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VECTOR VALUED COMMUTATORS ON NON-HOMOGENEOUS SPACES

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Abstract. Let μ be a Borel measure on $\mathbb{R}^d$ which may be non doubling. The only condition that μ must satisfy is $\mu(Q) \leq c_0 l(Q)^n$ for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes and for some fixed n with $0 < n \leq d$. This paper is to develop the vector valued commutator theory in the context of the non-homogeneous spaces. As an application, the boundedness of the maximal commutator of any Calderón-Zygmund operator on the non-homogeneous space with a $RBMO(\mu)$ function introduced by Tolsa in [9] is obtained.

1. INTRODUCTION

Let μ be a non-negative n-dimensional Borel measure on R^d, that is, a measure satisfying

$$\mu(Q) \leq c_0 l(Q)^n$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \leq d$. Throughout this paper, all cubes we shall consider will be those with sides parallel to the coordinate axes. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$. Moreover, $Q(x, r)$ will be the cube centered at x with side length r.

The classical theory of harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^n, \mu)$ has been developed under the assumption that the underlying measure μ satisfies the doubling property, i.e., there exists a constant $c > 0$ such that $\mu(B(x, 2r)) \leq c\mu(B(x, r))$ for every $x \in \mathbb{R}^n$ and $r > 0$. However, some recent

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results on Calderón-Zygmund operators ([4,5,7,8]) and functions of bounded mean oscillation ([3,9]) show that it should be possible to dispense with the doubling condition for most of the classical theory. The purpose of this paper is to develop the vector valued commutator theory in this new setting.

Let us introduce some notations and definitions. Given a Banach space E we will denote by $L^p_E(\mu)$ the Bochner-Lebesgue space of $E-$valued strongly measurable functions such that

$$\int_{\mathbb{R}^d} \|f(x)\|^p_E d\mu < +\infty.$$

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, we set

$$K_{Q,\,R} = 1 + \sum_{k=1}^{N_{Q,\,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q,\,R}$ is the first integer k such that $l(2^k Q) \geq l(R)$. $K_{Q,\,R}$ was introduced by Tolsa in [9].

Given β_d (depending on d) big enough (for example, $\beta_d > 2^n$), we say that some cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \leq \beta_d \mu(Q)$.

Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer ≥ 0 such that $2^N Q$ is doubling. We denote this cube by $\tilde{Q}$.

Let $\eta > 1$ be some fixed constant. We say that a locally integrable function $b(x)$ is in $RBMO(\mu)$ if there exists some constant c_1 such that for any cube Q,

(1)
$$\frac{1}{\mu(\eta Q)} \int_Q |b - m_{\tilde{Q}} b| d\mu \leq c_1$$

and

(2)
$$|m_Q b - m_R b| \leq c_1 K_{Q,\,R} \text{ for any two doubling cubes } Q \subset R,$$

where $m_Q b = \dfrac{1}{\mu(Q)} \displaystyle\int_Q b d\mu$. The minimal constant c_1 is the $RBMO(\mu)$ norm of b, and it will be denoted by $\|b\|_*$.

Let $E,\,F$ be a couple of Banach spaces. $\mathcal{L}(E,\,F)$ will denote the set of bounded linear operators from E to F. We say a kernel $k(x,\,y) : \mathbb{R}^d \times \mathbb{R}^d \backslash \{(x,\,y) : x = y\} \to \mathcal{L}(E,\,F)$ is a (vector-valued) $n-$dimensional singular integral kernel if

(1) $\|k(x,\,y)\|_{\mathcal{L}(E,\,F)} \leq \frac{A}{|x-y|^n}$ if $x \neq y$,

(2) and there exists $0 < \delta \leq 1$ such that

$$\|k(x,\,y) - k(x',\,y)\|_{\mathcal{L}(E,F)} + \|k(y,\,x) - k(y,\,x')\|_{\mathcal{L}(E,F)} \leq \frac{A|x-x'|^\delta}{|x-y|^{n+\delta}}$$

if $|x-y| > 2|x-x'|$.

