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AN ATOMIC DECOMPOSITION FOR THE HARDY-SOBOLEV SPACE

Zengjian Lou and Shouzhi Yang

Abstract. We define a Hardy-Sobolev space and give its atomic decomposition. As an application of the decomposition we prove a div-curl lemma.

1. Introduction and Preliminaries

The *Hardy space* $H^1(\mathbb{R}^n)$ is the space of locally integrable functions f for which

$$M(f)(x) = \sup_{t>0} |\psi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$, where $\psi \in \mathcal{D}(\mathbb{R}^n)$ (the space of infinitely differentiable functions with compact supports), $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t}),\ t > 0,\ \int_{\mathbb{R}^n} \psi(x)\, dx = 1$, supp $\psi \subset B(0,1)$, a ball centered at the origin with radius 1. The norm of $H^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^1(\mathbb{R}^n)} = \|M(f)\|_{L^1(\mathbb{R}^n)}.$$

Among many characterizations of Hardy spaces, the atomic decomposition is an important one. An $L^2(\mathbb{R}^n)$ function a is an $H^1(\mathbb{R}^n)$-*atom* if there exists a ball $B = B_a$ in $\mathbb{R}^n$ satisfying:

(1) supp $a \subset B$;

(2) $\|a\|_{L^2(B)} \le |B|^{-1/2}$;

(3) $\int_B a(x)\, dx = 0$.

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The basic result about atoms is the following atomic decomposition theorem (see [3] and [9]): A function f on $\mathbb{R}^n$ belongs to $H^1(\mathbb{R}^n)$ if and only if f has a decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $H^1(\mathbb{R}^n)$-atoms and

$$\sum_{k=0}^{\infty} |\lambda_k| \leq C\|f\|_{H^1(\mathbb{R}^n)}.$$

The *tent space* $\mathcal{N}^p(\mathbb{R}^{n+1}_+)$ $(1 \leq p < \infty)$ is the space of all measurable functions F on $\mathbb{R}^{n+1}_+$ for which $S(F) \in L^p(\mathbb{R}^n)$, where $S(F)$ is the square function defined by

$$S(F)(x) = \left(\int_{\Gamma(x)} |F(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

$\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$ is the cone whose vertex at $x \in \mathbb{R}^n$. The norm of $F \in \mathcal{N}^p(\mathbb{R}^{n+1}_+)$ is defined by

$$\|F\|_{\mathcal{N}^p(\mathbb{R}^{n+1}_+)} = \|S(F)\|_{L^p(\mathbb{R}^n)}.$$

An $\mathcal{N}^p(\mathbb{R}^{n+1}_+)$-*atom* is a function α supported in a tent $T(B) = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x - x_0| \leq r - t\}$ of a ball $B = B(x_0, r)$ in $\mathbb{R}^n$, for which

$$\int_{T(B)} |\alpha(x,t)|^2 \, \frac{dxdt}{t} \leq |B|^{1-2/p}.$$

In [5], Coifman, Meyer and Stein proved the following atomic decomposition theorem: any $F \in \mathcal{N}^p(\mathbb{R}^{n+1}_+)$ can be written as

$$F = \sum_{k=0}^{\infty} \lambda_k \alpha_k,$$

where the α_k are $\mathcal{N}^p(\mathbb{R}^{n+1}_+)$-atoms and

$$\sum_{k=0}^{\infty} |\lambda_k| \leq C\|F\|_{\mathcal{N}^p(\mathbb{R}^{n+1}_+)}.$$

Let $\mathcal{D}'(\mathbb{R}^n)$ denote the dual of $\mathcal{D}(\mathbb{R}^n)$, often called the space of distributions. For $f \in \mathcal{D}'(\mathbb{R}^n)$, its gradient is defined, in the sense of distributions, by

$$< \nabla f, \varphi >= - \int_{\mathbb{R}^n} f \text{ div } \varphi \, dx$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$. For $f = (f_1, \cdots, f_n) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$, we say that curl $f = 0$ on $\mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} \left(f_j \frac{\partial \varphi}{\partial x_i} - f_i \frac{\partial \varphi}{\partial x_j} \right) dx = 0, \qquad \varphi \in \mathcal{D}(\mathbb{R}^n),\ i,\ j = 1, \cdots, n.$$

