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STRONG CONVERGENCE TO COMMON FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAP

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Abstract. Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex and bounded subset of E. Let $T_1, T_2, \cdots T_r : C \to C$ be a finite family of asymptotically nonexpansive mappings. In this paper, we suggest and analyze an iterative algorithm for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^r$. We show the convergence of the proposed algorithm to a common fixed point $p \in \cap_{i=1}^r F(T_i)$ which is the unique solution of some variational inequality. Our results can be considered as an refinement and improvement of many known results.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E. Recall that a mapping $f : C \to C$ is called a contraction if there exists a constant $\gamma \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \gamma\|x - y\|, \forall x, y \in C$ and a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let $T_1, T_2, \cdots, T_r$ be a finite family of nonexpansive mappings satisfying that the set $F = \cap_{i=1}^r F(T_i)$ of common fixed points of $T_1, T_2, \cdots, T_r$ is nonempty. The problem of finding a common fixed point of a finite family of nonexpasnive mappings has been investigated by many researchers; see, for example, Atsushiba and Takahashi [1], Bauschke [2], Lions [3], Shimizu and Takahashi [4], Takahashi, Tamura and Toyoda [5], Zeng, Cubiotti and Yao [6]. Especially, in 2005, Kimura, Takahashi and Toyoda [7] deal with an iteration scheme for a finite family of nonexpansive mappings which is more general

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than that of Wittmann's result [8]. They proved the following strong convergence theorem.

Theorem KTT. (see [7, Theorem 4]) *Let E be a uniformly convex Banach space whose norm is uniformly Gateaux differentiable and let C be a closed convex subset of E. Let $T_1, T_2, \cdots, T_r$ be nonexpansive mappings of C into itself such that the set $F = \cap_{i=1}^r F(T_i)$ of common fixed points of $T_1, T_2, \cdots, T_r$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ which satisfy the following control conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*
(ii) $\lim_{n\to\infty} \beta_n^i = \beta^i$ *and* $\sum_{i=1}^r \beta_n^i = 1, n \in N$ *for some* $\beta^i \in (0,1)$*,* $i \in \{1,,\cdots,r\}$*;*
(iii) $\sum_{n=1}^{\infty}\sum_{i=1}^r |\beta_{n+1}^i - \beta_n^i| < \infty$.

Let $x \in C$ and define a sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\sum_{i=1}^r \beta_n^i T_i x_n, \quad n \in N. \tag{1}$$

Then $\{x_n\}$ converges strongly to a common fixed point $p \in \cap_{i=1}^r F(T_i)$.

Recall also that a mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all integers $n \ge 0$ and all $x, y \in C$. The mapping $T : C \to C$ is called uniformly L-Lipschitzian $(L > 0)$ if $\|T^n x - T^n y\| \le L\|x - y\|$, $\forall x, y \in C$ and for all $n \ge 0$. It is clear that every nonexpansive mapping is asymptotically nonexpansive. The converse does not hold. The asymptotically nonexpansive mapping as an important generalization of nonexpansive mapping has been studied by many authors; you may see [9-20, 23-30].

Inspired and motivated by the result of Kimura, Takahashi and Toyoda [7], in this paper, we suggest and analyze an iterative algorithm for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^r$ as follows:

Let C be a nonempty closed convex subset of a real Banach space E, $\{T_i\}_{i=1}^r : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{k_n^{(i)}\}_{i=1}^r$. Let $\{t_n\} \subset (0,1)$, α and β be two positive numbers such that $\alpha + \beta = 1$ and f be a contraction on C, a sequence $\{z_n\}$ iteratively defined by $z_0 \in C$ and

$$z_{n+1} = (1 - \frac{t_n}{k_n}) f(z_n) + \frac{\alpha t_n}{k_n} z_n + \frac{\beta t_n}{k_n}\sum_{i=1}^r \tau_i T_i^n z_n, \tag{2}$$

where $\{\tau_i\}_{i=1}^r$ are positive numbers in $(0,1)$ satisfying $\sum_{i=1}^r \tau_i = 1$ and $k_n = \max\{k_n^{(i)}, i = 1, 2, \cdots, r\}$.

Remark 1.1. From [24, Proposition 1], if $\{T_i\}_{i=1}^r : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{k_n^{(i)}\}_{i=1}^r$, then we can obtain $\{k_n\}$ such that

$$\|T_i^n x - T_i^n y\| \leq k_n \|x - y\|, \forall n \geq 1, x, y \in C, i = 1, 2, \cdots, r,$$

where $k_n = \max\{k_n^{(i)}, i = 1, 2, \cdots, r\}$. In the sequel, we will assume that $\{T_i\}_{i=1}^r : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{k_n\}$.

