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WEAKLY COMPLETELY CONTINUOUS SUBSPACES OF OPERATOR IDEALS

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Abstract. By introducing the concept of weakly completely continuous subspaces of operator ideals, it will be given some characterizations of this concept, specially in terms of relative weak compactness of all point evaluations related to that subspace. Also it is shown that the only Banach spaces such that all closed subspace of an operator ideal between them has this property, are reflexive Banach spaces.

1. Introduction

Let X and Y be two Banach spaces and by the meaning of [2] or [8], let $\mathcal{U}$ be an arbitrary Banach operator ideal. A linear subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ of the component $\mathcal{U}(X,Y)$ of operator ideal $\mathcal{U}$ is called strongly completely continuous (in short, scc) in $\mathcal{U}(X,Y)$ (resp. in $K(X,Y)$) if for each two Banach spaces W and Z and any compact operators $R : Y \to W$ and $S : Z \to X$, the left and right multiplication operators L_R and R_S as operators from $\mathcal{M}$ into $\mathcal{U}(X,W)$ and $\mathcal{U}(Z,Y)$ (resp. into $K(X,W)$ and $K(Z,Y)$) respectively, are compact, where $L_R(T) = RT$ and $R_S(T) = TS$, for $T \in \mathcal{M}$. But if under the same conditions, the operators L_R and R_S are weakly compact, then we say that $\mathcal{M}$ is weakly completely continuous (in short, wcc) in $\mathcal{U}(X,Y)$ (resp. in $K(X,Y)$).

Note that by [7] (see also [2, p. 482]), for any Hilbert space H, a bounded linear operator $R \in L(H)$ is compact if and only if the left multiplication operator $L_R : L(H) \to L(H)$ is weakly compact. This shows that it is not possible to remove or weaken the compactness of the operators R and S in the definitions of (scc) and (wcc).

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In [5], the authors introduced the concept of (scc) subspaces of operator ideals, which extends complete continuity of closed subalgebras of $K(X)$ of all compact operators on a Banach space X, and obtained some characterizations of this concept. For instance, if $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is (scc) in $\mathcal{U}(X,Y)$, then all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively compact in Y and X^* respectively, where $x \in X$ and $y^* \in Y^*$, $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}_1}(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$, and $\mathcal{M}_1$ is the closed unit ball of $\mathcal{M}$. The converse of this assertion is also valid if $\mathcal{U}$ is a closed operator ideal.
Here, we will prove that the same conclusions, among some improvements, hold for (wcc) subspaces instead of (scc) subspaces.
The main motivation of these concepts has been some recent work [1], [6], [10] and [12], dealing with some necessary or sufficient conditions for the Schur or the Dunford- Pettis property of subspaces of some special operator ideals. So the notions of (scc) and (wcc) for subspaces $\mathcal{M} \subseteq \mathcal{U}(X,Y)$, may help to clarify that setting.

Throughout of this article, X, Y, Z and W are arbitrary Banach spaces, the closed unit ball of a Banach space X is denoted by X_1; X^* is the dual of X and T^* refers to the adjoint of the operator T. $\mathcal{U}$ is an arbitrary (Banach) operator ideal and $\mathcal{U}(X,Y)$ is applied for the component of $\mathcal{U}$. We use $\|T\|$ and $A(T)$ for operator norm and ideal norm of any operator $T \in \mathcal{U}$ respectively. For arbitrary Banach spaces X and Y we use $L(X,Y)$, $K(X,Y)$ and $I(X,Y)$ for the components of operator ideals of all bounded linear, compact and integral operators between Banach spaces X and Y respectively. Our notations are standard and we refer the reader to [2] and [3] for another undefined notations and terminologies.

2. Main Results

For Banach spaces X, Y, Z and W and bounded operators $R : Y \to W$ and $S : Z \to X$, define the wedge product operator $S \wedge R$ from $\mathcal{U}(X,Y)$ into $\mathcal{U}(Z,W)$ by $(S \wedge R)T = RTS$. In [9], [11] and [13], the authors investigated the weak compactness of the wedge product operators on some operator ideals and C^*- algebras. In particular:

Theorem 2.1. (Saksman- Tylli and Racher). *Let X, Y, Z and W be four Banach spaces and let $R : Y \to W$ and $S : Z \to X$ be two weakly compact operators. If R or S is compact, then the wedge product operator $S \wedge R$ is a weakly compact operator from $L(X,Y)$(resp. $I(X,Y)$) into $L(Z,W)$(resp. $I(Z,W)$).*