A bounded linear operator T from $L^2_E(\mu)$ to $L^2_F(\mu)$ is said to be a vector-valued Calderón-Zygmund operator with $n-$dimensional singular integral kernel k if for every compactly supported function $f \in L^2_E(\mu)$,

$$Tf(x) = \int_{\mathbb{R}^d} k(x,\, y) f(y) d\mu(y), \text{ for } x \notin \operatorname{supp} f.$$

For $r > 0$, we define the truncated operators by

$$T_r f(x) = \int_{\mathbb{R}^d \setminus B(x,r)} k(x,\, y) f(y) d\mu(y)$$

and define the maximal operator associated with T as follows

$$T_* f(x) = \sup_{r>0} \|T_r f(x)\|_F.$$

2. Cotlar Type Inequality and Boundedness of T_*

Now we are ready to prove the boundedness of the maximal operator T_*. This follows immediately from

Theorem 1. *Let $f \in L^2_E(\mu)$. For any $\beta > 1$ and $x \in supp\mu$,*

$$T_* f(x) \le 4 \cdot 9^n \tilde{M}(\|Tf\|_F)(x) + B(\beta) \tilde{M}_\beta(\|f\|_E)(x)$$

where the constant $B(\beta) > 0$ depends on the parameter $\beta > 1$, the dimension n, the constants δ and A in the definition of the singular integral kernel k, and the norm $\|T\|_{L^2_E \to L^2_F}$ only. $\tilde{M} f(x) = \sup_{r>0} \dfrac{1}{\mu(B(x,\, 3r))} \int_{B(x,r)} |f(y)| d\mu(y)$ and $\tilde{M}_\beta f(x) = [\tilde{M}(|f|^\beta)(x)]^{1/\beta}$.

Proof. We follow the ideas of [5, Theorem 7.1]. Let $x \in \operatorname{supp}\mu$ and $r > 0$. Consider the sequence of balls $B(x,\, r_j)$ with $r_j = 3^j r$ and set $\mu_j = \mu(B(x,\, r_j))$. We can choose $l \ge 1$, the smallest positive integer such that $\mu_l \le 2 \cdot 3^n \mu_{l-1}$. Put $R = r_{l-1} = 3^{l-1} r$. Then

$$\begin{aligned}
\|T_r f(x) - T_{3R} f(x)\|_F &\le \int_{B(x,3R)\setminus B(x,r)} \|k(x,\, y)\|_{\mathcal{L}(E,F)} \|f(y)\|_E d\mu(y) \\
&= \sum_{j=1}^{l} \int_{B(x,r_j)\setminus B(x,r_{j-1})} \|k(x,\, y)\|_{\mathcal{L}(E,F)} \|f(y)\|_E d\mu(y) \\
&\le \sum_{j=1}^{l} A r_{j-1}^{-n} \int_{B(x,r_j)} \|f(y)\|_E d\mu(y).
\end{aligned}$$

Note that $r_{j-1} = 3^{j-1-l} r_l$ and

$$\mu_j \le (2 \cdot 3^n)^{j-l+1} \mu_l \quad \text{for} \quad 1 \le j \le l.$$

Hence

$$\begin{aligned}\|T_r f(x) - T_{3R} f(x)\|_F &\le \sum_{j=1}^{l} A r_{j-1}^{-n} \mu_j \tilde{M}(\|f\|_E)(x) \\ &\le 2 \cdot 9^n A \tilde{M}(\|f\|_E)(x) \frac{\mu_l}{r_l^n} \sum_{j=1}^{l} 2^{j-l} \le 4 \cdot 9^n A \tilde{M}(\|f\|_E)(x).\end{aligned}$$

So, we need only to estimate $T_{3R} f(x)$. We consider the average

$$V_R(x) := \frac{1}{\mu(B(x,\, R))} \int_{B(x,\, R)} T f d\mu,$$

which is bounded by $\frac{\mu(B(x, 3R))}{\mu(B(x,R))} \tilde{M}(\|Tf\|_F)(x) \le 4 \cdot 9^n \tilde{M}(\|Tf\|_F)(x)$. On the other hand,

$$\begin{aligned}V_R(x) &= \frac{1}{\mu(B(x,\, R))} \int_{B(x,R)} T(f \chi_{\mathbb{R}^d \setminus B(x,3R)}) d\mu \\ &\quad + \frac{1}{\mu(B(x,\, R))} \int_{B(x,R)} T(f \chi_{B(x,3R)}) d\mu = I + II,\end{aligned}$$

and $\|T_{3R} f(x) - V_R(x)\|_F \le \|T_{3R} f(x) - I\|_F + \|II\|_F$. By using the second condition on the kernel,

$$\begin{aligned}&\|T_{3R} f(x) - I\|_F \\ &\le \frac{1}{\mu(B(x,\, R))} \int_{B(x,R)} \int_{\mathbb{R}^d \setminus B(x,3R)} \|k(x,\, y) - k(z,\, y)\|_{\mathcal{L}(E,\, F)} \|f(y)\|_E d\mu(y) d\mu(z) \\ &\le \frac{1}{\mu(B(x,R))} \int_{B(x,R)} \sum_{j=1}^{\infty} \int_{B(x,3^{j+1}R) \setminus B(x,3^j R)} \frac{A|x-z|^\delta}{|y-x|^{n+\delta}} \|f(y)\|_E d\mu(y) d\mu(z) \\ &\le \frac{1}{\mu(B(x,\, R))} \int_{B(x,R)} \sum_{j=1}^{\infty} \frac{A R^\delta}{(3^j R)^{n+\delta}} \int_{B(x,3^{j+1}R)} \|f(y)\|_E d\mu(y) d\mu(z) \\ &\quad \le c(\delta) 3^n A \tilde{M}(\|f\|_E)(x).\end{aligned}$$