In this paper we will show the convergence of the proposed algorithm (2) to a common fixed point $p \in \cap_{i=1}^r F(T_i)$ which is the unique solution of some variational inequality. Our results can be considered as an refinement and improvement of many known results.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

for $x \in E$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of the Banach space E. The space E is said to have a Gateaux differentiable norm if the limit

$$\lim_{n\to\infty} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for each $x, y \in S$, and we call E smooth; and E is said to have a uniformly Gateaux differentiable norm if for each $y \in S$ the limit (3) is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit (3) exists uniformly for $(x, y) \in S \times S$. It is well known that if E is smooth then any duality mapping on E is single-valued, and if E has a uniformly Gateaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

Let C a nonempty closed convex and bounded subset of the Banach space E and let the diameter of C be defined by $d(C) = \sup\{\|x - y\| : x, y \in C\}$. For each $x \in C$, let $r(x, C) = \sup\{\|x - y\| : y \in C\}$ and let $r(C) = \inf\{r(x, C) : x \in C\}$

denote the Chebyshev radius of C relative to itself. The normal structure coefficient $N(E)$ of E is defined by

$$N(E) = \inf \left\{ \frac{d(C)}{r(C)} : C \text{ is a closed convex and bounded subset of } E \text{ with } d(C) > 0 \right\}.$$

A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have the uniform normal structure.

A mapping $T : C \to C$ is called uniformly asymptotically regular (in short u.a.r.) if for each $\epsilon > 0$ there exists $n_0 \in N$ such that

$$\|T^{n+1}x - T^n x\| \leq \epsilon,$$

for all $n \geq n_0$ and $x \in C$ and it is called uniformly asymptotically regular with sequence $\{\epsilon_n\}$ (in short u.a.r.s.) if

$$\|T^{n+1}x - T^n x\| \leq \epsilon_n,$$

for all integers $n \geq 1$ and all $x \in C$, where $\epsilon_n \to 0$ as $n \to \infty$.

Remark 2.1. It is clear that every nonexpansive mapping is u.a.r.s.

We let LIM be a Banach limit. Recall that LIM $\in (l^\infty)^*$ such that $\|\text{LIM}\| = 1$, $\liminf_{n\to\infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$, and $\text{LIM}_n a_n = \text{LIM}_n a_{n+1}$ for all $\{a_n\} \in l^\infty$. Let $\{x_n\}$ be a bounded sequence of E. Then we can define the real-valued continuous convex function g on E by $g(z) = LIM_n\|x_n - z\|^2$ for all $z \in C$.

We will need the following lemmas for proving our main results.

Lemma 2.1. ([12]) *Suppose E is a Banach space with uniformly normal structure, C is a nonempty bounded subset of E, and $T : C \to C$ is a uniformly L-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also that there exists a nonempty bounded closed convex subset M of C with the following property (P):*

$$x \in M \ \textit{implies} \ \omega_\omega(x) \subset M,$$

where $\omega_\omega(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\{y \in E : y = weak - \lim_j T^{n_j}x \ \ for some \ \ n_j \to \infty\}.$$

Then T has a fixed point in M.

Lemma 2.2. ([16]) *Let E be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,$$

for all $x, y \in E$ and $\forall j(x+y) \in J(x+y)$.

Lemma 2.3. ([21]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4. ([22]) *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers such that

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,*

(ii) *$\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=0}^{\infty} \alpha_n\beta_n$ is convergent.*

Then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

In this section, we will prove our main results. For the sake of convenience, we prove the conclusions only for the case of $r = 2$ and then the other cases can be proved by the same way. First we give the following result which will be used to prove Theorem 3.5.

Theorem 3.1. *Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex and bounded subset of E. Let $T_1, T_2 : C \to C$ be two asymptotically nonexpansive mappings with sequences $\{k_n\} \subset [1,\infty)$ satisfying $\max\{k_n, n \geq 0\} < N(E)^{\frac{1}{2}}$. Let $f : C \to C$ be a contraction with constant $\gamma \in [0,1)$. Let $\{t_n\} \subset (0, \frac{(1-\gamma)k_n}{k_n-\gamma})$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. Let τ_1 and τ_2 be two positive numbers such that $\tau_1 + \tau_2 = 1$. Suppose $F(S) = F(T_1)\bigcap F(T_2) \neq \emptyset$, where $S = \tau_1 T_1 + \tau_2 T_2$. Then for each integer $n \geq 0$, there exists a unique $x_n \in C$ such that*

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}(\tau_1 T_1^n x_n + \tau_2 T_2^n x_n).$$

Further, if T_1 *and* T_2 *satisfy* $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0$*, then the sequence* $\{x_n\}$ *converges strongly to the unique solution of the variational inequality:*

$$p \in F(T_1) \cap F(T_2) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \le 0, \forall x^* \in F(T_1) \cap F(T_2).$$