Let $H^1(\mathbb{R}^n, \mathbb{R}^n)$ denote the Hardy space of functions $f = (f_1, \cdots, f_n)$ each of whose components f_l is in $H^1(\mathbb{R}^n)$ $(l = 1, \cdots, n)$ with norm

$$\|f\|_{H^1(\mathbb{R}^n, \mathbb{R}^n)} = \sum_{l=1}^{n} \|f_l\|_{H^1(\mathbb{R}^n)}.$$

In this paper, we investigate the space of f in $\mathcal{D}'(\mathbb{R}^n)$ whose gradient ∇f is in $H^1(\mathbb{R}^n, \mathbb{R}^n)$. We call it Hardy-Sobolev space and thus set

$$H^{1,1}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : \nabla f \in H^1(\mathbb{R}^n, \mathbb{R}^n)\}$$

with the semi-norm of $f \in H^{1,1}(\mathbb{R}^n)$

$$\|f\|_{H^{1,1}(\mathbb{R}^n)} = \|\nabla f\|_{H^1(\mathbb{R}^n, \mathbb{R}^n)}$$

(see [2] for more information on a slight different Hardy-Sobolev space). We call a function $a \in L^2(\mathbb{R}^n)$ an $H^{1,1}(\mathbb{R}^n)$*-atom* if there exists a ball B in $\mathbb{R}^n$ such that

(1) supp $a \subset B$;
(2) $\|a\|_{L^2(B)} \leq r(B)|B|^{-1/2}$, where $r(B)$ denotes the radius of B;
(3) ∇a is an $H^1(\mathbb{R}^n, \mathbb{R}^n)$-atom.

It is easy to see that if a is an $H^{1,1}(\mathbb{R}^n)$-atom, then $a \in H^{1,1}(\mathbb{R}^n)$. Since f is in $H^{1,1}(\mathbb{R}^n)$ if and only if $f + C$ is in $H^{1,1}(\mathbb{R}^n)$ (C is a constant), we consider all functions $f + C$ are same as f. As a main theorem of the paper we show that any f in $H^{1,1}(\mathbb{R}^n)$ can be decomposed into a sum of $H^{1,1}(\mathbb{R}^n)$-atoms. As an application of the decomposition we prove a div-curl lemma.

Throughout the paper, unless otherwise specified, C denotes a constant independent of functions and domains related to the inequalities. Such C may differ at different occurrences.

2. Atomic Decomposition

The main result of the paper is the following atomic decomposition theorem.

Theorem 1. *A distribution f on $\mathbb{R}^n$ is in $H^{1,1}(\mathbb{R}^n)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k*'s are* $H^{1,1}(\mathbb{R}^n)$*-atoms and* $\sum_{k=0}^{\infty}|\lambda_k|<\infty$*. Furthermore,*

$$\|f\|_{H^{1,1}(\mathbb{R}^n)} \sim \inf\left(\sum_{k=0}^{\infty}|\lambda_k|\right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality are absolute constants.

For the proof of Theorem 1, we need two lemmas.

Lemma 1. *If* $g \in H^1(\mathbb{R}^n,\mathbb{R}^n)$ *and* curl $g=0$*, then* g *has a decomposition*

$$g=\sum_{k=0}^{\infty}\lambda_k b_k,$$

where the b_k*'s are* $H^1(\mathbb{R}^n,\mathbb{R}^n)$*-atoms satisfying* curl $b_k=0$ *and*

$$\sum_{k=0}^{\infty}|\lambda_k| \le C\|g\|_{H^1(\mathbb{R}^n,\mathbb{R}^n)}$$

Proof. From Lemma 1.1 in [6], there exists a function $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that

(1) supp $\varphi \subset B(0,1)$;

(2) $\varphi \in C^\infty(\mathbb{R}^n)$;

(3) $\int_0^\infty t|\xi|^2\hat{\varphi}(t\xi)^2\, dt=1,\ \xi\in\mathbb{R}^n\setminus\{0\}$.

For $g \in H^1(\mathbb{R}^n,\mathbb{R}^n)$, define

$$F(x,t)=t\ \mathrm{div}\Big(g*\varphi_t(x)\Big),\quad x\in\mathbb{R}^n,\quad t>0.$$

Then

$$F(x,t)=t\ \mathrm{div}\big(g_1*\varphi_t(x),\cdots,g_n*\varphi_t(x)\big)=\sum_{l=1}^{n}g_l*(\partial_l\varphi)_t(x),$$

where $g_l, l=1,\cdots,n$, is the component of g.

