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ABSTRACT CAUCHY PROBLEMS FOR QUASI-LINEAR EVOLUTION EQUATIONS WITH NON-DENSELY DEFINED OPERATORS

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Abstract. In this paper we study the abstract Cauchy problem for quasi-linear evolution equation $u'(t) = A(u(t))u(t)$, where $\{A(w); w \in W\}$ is a family of closed linear operators in a real Banach space X such that $D(A(w)) = Y$ for $w \in W$, and W is an open subset of another Banach space Y which is continuously embedded in X. The purpose of this paper is not only to establish a 'global' well-posedness theorem without assuming that Y is dense in X but also to propose a new type of dissipativity condition which is closely related with the continuous dependence of solutions on initial data.

1. Introduction

This paper is devoted to the abstract Cauchy problem for the quasi-linear evolution equation

$$\text{(QE; } u_0\text{)} \qquad \begin{cases} u'(t) = A(u(t))u(t) & \text{for } t \in [0, \tau), \\ u(0) = u_0, \end{cases}$$

where $\{A(w); w \in W\}$ is a family of closed linear operators in a real Banach space X such that $D(A(w)) \supset Y$ for $w \in W$, and W is an open subset of another Banach space Y which is continuously embedded in X.

The study of 'local' well-posedness of the Cauchy problem (QE; u_0) was initiated by Kato [9] in the case where X and Y are reflexive and Y is dense in X.

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After his pioneering work, Sanekata [15] successfully eliminated the reflexivity condition and his results sheded some new light on the problems for partial differential equations in spaces of continuous functions. (See also Kato [11].)

It is necessary to improve their results, in order to get solutions in the classical sense of partial differential equations with Dirichlet or periodic boundary conditions. In fact, the domains of such differential operators are not generally dense in the underlying spaces, so that the lack of density of Y in X occurs in the abstract setting. It is the paper due to Da Prato and Sinestrari [3] that first gave some interesting results on the inhomogeneous abstract Cauchy problem for a closed linear operator A in X satisfying the Hille-Yosida condition with the exception of the density of the domain of A. Their results have been recently extended to various types of equations by several authors. (See [1] for the integrated semigroup method, [4] for the nonautonomous case, [18] for the semilinear case, and [13, 14] for integrodifferential equations and [2] for abstract second order equations with Wentzell boundary conditions. Related topics can be found in the paper due to Sinestrari [16].)

The purpose of this paper is not only to establish a 'global' well-posedness theorem without the assumption that Y is dense in X but also to propose a new type of dissipativity condition which is closely related with the continuous dependence of solutions on initial data (see the paragraph before Proposition 2.7). Our motivation is based on the following consideration: Let $u_0, \hat{u}_0 \in D$, where D is assumed to be the set of all initial data satisfying that there exists a curve c lying in D such that $c(0) = u_0$ and $c(1) = \hat{u}_0$ and such that for each $\theta \in [0,1]$, the difference equation $(c_\lambda(\theta) - c(\theta))/\lambda = A(c_\lambda(\theta))c_\lambda(\theta)$ has a solution $c_\lambda(\theta)$. Let $u_\lambda = c_\lambda(0)$ and $\hat{u}_\lambda = c_\lambda(1)$. Then we have

$$(1.1) \qquad (\dot{c}_\lambda(\theta) - \dot{c}(\theta))/\lambda = A(c_\lambda(\theta))\dot{c}_\lambda(\theta) + (dA(c_\lambda(\theta))\dot{c}_\lambda(\theta))c_\lambda(\theta)$$

where $dA(w)\xi = \lim_{h\to 0}(A(w + h\xi) - A(w))/h$ and the limit is taken in some sense. If there exists a family $\{\|\cdot\|_w; w \in D\}$ of equivalent norms in X such that $(\|\dot{c}_\lambda(\theta)\|_{c_\lambda(\theta)} - \|\dot{c}(\theta)\|_{c(\theta)})/\lambda \le \omega\|\dot{c}_\lambda(\theta)\|_{c_\lambda(\theta)}$, or

$$(1.2) \qquad \|\dot{c}_\lambda(\theta)\|_{c_\lambda(\theta)} \le \exp(\omega\lambda)\|\dot{c}(\theta)\|_{c(\theta)},$$

then we have $\|u_\lambda - \hat{u}_\lambda\| \le M\exp(\omega\lambda)\|u_0 - \hat{u}_0\|$, by using the metric $V(x,y) := \inf\{\int_0^1 \|\dot{c}(\theta)\|_{c(\theta)}\, d\theta;\ c(0) = x, c(1) = y\}$ which is equivalent to the metric induced by the original norm $\|\cdot\|$ in X. In discussing the continuous dependence of solutions on initial data, it is therefore natural to assume the existence of a family $\{\|\cdot\|_w; w \in D\}$ of equivalent norms in X satisfying (1.2). This consideration leads us to the dissipativity condition (D1)-(D2), by noticing that the first term on the right-hand side of (1.1) is the principal part and $\dot{c}_\lambda(\theta)$ is written as $\dot{c}_\lambda(\theta) = (I - \lambda A(c_\lambda(\theta))^{-1}(\dot{c}(\theta) + \lambda(dA(c_\lambda(\theta))\dot{c}_\lambda(\theta))c_\lambda(\theta))$.

In Section 2 we introduce a range condition with growth condition, using a vector-valued functional φ. In case of concrete problems, such a functional φ is constructed according to the nature of nonlinear systems and is used to ensure the global existence of solutions as well as their asymptotic properties. In fact, we give an application of our main theorem to the global existence and exponential decay property of solutions of quasi-linear wave equations of Kirchhoff type with acoustic boundary conditions in Section 6. The construction and convergence of approximate solutions will be discussed in Sections 3 and 4 respectively. Section 5 contains the proof of the main theorem (Theorem 2.9). In the final section, we give an approach to the local solvability of quasi-linear wave equations with Wentzell boundary conditions in the space of continuous functions.

