^{\alpha+2} \big)' du|}{(1-|z|)^{\alpha+2}(1-|\zeta|)^{\alpha+2}} \\ \le\ &\sup_{w\in U} \frac{|\alpha+2|\,|\bar{w}|\big|\int_\zeta^z |1-u\bar{w}|^{\alpha+1} |du|\big|}{(1-|z|)^{\alpha+2}(1-|\zeta|)^{\alpha+2}} \\ \le\ &\frac{(\alpha+2)2^{\alpha+1}|z-\zeta|}{(1-|z|)^{\alpha+2}(1-|\zeta|)^{\alpha+2}}. \end{aligned}$$

Let

$$J_\alpha(z,\zeta)=\int_U\left|\frac{1}{(1-z\bar{w})^{\alpha+2}}-\frac{1}{(1-\zeta\bar{w})^{\alpha+2}}\right|dm_\alpha(w).$$

By Lemma 1 (b), we have that

$$\begin{aligned}J_\alpha(z,\zeta)&\le\int_U\frac{dm_\alpha(w)}{|1-z\bar{w}|^{\alpha+2}}+\int_U\frac{dm_\alpha(w)}{|1-\zeta\bar{w}|^{\alpha+2}}\\&\le b_\alpha\left(1+\ln\frac{1}{1-|z|^2}+\ln\frac{1}{1-|\zeta|^2}\right),\end{aligned}\tag{13}$$

for some positive constant b_α and for every $z,\zeta\in U$.

Replacing estimates (13) and (12) in (11), setting $z=\varphi(w)$ and $\zeta=\psi(w)$ in obtained inequality, then multiplying by $dm_\alpha(w)$, integrating over U, and using the definition of the operator norm in Banach spaces, we obtain (10).

From Theorem 5 we obtain the following corollary:

Corollary 3. *Let φ be an analytic self-map of U, $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of self-maps of U and assume that the following condition holds*

$$\lim_{n\to\infty}\int_U\frac{|\varphi_n(w)-\varphi(w)|(1-|w|^2)^\alpha}{(1-|\varphi_n(w)|)^{\alpha+2}(1-|\varphi(w)|)^{\alpha+2}}\ln\frac{e}{(1-|\varphi_n(w)|)(1-|\varphi(w)|)}dm(w)=0,$$

then

$$\lim_{n\to\infty}||C_{\varphi_n}-C_\varphi||_{op}=0.$$

Remark 5. Note that in Corollary 3 we do not request that the composition operator C_φ is a Hilbert-Schmidt.

Finally we would like to point out that using the fact that the set of all polynomials is dense in $\mathcal{A}^2_\alpha$ and the boundedness of $C_\varphi:\mathcal{A}^2_\alpha\to\mathcal{A}^2_\alpha$, the following pointwise convergence result can be proven. A similar result with detailed proof can be found in [7].

Theorem 6. *Let φ be an analytic self-map of U and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of U. If*

$$||\varphi_n-\varphi||_{2,\alpha}\to 0\quad\textit{as}\quad n\to\infty,$$

then for each $f\in\mathcal{A}^2_\alpha$,

$$||C_{\varphi_n}(f)-C_\varphi(f)||_{2,\alpha}\to 0\quad\textit{as}\quad n\to\infty.$$

Interested reader can extend some of the results in the paper for the case of the generalized weighted Bergman space ([4])

$$\mathcal{A}^2_W(U) = \Big\{ f \in H(U) \mid ||f||^2_W = \int_U |f(z)|^2 W(|z|) dm(z) < \infty \Big\},$$

with kernel functions

$$K_w(z) = \sum_{n=0}^{\infty} \frac{\bar{w}^n z^n}{||z^n||^2_W},$$

where $W(r)$ is a positive, continuous function on $(0,1)$ with $\int_0^1 W(r) r dr < \infty$.

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