From the proof of Theorem 6 (3) in [5] (see also Theorems 3 and 4 in Chapter III of [12]), the operator defined by

$$u \quad\to\quad S_\psi(u)$$

is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|S_\psi(u)\|_{L^1(\mathbb{R}^n)} \le C_\psi\|u\|_{H^1(\mathbb{R}^n)},$$

where $S_\psi(u)(x) = \left(\int_{\Gamma(x)} |u * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$, $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$, C_ψ denotes a constant depending on ψ. Thus $g_l \in H^1(\mathbb{R}^n)$ implies $S_{\partial_l\varphi}(g_l) \in L^1(\mathbb{R}^n)$ and

$$\|S_{\partial_l\varphi}(g_l)\|_{L^1(\mathbb{R}^n)} \le C_\varphi \|g_l\|_{H^1(\mathbb{R}^n)}.$$

That is $g_l * (\partial_l\varphi)_t \in \mathcal{N}^1(\mathbb{R}^{n+1}_+)$, further we have $F \in \mathcal{N}^1(\mathbb{R}^{n+1}_+)$ and

$$\|F\|_{\mathcal{N}^1(\mathbb{R}^{n+1}_+)} \le C_\varphi \|g\|_{H^1(\mathbb{R}^n,\mathbb{R}^n)}.$$

Using the atomic decomposition theorem for tent spaces, F has a decomposition

$$F = \sum_{k=0}^{\infty} \lambda_k \alpha_k$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \le C\|F\|_{\mathcal{N}^1(\mathbb{R}^{n+1}_+)},$$

where the α_k's are $\mathcal{N}^1(\mathbb{R}^{n+1}_+)$-atoms i.e. there exist balls B_k such that supp $\alpha_k \subset T(B_k)$ and

$$\int_{T(B_k)} |\alpha_k(x,t)|^2 \frac{dxdt}{t} \le \frac{1}{|B_k|}.$$

Define

$$b_k = -\int_0^\infty t\nabla\big(\alpha_k(\cdot,t) * \varphi_t\big) \frac{dt}{t} := (b_k^1, \cdots, b_k^n),$$

where $b_k^l = -\int_0^\infty \alpha_k(\cdot,t) * (\partial_l\varphi)_t \frac{dt}{t}$, $l = 1, \cdots, n$. It is obvious that curl $b_k = 0$ and easy to check that b_k satisfies the moment condition. Since supp $\alpha_k \subset T(B_k)$ and φ is supported in the unit ball, a simple computation shows that supp $b_k \subset B_k$. We next prove that b_k has also the size condition. Applying Theorem 6 in [5] again, the operator

$$\pi_\psi(\alpha) = \int_0^\infty \alpha(\cdot,t) * \psi_t \frac{dt}{t}$$

is bounded from $\mathcal{N}^2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$ for $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$ and

$$\|\pi_\psi(\alpha)\|_{L^2(\mathbb{R}^n)} \le C_\psi \|\alpha\|_{\mathcal{N}^2(\mathbb{R}^{n+1}_+)}.$$

Since α_k are $\mathcal{N}^1(\mathbb{R}^{n+1}_+)$-atoms, so $\alpha_k \in \mathcal{N}^2(\mathbb{R}^{n+1}_+)$. The boundedness of π_ψ

implies that $b_k^l \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned}
\|b_k^l\|_{L^2(\mathbb{R}^n)}^2 &= \|\pi_{\partial_l\varphi}(\alpha_k)\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq C_\varphi \|\alpha_k\|_{\mathcal{N}^2(\mathbb{R}_+^{n+1})}^2 \\
&= C_\varphi \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{n+1}} |\alpha_k(x,t)|^2 \chi(\frac{y-x}{t})\ \frac{dxdt}{t^{n+1}}\ dy \\
&\leq C_\varphi \int_{T(B_k)} |\alpha_k(x,t)|^2\ \frac{dxdt}{t} \\
&\leq C_\varphi\ |B_k|^{-1},
\end{aligned}$$