Proof. We note that $t_n < \frac{(1-\gamma)k_n}{k_n-\gamma}$ which implies $\delta_n = (1-\frac{t_n}{k_n})\gamma + t_n \in (0,1)$ for each integer $n \ge 0$, then the mapping $T_n : C \to C$ is defined for each $x \in C$ by $T_n x = (1-\frac{t_n}{k_n})f(x) + \frac{t_n}{k_n}(\tau_1 T_1^n x + \tau_2 T_2^n x)$ is a contraction. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}\|T_n x - T_n y\| &\le (1-\frac{t_n}{k_n})\|f(x)-f(y)\| + \frac{t_n}{k_n}\|(\tau_1 T_1^n x + \tau_2 T_2^n x)\\ &\quad -(\tau_1 T_1^n y + \tau_2 T_2^n y)\|\\ &= (1-\frac{t_n}{k_n})\|f(x)-f(y)\| + \frac{t_n}{k_n}\|(\tau_1 T_1^n x - \tau_1 T_1^n y)\\ &\quad +(\tau_2 T_2^n x - \tau_2 T_2^n y)\|\\ &\le (1-\frac{t_n}{k_n})\gamma\|x-y\| + \frac{t_n}{k_n}\{\tau_1 k_n\|x-y\| + \tau_2 k_n\|x-y\|\}\\ &\le [(1-\tfrac{t_n}{k_n})\gamma + t_n]\|x-y\|.\end{aligned}$$

It follows from Banach's contractive principle that there exists a unique x_n in C such that $T_n x_n = x_n$, that is,

$$x_n = (1-\frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}(\tau_1 T_1^n x_n + \tau_2 T_2^n x_n). \tag{4}$$

From the assumptions, we obtain $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. Define a function $g : C \to R^+$ by

$$g(z) = LIM_n\|x_n - z\|^2$$

for all $z \in C$. Since g is continuous and convex, $g(z) \to \infty$ as $\|z\| \to \infty$ and E is reflexive, g attains it infimum over C. Let $M = \{x \in C : g(x) = \inf_{z\in C} g(z)\}$. It is easy to see that M is nonempty, closed and bounded. From [12, Theorem 2, p.1348] we know that though M is not necessarily invariant under S, it does have the property (P). Therefore by Lemma 2.1, we obtain S has a fixed point in M. Let $p \in M \cap F(T_1) \cap F(T_2)$ and let $t \in (0,1)$. For any $x \in C$, we have $g(p) \le g((1-t)p + tx)$. Then, by Lemma 2.2, we have

$$0 \le \frac{g((1-t)p+tx) - g(p)}{t} \le -2LIM_n\langle x-p, j(x_n - p - t(x-p))\rangle.$$

This implies that

$$LIM_n\langle x-p, j(x_n-p-t(x-p))\rangle \le 0. \tag{5}$$

Since j is norm-to-weak* uniformly continuous on any bounded set, from (5), we have

$$LIM_n\langle x-p, j(x_n-p)\rangle \le 0, \forall x \in C.$$

In particular,

$$LIM_n\langle f(p)-p, j(x_n-p)\rangle \le 0.$$

On the other hand, from (4), we have

$$x_n-(\tau_1 T_1^n x_n+\tau_2 T_2^n x_n) = (1-\frac{t_n}{k_n})(f(x_n)-(\tau_1 T_1^n x_n+\tau_2 T_2^n x_n)),$$

which implies that

$$x_n-(\tau_1 T_1^n x_n+\tau_2 T_2^n x_n) = \frac{1-\frac{t_n}{k_n}}{\frac{t_n}{k_n}}(f(x_n)-x_n) = \frac{k_n-t_n}{t_n}(f(x_n)-x_n). \tag{6}$$

Note that

$$\begin{aligned}
&\langle x_n-(\tau_1 T_1^n x_n+\tau_2 T_2^n x_n), j(x_n-p)\rangle \\
&= \langle x_n-p, j(x_n-p)\rangle + \langle(\tau_1 T_1^n p+\tau_2 T_2^n p \\
&\quad -(\tau_1 T_1^n x_n+\tau_2 T_2^n x_n), j(x_n-p)\rangle \\
&= \|x_n-p\|^2+\tau_1\langle T_1^n p-T_1^n x_n, j(x_n-p)\rangle \\
&\quad +\tau_2\langle T_2^n p-T_2^n x_n, j(x_n-p)\rangle \\
&\ge -(k_n-1)\|x_n-p\|^2.
\end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\langle x_n-f(x_n), j(x_n-p)\rangle \le \frac{t_n(k_n-1)}{k_n-t_n}\|x_n-p\|^2,$$

which implies that

$$\limsup_{n\to\infty}\langle x_n-f(x_n), j(x_n-p)\rangle \le 0.$$

Observe that

$$(1-\gamma)\|x_n-p\|^2 \le \langle x_n-f(x_n), j(x_n-p)\rangle + \langle f(p)-p, j(x_n-p)\rangle.$$