As a corollary, if X and Y are two reflexive Banach spaces, then each operator ideal $\mathcal{U}(X,Y)$ and so each linear subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is (wcc) in $K(X,Y)$, since $\mathcal{U}(X,Y)$ is a linear subspace of $L(X,Y)$ and the identity operator on a reflexive space is weakly compact. Also for reflexive Banach spaces X and Y, the operator ideal $I(X,Y)$ of all integral operators is (wcc). It follows that each linear subspace $\mathcal{M} \subseteq I(X,Y)$ is (wcc) in $I(X,Y)$.

At the first characterization of weak complete continuity of subspaces $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ we have the following theorem:

Theorem 2.2. *Let X and Y be arbitrary Banach spaces and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ be a linear subspace. Then $\mathcal{M}$ is (wcc) in $\mathcal{U}(X,Y)$ if and only if for each two operators $R : Y \to W$ and $S : Z \to X$, the restricted wedge operator $S \wedge R : \mathcal{M} \to \mathcal{U}(Z,W)$ is weakly compact, whenever R or S is compact.*

Proof. Because of $S \wedge R = L_R \circ R_S = R_S \circ L_R$ and by the ideal property of weakly compact operators, the necessity condition of this theorem is straightforward. The sufficiency condition of theorem follows by choosing R compact and S the identity operator on X, respectively S compact and R the identity operator on Y. ■

By the same method, if one replace the weak compactness of $S \wedge R$ by (norm) compactness of this operator, then one obtains a similar characterization of (scc) subspaces of arbitrary operator ideal.

Now we will obtain another criteria for (wcc) subspaces of operator ideals with respect to relative weak compactness of all point evaluations related to that subspace, which are similar to theorems of [5].

Theorem 2.3. *Let X and Y^* have the approximation property and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ be a linear subspace. If all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact in Y and X^* respectively, then $\mathcal{M}$ is (wcc) in $\mathcal{U}(X,Y)$.*

Proof. Let $R : Y \to W$ be a compact operator. Since Y^* has the approximation property, there exists a finite rank sequence $R_n : Y \to W$ such that $\|R_n - R\| \to 0$ as $n \to \infty$.

We claim that each multiplication operator $L_{R_n} : \mathcal{M} \to \mathcal{U}(X,W)$ is weakly compact and for this, it is enough to consider the particular case $R_n = y^* \otimes w$, for $y^* \in Y^*$ and $w \in W$. Now define $U : X^* \to \mathcal{U}(X,W)$ by $Ux^* = x^* \otimes w$. Then for each $T \in \mathcal{M}$,

$$L_{R_n}T = R_nT = T^*y^* \otimes w = U(T^*y^*).$$

By assumption, $\widetilde{\mathcal{M}}_1(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$ and so $L_{R_n}\mathcal{M}_1 = U(\widetilde{\mathcal{M}}_1(y^*))$ is relatively weakly compact. This shows that each L_{R_n} is weakly compact. Since

$$\|L_{R_n} - L_R\| \leq \|R_n - R\| \to 0 \text{ as } n \to \infty,$$

the left multiplication operator $L_R : \mathcal{M} \to \mathcal{U}(X, W)$ is weakly compact. Similarly, for compact operator $S : Z \to X$, the approximation property of X yields a finite rank sequence $S_n : Z \to X$ such that $\|S_n - S\| \to 0$ as $n \to \infty$.

The fact that the dual of any finite rank operator $\sum_{i=1}^{m} z_i^* \otimes x_i$ is $\sum_{i=1}^{m} x_i \otimes z_i^*$, as an operator from X^* to Z^*, combined with the weak compactness of all point evaluations $\mathcal{M}_1(x)$ yields the weak compactness of multiplication operators $\widetilde{L}_{S_n^*} : \mathcal{M} \to \mathcal{U}(Y^*, Z^*)$ where $\widetilde{L}_{S_n^*} T = S_n^* T^*$, for all $T \in \mathcal{M}$. Note that, since each S_n^* is finite rank, the composition operators $S_n^* T^*$ belong to $\mathcal{U}(Y^*, Z^*)$, although T^* may not belong to $\mathcal{U}(Y^*, X^*)$. Hence each $\widetilde{\mathcal{M}_1 S_n} = S_n^* \widetilde{\mathcal{M}}_1$ is relatively weakly compact in $\mathcal{U}(Y^*, Z^*)$ and so one can deduce that each $\mathcal{M}_1 S_n$ is relatively weakly compact in $\mathcal{U}(Z, Y)$; in fact if we consider the particular case $S_n = z^* \otimes x$, then for each $T \in \mathcal{M}$,