where χ denotes the characteristic function in the unit ball. Therefore

$$\|b_k\|_{L^2(B_k,\mathbb{R}^n)} \leq C_\varphi\ |B_k|^{-1/2}.$$

Finally we prove $g = \sum_{k=0}^\infty \lambda_k b_k$. Since $g \in H^1(\mathbb{R}^n, \mathbb{R}^n)$ and curl $g = 0$, there exists a distribution f such that $g = \nabla f$. We have

$$\begin{aligned}
\sum_{k=0}^\infty \lambda_k b_k &= -\int_0^\infty \sum_{k=0}^\infty \lambda_k t \nabla\big(\alpha_k(\cdot,t) * \varphi_t\big)\ \frac{dt}{t} \\
&= -\int_0^\infty \nabla\big(F(\cdot,t) * \varphi_t\big)\ dt \\
&= -\int_0^\infty \nabla\Big\{\Big(t\ \mathrm{div}\big((\nabla f) * \varphi_t\big)\Big) * \varphi_t\Big\}\ dt.
\end{aligned}$$

So it is sufficient to show that

$$-\int_0^\infty \Big(t\ \mathrm{div}\big((\nabla f) * \varphi_t\big)\Big) * \varphi_t\ dt = f,$$

which follows from the condition (3) of φ satisfying, in fact

$$\begin{aligned}
&-\int_0^\infty \Big\{\Big(t\ \mathrm{div}\big((\nabla f) * \varphi_t\big)\Big) * \varphi_t\Big\}^\wedge(\xi)\ dt \\
&= -\int_0^\infty \Big\{t \sum_{l=1}^n \partial_l\Big((\partial_l f) * \varphi_t\Big)\Big\}^\wedge(\xi)\hat{\varphi}(t\xi)\ dt \\
&= -i\int_0^\infty t \sum_{l=1}^n \xi_l\Big((\partial_l f) * \varphi_t\Big)^\wedge(\xi)\hat{\varphi}(t\xi)\ dt = \int_0^\infty t\sum_{l=1}^n \xi_l^2 \hat{\varphi}(t\xi)^2 \hat{f}(\xi)\ dt \\
&= \int_0^\infty t|\xi|^2 \hat{\varphi}(t\xi)^2 \hat{f}(\xi)\ dt = \hat{f}(\xi),
\end{aligned}$$

where i is the image unit with $i^2 = -1$. Lemma 1 is proved. ■

Let Ω be a smooth domain. For $f \in L^2(\Omega, \mathbb{R}^n)$, we say that curl $f = 0$ on Ω if

$$\int_\Omega \left(f_j \frac{\partial \varphi}{\partial x_i} - f_i \frac{\partial \varphi}{\partial x_j} \right) dx = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$, $i,\ j = 1, \cdots, n$. For $f \in L^2(\Omega, \mathbb{R}^n)$ with curl $f = 0$ on Ω, define $\nu \times f|_{\partial\Omega}$ by

$$\int_{\partial\Omega} (\nu \times f) \cdot \varphi \, dx = \int_\Omega f \cdot \text{curl } \Phi \, dx$$

for all $\Phi \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $\varphi = \Phi|_{\partial\Omega}$, where ν denotes the outward unit normal vector. Note that the definition of $\nu \times f|_{\partial\Omega}$ is independent of the choice of the extensions Φ ([8, page 208]). Let $W^{1,2}(\Omega)$ denote the Sobolev space and $W_0^{1,2}(\Omega)$ be the space of functions in $W^{1,2}(\Omega)$ with zero boundary values (see [1]). The following lemma can be obtained from Theorem 3.3.3 in Chapter 3 of [11].

Lemma 2. *Let Ω be a bounded smooth contractible domain. If $u \in L^2(\Omega, \mathbb{R}^n)$ with* curl $u = 0$ *and $\nu \times u|_{\partial\Omega} = 0$, then there exists $v \in W_0^{1,2}(\Omega)$ such that $u = \nabla v$ and*

$$\|v\|_{W^{1,2}(\Omega)} \le C \|u\|_{L^2(\Omega, \mathbb{R}^n)},$$

where the constant C depends on the domain Ω. When Ω is a ball B, we have

$$\|v\|_{L^2(B)} \le C r(B) \|u\|_{L^2(B, \mathbb{R}^n)},$$

where C is independent of $u,\ v$ and B.