Whereas for the second term, by Hölder's inequality,

$$\begin{aligned}\|II\|_F &\le \frac{1}{\mu(B(x,\, R))} \|T\|_{L^\beta_E(\mu) \to L^\beta_F(\mu)} \mu(B(x,\, R))^{1/\beta'} \|f \chi_{B(x,3R)}\|_{L^\beta_E(\mu)} \\ &\le \|T\|_{L^\beta_E(\mu) \to L^\beta_F(\mu)} \Big(\frac{1}{\mu(B(x,\, R))} \int_{B(x,3R)} \|f(y)\|_E^\beta d\mu \Big)^{1/\beta}.\end{aligned}$$

According to our choice of R, we have

$$\mu(B(x, 3R)) = \mu_l \leq 2 \cdot 3^n \mu_{l-1} = 2 \cdot 3^n \mu(B(x, R)).$$

This allows us to conclude that the second term is bounded by

$$(2 \cdot 3^n)^{1/\beta} \|T\|_{L^\beta_E(\mu) \to L^\beta_F(\mu)} \tilde{M}_\beta(\|f\|_E)(x).$$

By taking the supremum on $r > 0$ we have the desired estimate.

3. The Vector Valued Commutator Theorems

Now we can state the main results in this paper.

Theorem 2. *Let E, F be Banach spaces. Let T be a vector-valued Calderón-Zygmund operator with an $n-$dimensional singular integral kernel $k(x, y)$. Given $b \in RBMO(\mu)$. Then the commutator C_b defined by*

$$C_b f(x) = b(x)Tf(x) - T(bf)(x)$$

is bounded from $L^p_E(\mu)$ into $L^p_F(\mu)$ for $1 < p < \infty$.

Theorem 2. *Let F be a Banach lattice and V a bounded linear operator from $L^p(\mu)$ into $L^p_F(\mu)$ for $1 < p < \infty$. Assume that there exists an $F-$valued function $w(x, y)$ satisfying:*

$(W1)$ *for any compactly supported function $f \in L^2(\mu)$,*

$$Vf(x) = \int_{\mathbb{R}^d} w(x, y) f(y) d\mu(y),$$

$(W2)$ *for every x and $y \in \mathbb{R}^d$, $w(x, y) \geq 0$, and for $x \neq y$,*

$$\|w(x, y)\|_F \leq \frac{A}{|x-y|^n},$$

$(W3)$ *there exists $0 < \delta \leq 1$ such that*

$$\|w(x, y) - w(x', y)\|_F + \|w(y, x) - w(y, x')\|_F \leq \frac{A|x-x'|^\delta}{|x-y|^{n+\delta}}$$

if $|x-y| > 2|x-x'|$. If $b \in RBMO(\mu)$, then the operator V_b^+ defined by

$$V_b^+ f(x) = \int_{\mathbb{R}^d} |b(x) - b(y)| w(x, y) f(y) d\mu(y)$$

is bounded from $L^p(\mu)$ into $L^p_F(\mu)$ for $1 < p < \infty$.

4. An Application of the Vector Valued Commutator Theorems

Let $b \in RBMO(\mu)$ and T be a Calderón-Zygmund operator with an $n-$dimensional singular integral kernel $k(x,\, y)$. We define

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} k(x,\, y) f(y) d\mu(y),$$

and

$$C_b^* f(x) = \sup_{\epsilon>0} |b(x) T_\epsilon f(x) - T_\epsilon(bf)(x)|.$$

Then the operator C_b^* is bounded on $L^p(\mu)$ for all $1 < p < \infty$.