$$TS_n = z^* \otimes Tx \text{ and } (TS_n)^* = S_n^* T^* = Tx \otimes z^*.$$

So

$$A(TS_n) = A((TS_n)^*) = \|Tx\|.\|z^*\|.$$

Finally, the relation $R_S \mathcal{M}_1 = \mathcal{M}_1 S \subseteq \|S - S_n\| (\mathcal{U}(Z, Y))_1 + \mathcal{M}_1 S_n$, shows that the right multiplication operator $R_S : \mathcal{M} \to \mathcal{U}(Z, Y)$ is weakly compact. ∎

If one assumes that all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact, then the same method proves the following theorem:

Theorem 2.4. *Let X and Y^* have the approximation property and $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ be a linear subspace. If all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in Y and X^* respectively, then $\mathcal{M}$ is (scc) in $\mathcal{U}(X, Y)$.*

In the following, we investigate some operator ideals such that the approximation assumptions of the theorem is unnecessary. In general, for arbitrary operator ideal $\mathcal{U}$ and Banach spaces X and Y, if $\mathcal{M} \subseteq \mathcal{U}(X, Y)$, the relative weak compactness of all point evaluations is sufficient for the (wcc) of $\mathcal{M}$ in $K(X, Y)$:

Theorem 2.5. *Let X and Y be arbitrary Banach spaces and $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ be a linear subspace. If all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact in Y and X^* respectively, then $\mathcal{M}$ is (wcc) in $K(X, Y)$.*

Proof. Let $R : Y \to V$ be a compact operator. Choose a Banach space W with the approximation property and an isometric embedding $J : V \to W$ (for instance, $W = l^\infty(V_1^*)$). Since $JR : Y \to W$ is compact and W has the approximation property, the proof of theorem 2.3 shows that the operator $L_{JR} : \mathcal{M} \to K(X, W)$ is weakly compact. Since $L_{JR} = L_J \circ L_R$ and $L_J : L(X, V) \to L(X, W)$ is an isometric embedding, the operator L_R is also weakly compact.

Similarly, for compact operator $S : Z \to X$, if $J : Z^* \to l^\infty((Z^{**})_1)$ is an isometric embedding and (S_n) is a finite rank approximating sequence for the compact operator JS^*, then the assumption of the relative weak compactness of all $\mathcal{M}_1(x)$ combined with the fact that each S_n is a finite sum of one rank operators $x \otimes z$ with $x \in X$ and $z \in l^\infty((Z^{**})_1)$, shows that each L_{S_n} as operator from $\widetilde{\mathcal{M}}$ is weakly compact. Hence L_{JS^*} and so L_{S^*} as operator from $\widetilde{\mathcal{M}}$ is weakly compact. This means that $\widetilde{\mathcal{M}_1 S} = S^*\widetilde{\mathcal{M}}_1$ and so $R_S\mathcal{M}_1 = \mathcal{M}_1 S$ is relatively weakly compact in $K(Z, X)$. ∎

If we replace the weak compactness of all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ by relative compactness of them, we have another proof for theorem 2.2 of [5]. The following corollary is a refinement of theorem 2.5 for closed operator ideal and it will be proved by the method of corollary 2.3 of [5]. Recall that an operator ideal $\mathcal{U}$ is closed if all components $\mathcal{U}(X, Y)$ is closed in $L(X, Y)$.