Now we turn to the proof of Theorem 1.

Proof. Necessity. For $f \in H^{1,1}(\mathbb{R}^n)$, let $g = \nabla f$. Then $g \in H^1(\mathbb{R}^n, \mathbb{R}^n)$ and curl $g = 0$. Applying Lemma 1, g can be written as

$$g = \sum_{k=0}^{\infty} \lambda_k b_k.$$

where b_k are $H^1(\mathbb{R}^n, \mathbb{R}^n)$-atoms with curl $b_k = 0$, and

$$\sum_{k=0}^{\infty} |\lambda_k| \le \|g\|_{H^1(\mathbb{R}^n, \mathbb{R}^n)} = \|f\|_{H^{1,1}(\mathbb{R}^n)}.$$

Since b_k are $H^1(\mathbb{R}^n, \mathbb{R}^n)$-atoms, there exist balls B_k such that supp $b_k \subset B_k$ and

$$\|b_k\|_{L^2(B_k, \mathbb{R}^n)} \le |B_k|^{-1/2}.$$

Combining this with curl $b_k = 0$, Lemma 2 implies that there exist $a_k \in W_0^{1,2}(B_k)$ such that $b_k = \nabla a_k$ and

$$\|a_k\|_{L^2(B_k)} \le Cr(B_k)\|b_k\|_{L^2(B_k,\mathbb{R}^n)} \le Cr(B_k)|B_k|^{-1/2}.$$

Hence a_k are $H^{1,1}(\mathbb{R}^n)$-atoms and

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

in the sense of distributions, where we considered $f + C$ as f.

Sufficiency. Suppose f can be written as a sum of $H^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$-atoms a_k. To prove $f \in \mathcal{D}'(\mathbb{R}^n)$, it is sufficient to show that the sum $\sum_{k=0}^{\infty} \lambda_k a_k$ is convergent in the sense of distributions. From $\sum_{k=0}^{\infty} |\lambda_k| < \infty$, we have

$$\sum_{k=m}^{m'} |\lambda_k| \to 0 \qquad \text{as} \quad m,\ m' \to \infty.$$

Combining this with the size condition of a_k, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with compact support K, we get

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} \left(\sum_{k=m}^{m'} \lambda_k a_k\right) \varphi\ dx\Big| &\le \sum_{k=m}^{m'} |\lambda_k| \Big|\int_{B_k \cap K} a_k \varphi\ dx\Big| \\
&\le \|\varphi\|_{L^\infty(K)} \sum_{k=m}^{m'} |\lambda_k| \|a_k\|_{L^2(B_k \cap K)} |B_k \cap K|^{1/2} \\
&\le \|\varphi\|_{L^\infty(K)} \sum_{k=m}^{m'} |\lambda_k| r(B_k) |B_k|^{-1/2} |B_k \cap K|^{1/2} \\
&\le \|\varphi\|_{L^\infty(K)} \max\{1, |K|^{1/2}\} \sum_{k=m}^{m'} |\lambda_k| \\
&\to 0 \quad \text{as} \ m,\ m' \to \infty.
\end{aligned}$$

The convergence of $\sum_{k=0}^{\infty} \lambda_k a_k$ is proved, so $f \in \mathcal{D}'(\mathbb{R}^n)$. Applying the atomic decomposition theorem for $H^1(\mathbb{R}^n)$, we have $\nabla f \in H^1(\mathbb{R}^n, \mathbb{R}^n)$ and

$$\|f\|_{H^{1,1}(\mathbb{R}^n)} = \|\nabla f\|_{H^1(\mathbb{R}^n,\mathbb{R}^n)} \le C \sum_{k=0}^{\infty} |\lambda_k|.$$

That is $f \in H^{1,1}(\mathbb{R}^n)$. The proof of Theorem 1 is finished. ∎

Remark 1. In [10], Peng defined Hardy-Sobolev spaces H_k^p as spaces of f in Hardy spaces H^p with $D^\alpha f \in H^p$ ($|\alpha| \le k$) and obtained some analogous results to those for Sobolev spaces.

3. An Application: Div-curl Lemma

In [4, Theorem 2], Coifman, Lions, Meyer and Semmes proved the following well-known Div-curl Lemma: Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with curl $f = 0$ and $e \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ with div $e = 0$ on $\mathbb{R}^n$. Then $e \cdot f \in H^1(\mathbb{R}^n)$. We now consider the case of $p = 1$, as an application of Theorem 1 we give the endpoint version of the div-curl lemma.