Following the idea of [6], we take $\phi,\, \psi \in C^\infty([0,\, \infty))$ such that $|\phi'(t)| \le ct^{-1}$, $|\psi'(t)| \le ct^{-1}$ and

$$\chi_{[2,\,\infty)} \le \phi \le \chi_{[1,\,\infty)},\ \chi_{[1,\,2]} \le \psi \le \chi_{[1/2,\,3]}.$$

We consider the operators

$$\Phi f(x) = \{\phi_\epsilon f(x)\}_{\epsilon>0} = \left\{ \int k(x,\, y) \phi(\frac{|x-y|}{\epsilon}) f(y) d\mu(y) \right\}_{\epsilon>0}$$

and

$$\Psi f(x) = \{\psi_\epsilon f(x)\}_{\epsilon>0} = \left\{ \int |k(x,\, y)| \psi(\frac{|x-y|}{\epsilon}) f(y) d\mu(y) \right\}_{\epsilon>0},$$

with kernels given by

$$\{\phi_\epsilon(x,\, y)\}_{\epsilon>0} = \left\{ k(x,\, y) \phi(\frac{|x-y|}{\epsilon}) \right\}_{\epsilon>0}$$

and

$$\{\psi_\epsilon(x,\, y)\}_{\epsilon>0} = \left\{ |k(x,\, y)| \psi(\frac{|x-y|}{\epsilon}) \right\}_{\epsilon>0}.$$

The kernel of Φ as $l^\infty(\mathbb{R})-$valued function is an $n-$dimensional singular integral kernel. Analogously, it can be shown that the kernel of Ψ satisfies (W2) and (W3) of Theorem 3. By the vector valued Calderón-Zygmund theory on the non-homogeneous spaces, see [1, 2], Φ and Ψ are bounded linear operators from $L^p(\mu)$ into $L^p_{l^\infty}(\mu)$ for all $1 < p < \infty$. Therefore Φ satisfies the hypotheses of Theorem 2 and Ψ satisfies the hypotheses of Theorem 3. Then by Theorems 2 and 3 the operators

$$\Phi_b f(x) = \{b(x)\phi_\epsilon f(x) - \phi_\epsilon(bf)(x)\}_{\epsilon>0}$$

and

$$\Psi_b^+ f(x) = \left\{ \int |b(x) - b(y)| \psi_\epsilon(x, y) f(y) d\mu(y) \right\}_{\epsilon > 0}$$

are bounded from $L^p(\mu)$ into $L^p_{l^\infty}(\mu)$ for all $1 < p < \infty$.
Now, we consider the operator

$$\tilde{T}_b f(x) = \{b(x) T_\epsilon f(x) - T_\epsilon(bf)(x)\}_{\epsilon > 0}.$$

The difference operator

$$\begin{aligned} U_b f(x) &= \Phi_b f(x) - \tilde{T}_b f(x) \\ &= \left\{ \int (b(x) - b(y)) \Big[\phi(\frac{|x-y|}{\epsilon}) - \chi_{[1,\infty)}(\frac{|x-y|}{\epsilon}) \Big] k(x, y) f(y) d\mu(y) \right\}_{\epsilon > 0} \end{aligned}$$

satisfies, for a certain ψ as above, that

$$\begin{aligned} \|U_b f(x)\|_{l^\infty} &\le \sup_{\epsilon > 0} \int |b(x) - b(y)| |k(x, y)| \psi(\frac{|x-y|}{\epsilon}) |f(y)| d\mu(y) \\ &= \|\Psi_b^+ f(x)\|_{l^\infty} \end{aligned}$$

and therefore U_b is bounded from $L^p(\mu)$ into $L^p_{l^\infty}(\mu)$ and, consequently, $\tilde{T}_b$ is bounded from $L^p(\mu)$ into $L^p_{l^\infty}(\mu)$, that is to say C_b^* is bounded on $L^p(\mu)$.

5. The Proof of the Vector Valued Commutator Theorems

Before proving the theorems, we need another equivalent norm for $RBMO(\mu)$ and some lemmas.

Suppose that for a measurable function $b(x)$ there exists some c_2 and for each cube Q, there exists real number b_Q such that

$$\sup_Q \frac{1}{\mu(\eta Q)} \int_Q |b - b_Q| d\mu \le c_2 \tag{3}$$

and

$$|b_Q - b_R| \le c_2 K_{Q,R} \text{ for any two cubes } Q \subset R. \tag{4}$$

Then, we write $\|b\|_{**} = \inf c_2$, where the infimum is taken over all the constants c_2 and all the numbers $\{b_Q\}$ satisfying (3) and (4). By Lemma 2.8 in [9], the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are equivalent for a fixed $\eta > 1$.

In [9], Tolsa defined a sharp maximal operator $M^\# f(x)$ such that

$$f \in RBMO(\mu) \Longleftrightarrow M^\# f \in L^\infty(\mu).$$

In order to prove the theorems, we need to introduce the vector valued version $M_E^{\#}f(x)$. We define

$$M_E^{\#}f(x) = \sup_{x\in Q} \frac{1}{\mu(\frac{3}{2}Q)} \int_Q \|f - m_{\tilde{Q}}f\|_E d\mu + \sup_{\substack{x \in Q \subset R \\ Q,\, R \text{ doubling}}} \frac{\|m_Q f - m_R f\|_E}{K_{Q,R}}.$$

We also consider the non-centered doubling maximal operator N_E:

$$N_E f(x) = \sup_{\substack{x \in Q \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q \|f\|_E d\mu.$$

By the Remark 2.3 of [9], for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} b(y)d\mu(y) = b(x).$$

So, $\|f(x)\|_E \le N_E f(x)$ for $\mu-$ a.e. $x \in \mathbb{R}^d$. Moreover, it is easy to show that N_E is bounded from $L^1_E(\mu)$ into $L^{1,\infty}(\mu)$ and from $L^p_E(\mu)$ into $L^p(\mu)$ for $p \in (1, \infty]$.