Corollary 2.6. *Let $\mathcal{U}$ be a closed operator ideal and $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ be a linear subspace such that all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact. Then $\mathcal{M}$ is (wcc) in $\mathcal{U}(X, Y)$.*

The proof of theorem 2.5 is based on the fact that for an isometric embedding $J : V \to W$, the left multiplication operator $L_J : L(X, V) \to L(X, W)$ is also an embedding. In general, if the operator ideal $\mathcal{U}$ is injective, then the conclusion of theorems 2.3 and 2.4 hold for that operator ideal, without any approximation assumption.
An operator ideal $\mathcal{U}$ is said to be injective if for each Banach spaces X, V and W and each isometric embedding $J : V \to W$, the operator $L_J : \mathcal{U}(X, V) \to \mathcal{U}(X, W)$ is also an (isometric) embedding, and furthermore, an operator $T \in L(X, V)$ belongs to $\mathcal{U}$ if $JT \in \mathcal{U}$. Many usual operator ideals are injective. For instance, the (weakly) compact operator, the (Weakly) Banach- Saks operators, the unconditionally converging operators and the p- summing operators between Banach spaces, with $1 \leq p < \infty$, are standard examples. For additional examples see [2], [4] and [8]. So we have the following theorem:

Theorem 2.7. *Let $\mathcal{U}$ be an injective operator ideal, X and Y be arbitrary Banach spaces and $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ be a linear subspace. If all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact (resp. norm compact) in Y and X^* respectively, then $\mathcal{M}$ is (wcc) (resp. (scc)) in $\mathcal{U}(X, Y)$.*

The following theorem shows that the converse of the above theorems is also valid in every operator ideal $\mathcal{U}$ which is analogous of theorem 2.5 of [5]:

Theorem 2.8. *Let $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ be a linear subspace such that for some Banach spaces W and Z, each finite rank operators $R : Y \to W$ and $S : Z \to X$; L_R and R_S, as operators from $\mathcal{M}$ into $\mathcal{U}(X,W)$ and $\mathcal{U}(Z,Y)$ respectively, be weakly compact. Then all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact.*

Proof. Suppose that $x \in X$ be arbitrary and consider the operator $\phi_x : \mathcal{M} \to Y$ via $\phi_x T = Tx$. Fix $z^* \in Z_1^*$ and define the isometric embedding $J : Y \to \mathcal{U}(Z,Y)$ by $Jy = z^* \otimes y$. If $S = z^* \otimes x$, then clearly $R_S = J\phi_x$ and by assumption is weakly compact. Since J is an isometric embedding, the operator ϕ_x is weakly compact. This means that $\mathcal{M}_1(x)$ is relatively weakly compact in Y. The same argument shows that all $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact in X^*. ■

Corollary 2.9. *Let X and Y be Banach spaces and $\mathcal{U}$ be an operator ideal that satisfy one of the following assertions*

1 *X and Y^* have the approximation property,*

2 *$\mathcal{U}$ is a closed operator ideal or,*

3 *$\mathcal{U}$ is an injective operator ideal.*

If $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a normed linear subspace, then the following are equivalent:

(a) *All of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact in Y and X^* respectively.*

(b) *$\mathcal{M}$ is (wcc) in $\mathcal{U}(X,Y)$.*

(c) *$\mathcal{M}$ is (wcc) in $K(X,Y)$.*

(d) *For some Banach spaces W and Z and any finite rank operators $R : Y \to W$ and $S : Z \to X$; L_R and R_S, as operators from $\mathcal{M}$ into $\mathcal{U}(X,W)$ and $\mathcal{U}(Z,Y)$ (or into $K(X,W)$ and $K(Z,Y)$) respectively, are weakly compact.*

Corollary 2.10. *For Banach spaces X and Y the following are equivalent:*

(a) *X and Y are reflexive Banach spaces.*

(b) *$\mathcal{U}(X,Y)$ is (wcc) in $K(X,Y)$, for all operator ideals $\mathcal{U}$.*

(c) *$\mathcal{U}(X,Y)$ is (wcc) in $K(X,Y)$, for some operator ideal $\mathcal{U}$.*

Proof. By the remark following theorem 2.1, (a) implies (b) and the implication (b)$\to$ (c) is clear. Now assume that (c) holds. Choose $x \in X$ and $x^* \in X^*$ such that $\|x^*\| = x^*(x) = 1$ and let $\mathcal{M} = \{x^* \otimes y : y \in Y\}$. Since $\mathcal{M}$ is (wcc) in $K(X,Y)$, by theorem 2.8, $\mathcal{M}_1(x) = Y_1$ is relatively weakly compact. So Y is reflexive.

Similarly, if we choose $y \in Y$ and $y^* \in Y^*$ such that $\|y\| = y^*(y) = 1$ and $\mathcal{M} = \{x^* \otimes y : x^* \in X^*\}$, then by theorem 2.8, $\widetilde{\mathcal{M}}_1(y^*) = X_1^*$ is relatively weakly compact and so X^* is reflexive. ∎

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