Thus we have

$$LIM_n\|x_n - p\| = 0.$$

Consequently, by the same argument as that in the proof of [22, Theorem 3.1], Theorem 3.1 is easily proved. ∎

Proposition 3.2. *Suppose E is a real Banach space. Let C be a nonempty closed convex and bounded subset of E. Let $T_1, T_2 : C \to C$ be two asymptotically nonexpansive mappings with sequences $\{k_n\} \subset [1,\infty)$ satisfying $\max\{k_n, n \geq 0\} < N(E)^{\frac{1}{2}}$. Let $f : C \to C$ be a contraction with constant $\gamma \in [0,1)$. Let $\{t_n\} \subset (0,1)$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. Let α, β, τ_1 and τ_2 be four positive numbers such that $\alpha+\beta = 1$ and $\tau_1+\tau_2 = 1$. Suppose $F(S) = F(T_1)\bigcap F(T_2) \neq \emptyset$, where $S = \tau_1 T_1 + \tau_2 T_2$. For an arbitrary $z_0 \in C$, let the sequence $\{z_n\}$ be iteratively defined by*

$$z_{n+1} = (1 - \frac{t_n}{k_n})f(z_n) + \frac{\alpha t_n}{k_n} z_n + \frac{\beta t_n}{k_n}(\tau_1 T_1^n z_n + \tau_2 T_2^n z_n).$$

If T_1 and T_2 are u.a.r.s, then $\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0$.

Proof. Set $\alpha_n = \frac{t_n}{k_n}$, then $\alpha_n \to 1$ as $n \to \infty$. Define

$$z_{n+1} = \alpha\alpha_n z_n + (1 - \alpha\alpha_n)y_n.$$

Observe that

$$\begin{aligned}
y_{n+1} - y_n &= \frac{z_{n+2} - \alpha\alpha_{n+1}z_{n+1}}{1-\alpha\alpha_{n+1}} - \frac{z_{n+1} - \alpha\alpha_n z_n}{1-\alpha\alpha_n}\\
&= \frac{(1-\alpha_{n+1})f(z_{n+1}) + \beta\alpha_{n+1}(\tau_1 T_1^{n+1}z_{n+1} + \tau_2 T_2^{n+1}z_{n+1})}{1-\alpha\alpha_{n+1}}\\
&\quad - \frac{(1-\alpha_n)f(z_n) + \beta\alpha_n(\tau_1 T_1^n z_n + \tau_2 T_2^n z_n)}{1-\alpha\alpha_n}\\
&= \frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}[f(z_{n+1}) - f(z_n)] + (\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}} - \frac{1-\alpha_n}{1-\alpha\alpha_n})f(z_n)\\
&\quad + \frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}(T_1^{n+1}z_{n+1} - T_1^{n+1}z_n) + \frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}(T_1^{n+1}z_n - T_1^n z_n)\\
&\quad + (\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}} - \frac{\beta\alpha_n\tau_1}{1-\alpha\alpha_n})T_1^n z_n + \frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}(T_2^{n+1}z_{n+1} - T_2^{n+1}z_n)\\
&\quad + \frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}(T_2^{n+1}z_n - T_2^n z_n) + (\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}} - \frac{\beta\alpha_n\tau_2}{1-\alpha\alpha_n})T_2^n z_n.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\|-\|z_{n+1}-z_n\| \\
&\le \frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}\gamma\|z_{n+1}-z_n\|+\left|\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{1-\alpha_n}{1-\alpha\alpha_n}\right|\|f(z_n)\| \\
&\quad+\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}\|T_1^{n+1}z_{n+1}-T_1^{n+1}z_n\|+\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}\|T_1^{n+1}z_n-T_1^nz_n\| \\
&\quad+\left|\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}-\frac{\beta\alpha_n\tau_1}{1-\alpha\alpha_n}\right|\|T_1^nz_n\|+\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}\|T_2^{n+1}z_{n+1}-T_2^{n+1}z_n\| \\
&\quad+\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}\|T_2^{n+1}z_n-T_2^nz_n\|+\left|\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}-\frac{\beta\alpha_n\tau_2}{1-\alpha\alpha_n}\right|\|T_2^nz_n\| \\
&\quad-\|z_{n+1}-z_n\| \\
&\le \left|\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{1-\alpha_n}{1-\alpha\alpha_n}\right|\|f(z_n)\|+\left|\frac{\beta\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{\beta\alpha_n}{1-\alpha\alpha_n}\right|\|\tau_1T_1^nz_n\| \\
&\quad+\left|\frac{\beta\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{\beta\alpha_n}{1-\alpha\alpha_n}\right|\|\tau_2T_2^nz_n\|+\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}\|T_1^{n+1}z_n-T_1^nz_n\| \\
&\quad+\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}\|T_2^{n+1}z_n-T_2^nz_n\|+\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}k_{n+1}\|z_{n+1}-z_n\| \\
&\quad+\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}k_{n+1}\|z_{n+1}-z_n\|+\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}\gamma\|z_{n+1}-z_n\|-\|z_{n+1}-z_n\| \\
&\le \left|\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{1-\alpha_n}{1-\alpha\alpha_n}\right|\{\|f(z_n)\|+\|\tau_1T_1^nz_n\|+\|\tau_2T_2^nz_n\|\} \\
&\quad+\frac{\beta\alpha_{n+1}\tau_1}{1-\alpha\alpha_{n+1}}\|T_1^{n+1}z_n-T_1^nz_n\|+\frac{\beta\alpha_{n+1}\tau_2}{1-\alpha\alpha_{n+1}}\|T_2^{n+1}z_n-T_2^nz_n\| \\
&\quad+\left\{\frac{\beta\alpha_{n+1}k_{n+1}}{1-\alpha\alpha_{n+1}}+\frac{(1-\alpha_{n+1})\gamma}{1-\alpha\alpha_{n+1}}-1\right\}\|z_{n+1}-z_n\|.
\end{aligned} \tag{8}
$$