Lemma 1. *Let $f(x)$ be an $E-$valued strongly measurable function with $\int f d\mu = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$, if $\inf(1, N_E f) \in L^p(\mu)$, then we have*

$$\|N_E f\|_{L^p(\mu)} \le c\|M_E^{\#} f\|_{L^p(\mu)}.$$

For the scalar case, this is the theorem 6.2 of [9]. By some modifications one can obtain the proof of the lemma. We omit the proof here for brevity.

Proof of the theorem 2. For all $p \in (1, \infty)$, we will show the following sharp maximal function estimate

$$M_F^{\#}(C_b f)(x) \le c_p\|b\|_* \Big(M_{p,(9/8)}(\|f\|_E)(x) + M_{p,(3/2)}(\|Tf\|_F)(x) + T_* f(x)\Big),$$

where, for $\eta > 1$, $M_{p,(\eta)}$ is the non-centered maximal operator

$$M_{p,(\eta)} f(x) = \sup_{x\in Q} \Big(\frac{1}{\mu(\eta Q)} \int_Q |f|^p d\mu\Big)^{1/p},$$

and the operator $M_{p,(\eta)}$ is bounded on $L^r(\mu)$ for $r > p$.

By Theorem 1 and Lemma 1, if we take r such that $1 < r < p < \infty$, we can get

$$\|C_b f\|_{L^p_F(\mu)} \le \|N_F(C_b f)\|_{L^p(\mu)} \le c\|M^{\#}_F(C_b f)\|_{L^p(\mu)}$$
$$\le c\|b\|_*\Big(\|M_{r,(9/8)}(\|f\|_E)\|_{L^p(\mu)} + \|M_{r,(3/2)}(\|Tf\|_F)\|_{L^p(\mu)} + \|T_* f\|_{L^p(\mu)}\Big)$$
$$\le c\|b\|_*\|f\|_{L^p_E(\mu)}.$$

Now we remain to show the above sharp maximal function estimate.

Let $\{b_Q\}_Q$ be a family of numbers satisfying

$$\int_Q |b - b_Q| d\mu \le 2\mu(2Q)\|b\|_{**}$$

for any cube Q, and

$$|b_Q - b_R| \le 2K_{Q,R}\|b\|_{**}$$

for all cubes $Q \subset R$. For any cube Q, we denote

$$h_Q := m_Q\Big(T((b - b_Q)f\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q})\Big).$$

We will prove that

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|C_b f - h_Q\|_F d\mu \le c\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x) + M_{p,(3/2)}(\|Tf\|_F)(x)\Big) \tag{5}$$

for all x and Q with $x \in Q$, and

$$\|h_Q - h_R\|_F \le c\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x) + T_* f(x)\Big)K^2_{Q,R} \tag{6}$$

for all cubes $Q \subset R$ with $x \in Q$.

To get (5) for some fixed cube Q and x with $x \in Q$, we write $C_b f$ in the following way:

$$C_b f = (b - b_Q)Tf - T((b - b_Q)f_1) - T((b - b_Q)f_2), \tag{7}$$

where $f_1 = f\chi_{\frac{4}{3}Q}$ and $f_2 = f - f_1$. Let us estimate the term $(b - b_Q)Tf$:

$$\begin{aligned}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|(b - b_Q)Tf\|_F d\mu &\le \Big(\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |b - b_Q|^{p'} d\mu\Big)^{1/p'} \\ &\quad\times\Big(\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|Tf\|^p_F d\mu\Big)^{1/p} \\ &\le \Big(\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |b - b_Q|^{p'} d\mu\Big)^{1/p'} M_{p,(3/2)}(\|Tf\|_F)(x) \\ &\le c\|b\|_* M_{p,(3/2)}(\|Tf\|_F)(x).\end{aligned} \tag{8}$$