Theorem 2. *Let $f \in H^{1,1}(\mathbb{R}^n)$ and $e \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with* div $e = 0$ *on* $\mathbb{R}^n$. *Then $e \cdot \nabla f \in H^1(\mathbb{R}^n)$.*

Proof. If $f \in H^{1,1}(\mathbb{R}^n)$, Theorem 1 yields that f has the decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $H^{1,1}(\mathbb{R}^n)$-atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. Therefore, for $e \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$e \cdot \nabla f = \sum_{k=0}^{\infty} \lambda_k e \cdot \nabla a_k.$$

To prove $e \cdot \nabla f \in H^1(\mathbb{R}^n)$, we need only to show that $e \cdot \nabla a_k$ are $H^1(\mathbb{R}^n)$-atoms by the atomic decomposition theorem for $H^1(\mathbb{R}^n)$. Since a_k is an $H^{1,1}(\mathbb{R}^n)$-atom, there exists a ball B_k in $\mathbb{R}^n$ such that supp $\nabla a_k \subset B_k$ and $\|\nabla a_k\|_{L^2(B_k, \mathbb{R}^n)} \le |B_k|^{-1/2}$. Combining this with $e \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ implies that

$$\|e \cdot \nabla a_k\|_{L^2(\mathbb{R}^n)} \le C|B_k|^{-1/2},$$

where $C = \|e\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)}$. By a simple calculation and div $e = 0$, we get

$$e \cdot \nabla a_k = \text{div } (a_k e),$$

which yields the moment condition

$$\int_{\mathbb{R}^n} e \cdot \nabla a_k \, dx = 0.$$

We proved Theorem 2. ∎

Remark 2. If the condition: $f \in H^{1,1}(\mathbb{R}^n)$ is replaced by $f \in L^1(\mathbb{R}^n)$ and $\nabla f \in H^1(\mathbb{R}^n, \mathbb{R}^n)$, Theorem 2 was proved in [2, Theorem 21] by a different method.

Corollary. *Let $f \in H^1(\mathbb{R}^n, \mathbb{R}^n)$ with* curl $f = 0$ *on* $\mathbb{R}^n$ *and* $e \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *with* div $e = 0$ *on* $\mathbb{R}^n$. *Then $e \cdot f \in H^1(\mathbb{R}^n)$.*

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REFERENCES

1. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
2. P. Auscher, E. Russ and Ph. Tchamitchian, Hardy Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$, *J. Funct. Anal.*, **218** (2005), 54-109.
3. R. Coifman, A real variable characterization of H^p, *Studia Math.*, **51** (1974), 269-274.
4. R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures and Appl.*, **72** (1993), 247-286.
5. R. Coifman, Y. Meyer and E. M. Stein, Some new function spaces and their application to harmonic analysis, *J. Funct. Anal.*, **62** (1985), 304-335.
6. M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study on function space*, CBMS-AMS Regional Conf., Auburn Univ., 1989.
7. V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, theory and algorithms, Springer-Verlag, Berlin Heidelberg, 1986.
8. J. A. Hogan, C. Li, A. McIntosh and K. Zhang, Global higher integrability of Jacobians on bounded domains, *Ann. Inst. H. Poincaré Anal. Non linéaire*, **17**(2000), 193-217.
9. R. H. Latter, *A decomposition of* $H^p(\mathbb{R}^n)$ *in term of atoms*, Studia Math. **62** (1978), 92-101.
10. L. Z. Peng, Hardy-Sobolev spaces, *Beijing Daxue Xuebao* (*in Chinese*), **2** (1983), 26-41.
11. G. Schwarz, *Hodge Decomposition* **-** *A method for solving boundary value problems*, Lecture Notes in Mathematics Vol. 1607, Springer-Verlag, Berlin Heidelberg, 1985.
12. E. M. Stein, *Harmonic Analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, New Jersey, 1993.

Zengjian Lou and Shouzhi Yang
Department of Mathematics,
Shantou University,
Shantou Guangdong 515063,
P. R. China
E-mail: zjlou@stu.edu.cn
E-mail: szyang@stu.edu.cn