We note that

$$
\begin{aligned}
k_{n+1}-\gamma-(\beta k_{n+1}+\alpha-\gamma)&=(1-\beta)k_{n+1}-\alpha \\
&\ge 1-\beta-\alpha=0.
\end{aligned}
$$

It follows that

$$t_{n+1}\le\frac{(1-\gamma)k_{n+1}}{k_{n+1}-\gamma}\le\frac{(1-\gamma)k_{n+1}}{\beta k_{n+1}+\alpha-\gamma},$$

which implies that

$$
\begin{aligned}
&k_{n+1}t_{n+1}\beta+t_{n+1}\alpha-t_{n+1}\gamma\le(1-\gamma)k_{n+1} \\
\Rightarrow\quad &\beta k_{n+1}\alpha_{n+1}+\alpha_{n+1}\alpha-\alpha_{n+1}\gamma\le 1-\gamma \\
\Rightarrow\quad &\beta k_{n+1}\alpha_{n+1}+(1-\alpha_{n+1})\gamma\le 1-\alpha_{n+1}\alpha \\
\Rightarrow\quad &\frac{\beta k_{n+1}\alpha_{n+1}+(1-\alpha_{n+1})\gamma}{1-\alpha_{n+1}\alpha}\le 1.
\end{aligned} \tag{9}
$$

By the conditions, we note that

$$\lim_{n\to\infty}\{\frac{1-\alpha_{n+1}}{1-\alpha\alpha_{n+1}}-\frac{1-\alpha_n}{1-\alpha\alpha_n}\}=0.$$

From (8) and (9), we obtain

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|z_{n+1}-z_n\|)\leq 0.$$

Hence, by Lemma 2.3 we know

$$\lim_{n\to\infty}\|y_n-z_n\|=0,$$

consequently

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \qquad \blacksquare$$

Corollary 3.3. *Suppose E is a real Banach space. Let C be a nonempty closed convex subset of E. Let $T_1, T_2 : C \to C$ be two nonexpansive mappings. Suppose $F(T_1)\bigcap F(T_2)\neq\emptyset$. Let α, β, τ_1 and τ_2 be four positive numbers such that $\alpha+\beta=1$ and $\tau_1+\tau_2=1$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies $\lim_{n\to\infty}\alpha_n=1$. For an arbitrary $z_0\in C$, let the sequence $\{z_n\}_n$ be iteratively defined by*

$$z_{n+1}=(1-\alpha_n)f(z_n)+\alpha\alpha_n z_n+\beta\alpha_n(\tau_1T_1z_n+\tau_2T_2z_n).$$

Then $\lim_{n\to\infty}\|z_{n+1}-z_n\|=0$.

Proof. First we can prove that $\{z_n\}$ is bounded. To end this, by taking a fixed element $p\in F(T_1)\cap F(T_2)$, we have

$$\begin{aligned}
\|z_{n+1}-p\| &\leq (1-\alpha_n)\|f(z_n)-p\|+\alpha\alpha_n\|z_n-p\| \\
&\quad +\beta\alpha_n(\tau_1\|T_1z_n-p\|+\tau_2\|T_2z_n-p\|) \\
&\leq (1-\alpha_n)\|f(z_n)-f(p)\|+(1-\alpha_n)\|f(p)-p\|+\alpha\alpha_n\|z_n-p\| \\
&\quad +\beta\alpha_n(\tau_1\|z_n-p\|+\tau_2\|z_n-p\|) \\
&\leq (1-\alpha_n)\gamma\|z_n-p\|+(1-\alpha_n)\|f(p)-p\|+\alpha_n\|z_n-p\| \\
&= \{1-(1-\gamma)(1-\alpha_n)\}\|z_n-p\|+(1-\gamma)(1-\alpha_n)\frac{\|f(p)-p\|}{1-\gamma} \\
&\leq \max\{\|z_n-p\|,\frac{1}{1-\gamma}\|f(p)-p\|\}.
\end{aligned}$$