Now we are going to estimate the second term on the right hand side of (7). We take $s=\sqrt{p}$. Then we have

$$
\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|T((b-b_Q)f_1)\|_F d\mu \le \frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)}\|T((b-b_Q)f_1)\|_{L^s_F(\mu)}\\
&\le c\frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)}\|(b-b_Q)f_1\|_{L^s_E(\mu)}\\
&\le c\frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)}\Big(\int_{\frac{4}{3}Q}\|(b-b_Q)f_1\|_E^s d\mu\Big)^{1/s}\\
&\le c\frac{1}{\mu(\frac{3}{2}Q)^{1/s}}\Big(\int_{\frac{4}{3}Q}|b-b_Q|^{ss'}d\mu\Big)^{1/ss'}\Big(\int_{\frac{4}{3}Q}\|f\|_E^p d\mu\Big)^{1/p}\\
&\le c\|b\|_* M_{p,(9/8)}(\|f\|_E)(x).
\end{aligned}\tag{9}
$$

By (7), (8) and (9), to get (5) we only need to estimate the difference $\|T((b-b_Q)f_2)-h_Q\|_F$. For $y_1,\, y_2\in Q$ we have

$$
\begin{aligned}
&\|T((b-b_Q)f_2)(y_1)-T((b-b_Q)f_2)(y_2)\|_F\\
&\le c\int_{\mathbb{R}^d\setminus\frac{4}{3}Q}\frac{|y_2-y_1|^\delta}{|z-y_1|^{n+\delta}}|b(z)-b_Q|\|f(z)\|_E d\mu(z)\\
&\le c\sum_{k=1}^\infty\int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q}\frac{l(Q)^\delta}{|z-y_1|^{n+\delta}}\Big(|b(z)-b_{2^k\frac{4}{3}Q}|\\
&\quad+|b_Q-b_{2^k\frac{4}{3}Q}|\Big)\|f(z)\|_E d\mu(z)\\
&\le c\sum_{k=1}^\infty 2^{-k\delta}\frac{1}{l(2^kQ)^n}\int_{2^k\frac{4}{3}Q}|b(z)-b_{2^k\frac{4}{3}Q}|\|f(z)\|_E d\mu(z)\\
&\quad+c\sum_{k=1}^\infty k2^{-k\delta}\|b\|_*\frac{1}{l(2^kQ)^n}\int_{2^k\frac{4}{3}Q}\|f(z)\|_E d\mu(z)\\
&\le c\sum_{k=1}^\infty 2^{-k\delta}\Big(\frac{1}{\mu(2^k\frac{3}{2}Q)}\int_{2^k\frac{4}{3}Q}|b-b_{2^k\frac{4}{3}Q}|^{p'}d\mu\Big)^{1/p'}\\
&\quad\times\Big(\frac{1}{\mu(2^k\frac{3}{2}Q)}\int_{2^k\frac{4}{3}Q}\|f\|_E^p d\mu\Big)^{1/p}\\
&\quad+c\sum_{k=1}^\infty k2^{-k\delta}\|b\|_*\Big(\frac{1}{\mu(2^k\frac{3}{2}Q)}\int_{2^k\frac{4}{3}Q}\|f\|_E^p d\mu\Big)^{1/p}
\end{aligned}\tag{10}
$$

$$\leq c\sum_{k=1}^{\infty} 2^{-k\delta}\|b\|_* M_{p,(9/8)}(\|f\|_E)(x)$$
$$+c\sum_{k=1}^{\infty} k2^{-k\delta}\|b\|_* M_{p,(9/8)}(\|f\|_E)(x)$$
$$\leq c\|b\|_* M_{p,(9/8)}(\|f\|_E)(x),$$

where we used the fact that

$$|b_Q - b_{2^k\frac{4}{3}Q}| \leq 2K_{Q,2^k\frac{4}{3}Q}\|b\|_{**} \leq ck\|b\|_*.$$

Taking the mean over $y_2 \in Q$, we get

$$\|T((b-b_Q)f_2)(y_1) - h_Q\|_F = \|T((b-b_Q)f_2)(y_1) - m_Q(T((b-b_Q)f_2))\|_F$$
$$\leq c\|b\|_* M_{p,(9/8)}(\|f\|_E)(x).$$

Then

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|T((b-b_Q)f_2)(y_1) - h_Q\|_F d\mu(y_1) \leq c\|b\|_* M_{p,(9/8)}(\|f\|_E)(x), \tag{11}$$

and so (5) holds.

Now we have to check the regularity condition (6) for the elements $\{h_Q\}_Q$. Consider two cubes $Q \subset R$ with $x \in Q$. We denote $N = N_{Q,R} + 1$. We write the difference $\|h_Q - h_R\|_F$ in the following way:

$$\|m_Q(T((b-b_Q)f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q})) - m_R(T((b-b_R)f\chi_{\mathbb{R}^d\setminus\frac{4}{3}R}))\|_F$$
$$\leq \|m_Q(T((b-b_Q)f\chi_{2Q\setminus\frac{4}{3}Q}))\|_F + \|m_Q(T((b_Q-b_R)f\chi_{\mathbb{R}^d\setminus 2Q}))\|_F$$
$$+\|m_Q(T((b-b_R)f\chi_{2^NQ\setminus 2Q}))\|_F + \|m_R(T((b-b_R)f\chi_{2^NQ\setminus\frac{4}{3}R}))\|_F$$
$$+\|m_Q(T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})) - m_R(T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ}))\|_F$$
$$:= U_1 + U_2 + U_3 + U_4 + U_5.$$

Let us estimate U_1. For $y \in Q$ we have

$$\|T((b-b_Q)f\chi_{2Q\setminus\frac{4}{3}Q})(y)\|_F \leq \frac{c}{l(Q)^n}\int_{2Q}|b-b_Q|\|f\|_E d\mu$$
$$\leq \frac{c}{l(Q)^n}\Big(\int_{2Q}|b-b_Q|^{p'}d\mu\Big)^{1/p'}\Big(\int_{2Q}\|f\|_E^p d\mu\Big)^{1/p}$$

$$
\begin{aligned}
&\le c\Big(\frac{1}{\mu(3Q)}\int_{2Q}|b-b_Q|^{p'}d\mu\Big)^{1/p'}\Big(\frac{1}{\mu(\frac{9}{4}Q)}\int_{2Q}\|f\|_E^p d\mu\Big)^{1/p}\\
&\le c\|b\|_* M_{p,\,(9/8)}(\|f\|_E)(x).
\end{aligned}
$$

So we obtain $U_1 \le c\|b\|_* M_{p,\,(9/8)}(\|f\|_E)(x)$.

Let us consider the term U_2. For $x,\, y \in Q$, it is easily seen that

$$
\|T(f\chi_{\mathbb{R}^d\setminus 2Q})(y)\|_F \le T_*f(x) + cM_{p,\,(9/8)}(\|f\|_E)(x).
$$

Thus

$$
\begin{aligned}
U_2 &= \|\frac{1}{\mu(Q)}\int_Q (b_Q - b_R)T(f\chi_{\mathbb{R}^d\setminus 2Q})(y)d\mu\|_F\\
&\le cK_{Q,\,R}\|b\|_*\Big(T_*f(x) + M_{p,\,(9/8)}(\|f\|_E)(x)\Big).
\end{aligned}
$$

The term U_4 is easy to estimate. Some calculations similar to the ones for U_1 yield $U_4 \le c\|b\|_* M_{p,\,(9/8)}(\|f\|_E)(x)$.

Let us turn to estimate the term U_5. Operating as in (10), for any $y,\, z \in R$, we get

$$
\|T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})(y) - T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})(z)\|_F \le c\|b\|_* M_{p,\,(9/8)}(\|f\|_E)(x).
$$

Taking the mean over Q for y and over R for z, we obtain

$$
U_5 \le c\|b\|_* M_{p,\,(9/8)}(\|f\|_E)(x).
$$

Finally, we remain to deal with U_3. For $y \in Q$, we have

$$
\begin{aligned}
\|T((b-b_R)f\chi_{2^NQ\setminus 2Q})(y)\|_F &\le c\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\int_{2^{k+1}Q\setminus 2^kQ}|b-b_R|\|f\|_E d\mu\\
&\le c\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\Big(\int_{2^{k+1}Q}|b-b_R|^{p'}d\mu\Big)^{1/p'}\Big(\int_{2^{k+1}Q}\|f\|_E^p d\mu\Big)^{1/p}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\Big(\int_{2^{k+1}Q}|b-b_R|^{p'}d\mu\Big)^{1/p'}\\
&\le \Big(\int_{2^{k+1}Q}|b-b_{2^{k+1}Q}|^{p'}d\mu\Big)^{1/p'} + \mu(2^{k+1}Q)^{1/p'}|b_{2^{k+1}Q} - b_R|\\
&\le cK_{Q,\,R}\|b\|_*\mu(2^{k+2}Q)^{1/p'}.
\end{aligned}
$$

Thus

$$\begin{aligned}
&\|T((b-b_R)f\chi_{2^N Q\setminus 2Q})(y)\|_F \\
&\leq cK_{Q,R}\|b\|_* \sum_{k=1}^{N-1} \frac{\mu(2^{k+2}Q)^{1/p'}}{l(2^kQ)^n}\Big(\int_{2^{k+1}Q}\|f\|_E^p d\mu\Big)^{1/p} \\
&\leq cK_{Q,R}\|b\|_* \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^{k+2}Q)}{l(2^kQ)^n}\Big(\frac{1}{\mu(2^{k+2}Q)}\int_{2^{k+1}Q}\|f\|_E^p d\mu\Big)^{1/p} \\
&\leq cK_{Q,R}^2\|b\|_* M_{p,(9/8)}(\|f\|_E)(x).
\end{aligned}$$

Taking the mean over Q, we get

$$U_3 \leq cK_{Q,R}^2\|b\|_* M_{p,(9/8)}(\|f\|_E)(x).$$

So by the estimates on U_1, U_2, U_3, U_4 and U_5, the regularity condition (6) follows.