By induction, we get

$$\|z_n - p\| \leq \max\{\|z_0 - p\|, \frac{1}{1-\gamma}\|f(p) - p\|\},$$

for all $n \geq 0$. This shows that $\{z_n\}$ is bounded. From Remark 2.1, we know that T_1 and T_2 are u.a.r.s. It follows from Proposition 3.2 that we can conclude the desired result. ■

Remark 3.4. We would like to point out that the conclusion $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$ is very crucial for proving the strong convergence of $\{z_n\}$ in many literatures; please refer to [7, 22].

Theorem 3.5. *Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex and bounded subset of E. Let $T_1, T_2 : C \to C$ be two asymptotically nonexpansive mappings with sequences $\{k_n\} \subset [1, \infty)$ satisfying $\max\{k_n, n \geq 0\} < N(E)^{\frac{1}{2}}$. Let $f : C \to C$ be a contraction with constant $\gamma \in [0, 1)$. Let $\{t_n\} \subset (0, \sigma_n)$ be such that $\lim_{n\to\infty} t_n = 1, \sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$, where $\sigma_n = \min\{\frac{(1-\gamma)k_n}{k_n - \gamma}, \frac{1}{k_n}\}$. Let α, β, τ_1 and τ_2 be four positive numbers such that $\alpha + \beta = 1$ and $\tau_1 + \tau_2 = 1$. Suppose $F(S) = F(T_1) \bigcap F(T_2) \neq \emptyset$, where $S = \tau_1 T_1 + \tau_2 T_2$. For an arbitrary $z_0 \in C$, let the sequence $\{z_n\}$ be iteratively defined by*

$$z_{n+1} = (1 - \frac{t_n}{k_n})f(z_n) + \frac{\alpha t_n}{k_n} z_n + \frac{\beta t_n}{k_n}(\tau_1 T_1^n z_n + \tau_2 T_2^n z_n).$$

Then for each integer $n \geq 0$, there exists a unique $x_n \in C$ such that

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}(\tau_1 T_1^n x_n + \tau_2 T_2^n x_n).$$

Further, if T_1 and T_2 satisfy $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0$ for $i = 1, 2$, then the sequence $\{z_n\}$ converges strongly to the unique solution of the variational inequality:

$$p \in F(T_1) \cap F(T_2) \text{ such that } \quad \langle (I - f)p, j(p - x^*)\rangle \leq 0, \ \forall x^* \in F(T_1) \cap F(T_2).$$

Proof. From Theorem 3.1, we have that there exists a unique $x_m \in C$ such that

$$x_m = (1 - \frac{t_m}{k_m})f(x_m) + \frac{t_m}{k_m}(\tau_1 T_1^m x_m + \tau_2 T_2^n x_m).$$

Set $\alpha_m = \frac{t_m}{k_m}$ for all $m \geq 0$, then we get

$$x_m - z_n = (1 - \alpha_m)(f(x_m) - z_n) + \alpha_m(\tau_1 T_1^m x_m + \tau_2 T_2^m x_m - z_n). \tag{10}$$

Applying Lemma 2.2 to (10), we have an estimation as follows

$$\begin{aligned}
&\|x_m - z_n\|^2 \\
&\le \alpha_m^2 \|\tau_1 T_1^m x_m + \tau_2 T_2^m x_m - z_n\|^2 + 2(1-\alpha_m)\langle f(x_m) - z_n, j(x_m - z_n)\rangle \\
&\le \alpha_m^2 (\|\tau_1 T_1^m x_m + \tau_2 T_2^m x_m - \tau_1 T_1^m z_n - \tau_2 T_2^m z_n\| + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n \\
&\quad - z_n\|)^2 + 2(1-\alpha_m)[\langle f(x_m) - x_m, j(x_m - z_n)\rangle + \|x_m - z_n\|^2] \\
&\le \alpha_m^2 \{(\tau_1 k_m + \tau_2 k_m)\|x_m - z_n\| + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\|\}^2 \\
&\quad + 2(1-\alpha_m)(\langle f(x_m) - x_m, j(x_m - z_n)\rangle + k_m^2 \|x_m - z_n\|^2) \\
&\le \alpha_m^2 \{k_m^2 \|x_m - z_n\|^2 + 2k_m \|x_m - z_n\| \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\| \\
&\quad + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\|^2\} + 2(1-\alpha_m)(\langle f(x_m) - x_m, j(x_m - z_n)\rangle \\
&\quad + k_m^2 \|x_m - z_n\|^2) \\
&= (1-(1-\alpha_m))^2 k_m^2 \|x_m - z_n\|^2 + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\| \\
&\quad \times (2k_m \|x_m - z_n\| + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\|) \\
&\quad + 2(1-\alpha_m)(\langle f(x_m) - x_m, j(x_m - z_n)\rangle + k_m^2 \|x_m - z_n\|^2) \\
&\le (1 + (1-\alpha_m)^2) k_m^2 \|x_m - z_n\|^2 + (\tau_1 \|T_1^m z_n - z_n\| + \tau_2 \|T_2^m z_n - z_n\|) \\
&\quad \times (2k_m \|x_m - z_n\| + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\|) \\
&\quad + 2(1-\alpha_m)\langle f(x_m) - x_m, j(x_m - z_n)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x_m) - x_m, j(z_n - x_m)\rangle &\le \frac{[k_m^2 - 1 + k_m^2(1-\alpha_m)^2]}{1-\alpha_m} M \\
&\quad + \limsup_{n\to\infty} \frac{M(\tau_1 \|T_1^m z_n - z_n\| + \tau_2 \|T_2^m z_n - z_n\|)}{1-\alpha_m},
\end{aligned}$$