Let us see how from (5) and (6) one obtain the sharp maximal function estimate. From (5), if Q is a doubling cube and $x \in Q$, we have

$$\begin{aligned}
\|m_Q(C_bf)-h_Q\|_F &\leq \frac{1}{\mu(Q)}\int_Q \|C_bf-h_Q\|_F d\mu \\
&\leq c\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x)+M_{p,(3/2)}(\|Tf\|_F)(x)\Big).
\end{aligned} \tag{12}$$

Also, for any cube $Q \ni x$, $K_{Q,\tilde{Q}} \leq c$, and then by (5) and (6) we get

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|C_bf-m_{\tilde{Q}}(C_bf)\|_F d\mu \\
&\leq \frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|C_bf-h_Q\|_F d\mu + \|h_Q-h_{\tilde{Q}}\|_F + \|h_{\tilde{Q}}-m_{\tilde{Q}}(C_bf)\|_F \\
&\leq c\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x)+M_{p,(3/2)}(\|Tf\|_F)(x)+T_*f(x)\Big).
\end{aligned} \tag{13}$$

On the other hand, for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \leq P_0$, where P_0 is the constant in Lemma 9.3 in [9], by (6) we have

$$\|h_Q-h_R\|_F \leq cK_{Q,R}\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x)+T_*f(x)\Big)P_0.$$

So by Lemma 9.3 in [9] we get

$$\|h_Q-h_R\|_F \leq cK_{Q,R}\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x)+T_*f(x)\Big)$$

for all doubling cubes $Q \subset R$ with $x \in Q$ and using (12) again, we obtain

$$\begin{aligned}&\|m_Q(C_b f) - m_R(C_b f)\|_F \\ &\quad \le cK_{Q,R}\|b\|_*\Big(M_{p,(9/8)}(\|f\|_E)(x) + M_{p,(3/2)}(\|Tf\|_F)(x) + T_* f(x)\Big).\end{aligned}$$

From this estimate and (13), we get the sharp maximal function estimate.

Proof of the theorem 3.. Let $\{b_Q\}_Q$ be a family of numbers satisfying

$$\int_Q |b - b_Q| d\mu \le 2\mu(2Q)\|b\|_{**}$$

for any cube Q, and

$$|b_Q - b_R| \le 2K_{Q,R}\|b\|_{**}$$

for all cubes $Q \subset R$. For any cube Q and positive function f , we denote

$$w_Q := m_Q\Big(V(|b - b_Q| f\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q})\Big).$$

We can analogously prove that for all $p \in (1, \infty)$,

$$\begin{aligned}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \|V_b^+ f - w_Q\|_F d\mu \le c\|b\|_*\Big(&M_{p,(9/8)}f(x) \\ &+ M_{p,(3/2)}(\|Vf\|_F)(x)\Big)\end{aligned} \tag{14}$$

for all x and Q with $x \in Q$, and

$$\|w_Q - w_R\|_F \le c\|b\|_*\Big(M_{p,(9/8)}f(x) + \|Vf(x)\|_F\Big)K^2_{Q,R} \tag{15}$$

for all cubes $Q \subset R$ with $x \in Q$. By (14) and (15), we obtain the following sharp maximal function estimate

$$M_F^{\#}(V_b^+ f)(x) \le c_p\|b\|_*\Big(M_{p,(9/8)}f(x) + M_{p,(3/2)}(\|Vf\|_F)(x) + \|Vf(x)\|_F\Big).$$

Then, if we take r such that $1 < r < p < \infty$, we can get

$$\begin{aligned}&\|V_b^+ f\|_{L^p_F(\mu)} \le \|N_F(V_b^+ f)\|_{L^p(\mu)} \le c\|M_F^{\#}(V_b^+ f)\|_{L^p(\mu)} \\ &\quad \le c\|b\|_*\Big(\|M_{r,(9/8)}f\|_{L^p(\mu)} + \|M_{r,(3/2)}(\|Vf\|_F)\|_{L^p(\mu)} + \|Vf\|_{L^p_F(\mu)}\Big) \\ &\quad \le c\|b\|_*\|f\|_{L^p(\mu)}.\end{aligned}$$

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