where M is a constant such that

$$M \ge \max\{\frac{\|x_m - z_n\|^2}{2}, \frac{2k_m \|x_m - z_n\| + \|\tau_1 T_1^m z_n + \tau_2 T_2^m z_n - z_n\|}{2}\},$$

$\forall m \ge 0, \forall n \ge 0$. So that

$$\limsup_{n\to\infty} \langle f(x_m) - x_m, j(z_n - x_m)\rangle \le \frac{[k_m^2 - 1 + k_m^2(1-\alpha_m)^2]}{1-\alpha_m} M. \tag{11}$$

By Theorem 3.1, $x_m \to p \in F(T_1) \cap F(T_2)$, which solves the variational inequality

$$p \in F(T_1) \cap F(T_2) \text{ such that } \langle (I - f)p, j(p - x^*)\rangle \le 0, \ \forall x^* \in F(T_1) \cap F(T_2).$$

Since j is norm to weak* continuous on any bounded set, letting $m \to \infty$ in (11), we obtain that

$$\limsup_{n\to\infty} \langle f(p) - p, j(z_n - p)\rangle \le 0.$$

From Lemma 2.2, we have

$$
\begin{aligned}
&\|z_{n+1}-p\|^2\\
&=\|(1-\alpha_n)(f(z_n)-p)+\alpha\alpha_n(z_n-p)+\beta\alpha_n(\tau_1T_1^nz_n+\tau_2T_2^nz_n-p)\|^2\\
&\le\|\beta\alpha_n(\tau_1T_1^nz_n+\tau_2T_2^nz_n-p)+\alpha\alpha_n(z_n-p)\|^2\\
&\quad+2(1-\alpha_n)\langle f(z_n)-p,j(z_{n+1}-p)\rangle\\
&\le\beta^2\alpha_n^2\|\tau_1T_1^nz_n+\tau_2T_2^nz_n-p\|^2+2\alpha\beta\alpha_n^2\|\tau_1T_1^nz_n+\tau_2T_2^nz_n-p\|\|z_n-p\|\\
&\quad+\alpha^2\alpha_n^2\|z_n-p\|^2+2(1-\alpha_n)\langle f(z_n)-f(p),j(z_{n+1}-p)\rangle\\
&\quad+2(1-\alpha_n)\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&\le\beta^2\alpha_n^2[\tau_1\|T_1^nz_n-p\|+\tau_2\|T_2^nz_n-p\|]^2+2\alpha\beta\alpha_n^2[\tau_1\|T_1^nz_n-p\|\\
&\quad+\tau_2\|T_2^nz_n-p\|]\|z_n-p\|+\alpha^2\alpha_n^2\|z_n-p\|^2+2(1-\alpha_n)\|f(z_n)-f(p)\|\\
&\quad\times\|z_{n+1}-p\|+2(1-\alpha_n)\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&\le(\beta^2k_n^2+2\beta\alpha k_n+\alpha^2)\alpha_n^2\|z_n-p\|^2+2(1-\alpha_n)\gamma\|z_n-p\|\|z_{n+1}-p\|\\
&\quad+2(1-\alpha_n)\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&\le\alpha_n^2k_n^2\|z_n-p\|^2+\gamma(1-\alpha_n)(\|z_n-p\|^2+\|z_{n+1}-p\|^2)\\
&\quad+2(1-\alpha_n)\langle f(p)-p,j(z_{n+1}-p)\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|z_{n+1}-p\|^2&\le\frac{[t_n^2+(1-\alpha_n)\gamma]}{1-(1-\alpha_n)\gamma}\|z_n-p\|^2\\
&\quad+\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\gamma}\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&=\{1-\frac{[1-2(1-\alpha_n)\gamma-t_n^2]}{1-(1-\alpha_n)\gamma}\}\|z_n-p\|^2\\
&\quad+\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\gamma}\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&=(1-\lambda_n)\|z_n-p\|^2+\lambda_n\delta_n.
\end{aligned}
$$

where $\lambda_n=\frac{[1-2(1-\alpha_n)\gamma-t_n^2]}{1-(1-\alpha_n)\gamma}$ and

$$
\begin{aligned}
\delta_n&=\frac{2(1-\alpha_n)}{1-2(1-\alpha_n)\gamma-t_n^2}\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&=\frac{2(1-\frac{t_n}{k_n})}{1-2(1-\frac{t_n}{k_n})\gamma-t_n^2}\langle f(p)-p,j(z_{n+1}-p)\rangle\\
&=\frac{2}{k_n(k_n+t_n)-2\gamma-k_n(k_n+1)\frac{k_n-1}{k_n-t_n}}\langle f(p)-p,j(z_{n+1}-p)\rangle.
\end{aligned}
$$

It is easily observed that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence the conditions in Lemma 2.4 are satisfied and so we can conclude our conclusion. ∎

By the same argument as that in the proof of Theorem 3.5, we can extend Theorem 3.5 to a finite family of asymptotically nonexpansive mappings. Since the proof is similar to that of the above result, therefore, is omitted.

Theorem 3.6. *Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex and bounded subset of E. Let $T_1, T_2 \cdots, T_r : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{k_n\} \subset [1, \infty)$ satisfying $\max\{k_n, n \geq 0\} < N(E)^{\frac{1}{2}}$. Let $f : C \to C$ be a contraction with constant $\gamma \in [0, 1)$. Let $\{t_n\} \subset (0, \sigma_n)$ be such that $\lim_{n\to\infty} t_n = 1, \sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$, where $\sigma_n = \min\{\frac{(1-\gamma)k_n}{k_n - \gamma}, \frac{1}{k_n}\}$. Let α, β and $\{\tau_i\}_{i=1}^r$ be positive numbers such that $\alpha + \beta = 1$ and $\sum_{i=1}^r \tau_i = 1$. Suppose $F(S) = \cap_{i=1}^r F(T_i) \neq \emptyset$, where $S = \sum_{i=1}^r \tau_i T_i$. For an arbitrary $z_0 \in C$, let the sequence $\{z_n\}$ be iteratively defined by*

$$z_{n+1} = (1 - \frac{t_n}{k_n})f(z_n) + \frac{\alpha t_n}{k_n} z_n + \frac{\beta t_n}{k_n} \sum_{i=1}^{r} \tau_i T_i^n z_n.$$

Then for each integer $n \geq 0$, there exists a unique $x_n \in C$ such that

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} \sum_{i=1}^{r} \tau_i T_i^n x_n.$$

Further, if $\{T_i\}_{i=1}^r$ satisfy $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0$ for all $i = 1, 2, \cdots, r$, then the sequence $\{z_n\}$ converges strongly to the unique solution of the variational inequality:

$$p \in \cap_{i=1}^r F(T_i) \text{ such that } \quad \langle (I - f)p, j(p - x^*) \rangle \leq 0,\ \forall x^* \in \cap_{i=1}^r F(T_r).$$

Remark 3.7. Since every nonexpansive mapping is asymptotically nonexpansive, our theorem 3.6 holds for the case when $\{T_i\}_{i=1}^r$ are nonexpansive. In this case, from corollary 3.3, the boundedness requirement on C can be removed from the above result, you may consult [22]. On the other hand, by the same argument as that in the proof of Theorem 3.5 and [7, Theorem 5], we can obtain the following corollary which can be viewed as an improvement of [7, Theorem 5].

Corollary 3.8. *Suppose E is a real uniformly convex Banach space which has a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex* *subset of E. Let $T_1, T_2, \cdots, T_r : C \to C$ be a finite family of nonexpansive mappings. Let $f : C \to C$ be a contraction with constant $\gamma \in [0, 1)$. Suppose $\cap_{i=1}^r F(T_i) \neq \emptyset$. Let $\{\tau_i\}_{i=1}^r$ be positive numbers such that $\sum_{i=1}^r \tau_i = 1$. Let α and β be two positive numbers satisfying $\alpha + \beta = 1$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies $\lim_{n\to\infty} \alpha_n = 1$ and $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$. For an arbitrary $z_0 \in C$, let the sequence $\{z_n\}$ be iteratively defined by*

$$z_{n+1} = (1 - \alpha_n)f(z_n) + \alpha\alpha_n z_n + \beta\alpha_n \sum_{i=1}^r \tau_i T_i z_n.$$

Then the sequence $\{z_n\}$ converges strongly to the unique solution of the variational inequality:

$$p \in \cap_{i=1}^r F(T_i) \text{ such that } \quad \langle (I - f)p, j(p - x^*)\rangle \leq 0,\ \forall x^* \in \cap_{i=1}^r F(T_i).$$

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