2. Assumptions and Main Results

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and Y is assumed to be continuously embedded in X. The norm closure of Y in X is denoted by $\overline{Y}$. The symbol $B(X, Y)$ stands for the Banach space of bounded linear operators on X to Y with usual operator norm $\|\cdot\|_{X,Y}$. The norm of $B(X, X)$ is denoted simply by $\|\cdot\|_X$. For $\tau > 0$ the norms of Banach spaces $C([0, \tau]; X)$ and $C^1([0, \tau]; X)$ are defined by $\|f\|_\infty = \sup\{\|f(t)\|_X; t \in [0, \tau]\}$ for $f \in C([0, \tau]; X)$ and $\|g\|_{C^1} = \|g\|_\infty + \|g'\|_\infty$ for $g \in C^1([0, \tau]; X)$, respectively. The notations $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, $\mathbb{R}_+ := [0, \infty)$ and $B_Y(r) := \{w \in Y; \|w\|_Y \leq r\}$ are used.

Let W be an open, convex subset of Y and D a closed subset of Y such that $D \subset W$. We make the following assumptions on $\{A(w); w \in W\}$ appearing in (QE; u_0).

(A1) $D(A(w)) = Y$ for each $w \in W$ and $\{A(w); w \in W\} \subset B(Y, X)$. Moreover, for each $r > 0$ there exists $c_A(r) \geq 1$ such that

$$c_A(r)^{-1}\|u\|_Y \leq \|u\|_X + \|A(w)u\|_X \leq c_A(r)\|u\|_Y$$

for $u \in Y$ and $w \in B_Y(r) \cap D$.

(A2) For each $w \in W$ there exists $dA(w) \in B(X, B(Y, X))$ such that

$$\lim_{t \to 0}(A(w + tz)y - A(w)y)/t = (dA(w)z)y \quad \text{in } X, \text{ for } y,\ z \in Y.$$

(A3) For each $r > 0$ there exists a nondecreasing function $\rho_{dA}(r; \sigma) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\sigma \downarrow 0} \rho_{dA}(r; \sigma) = 0$ such that

$$\|dA(w) - dA(z)\|_{X, B(Y,X)} \leq \rho_{dA}(r; \|w - z\|_X) \quad \text{for } w,\ z \in B_Y(r) \cap W.$$

(A4) For each $r > 0$ there exist $h_X(r) > 0$ and $M_X(r) \geq 1$ such that for $w \in B_Y(r) \cap D$ and $h \in (0, h_X(r)]$, the resolvent operator $(I - hA(w))^{-1}$ exists as a bounded linear operator on X and satisfies

$$\|(I - hA(w))^{-1}\|_X \leq M_X(r).$$

The following five basic lemmas will be often used in the following sections. Lemma 2.1 is proved by a density argument with the help of (A4).

Lemma 2.1. *Let $u \in D$. Then $\lim_{h\downarrow 0}(I - hA(u))^{-1}x = x$ in X, for $x \in \overline{Y}$.*
Lemma 2.2 follows easily from condition (A3).

Lemma 2.2. *For each $r > 0$ there exists $M_{dA}(r) > 0$ such that*

$$\|dA(w)\|_{X,B(Y,X)} \leq M_{dA}(r) \quad \text{for } w \in B_Y(r) \cap W.$$

Lemma 2.3. *For each $r > 0$ and w, $z \in B_Y(r) \cap W$ it holds that*

$$\|A(w) - A(z)\|_{Y,X} \leq M_{dA}(r)\|w - z\|_X.$$

Proof. Let $r > 0$ and w, $z \in B_Y(r) \cap W$. Since W is convex, we notice that $\theta w + (1-\theta)z \in B_Y(r) \cap W$ for $\theta \in [0, 1]$. By (A2) we have

$$A(w)y - A(z)y = \int_0^1 (dA(\theta w + (1-\theta)z)(w - z))y \, d\theta$$

for $y \in Y$. By Lemma 2.2, the desired inequality is obtained by estimating the above identity. ■

Lemma 2.4. *Let $r > 0$ and $h \in (0, h_X(r))$. Then it holds that*

$$\|(I - hA(w))^{-1}\|_{X,Y} \leq M_{X,Y}(r, h) \quad \textit{for } w \in B_Y(r) \cap D,$$

where $M_{X,Y}(r, h) = c_A(r)(M_X(r) + h^{-1}(M_X(r) + 1))$ and $M_X(r)$ is a constant specified in (A4)*.*

Proof. Let $r > 0$, $h \in (0, h_X(r))$ and $w \in B_Y(r) \cap D$. Then, by (A1) we have

$$\|(I - hA(w))^{-1}x\|_Y \leq c_A(r)(\|(I - hA(w))^{-1}x\|_X + \|A(w)(I - hA(w))^{-1}x\|_X)$$

for $x \in X$. Since $A(w)(I - hA(w))^{-1} = h^{-1}((I - hA(w))^{-1} - I)$, an application of condition (A4) to the above inequality gives the desired inequality. ■

Lemma 2.5. *Let $r > 0$, $h \in (0, h_X(r))$ and w, $\hat{w} \in B_Y(r) \cap D$. Then we have*

$$\|(I - hA(w))^{-1} - (I - hA(\hat{w}))^{-1}\|_{X,Y} \leq hM_{X,Y}(r, h)^2 M_{dA}(r)\|w - \hat{w}\|_X.$$

Proof. By Lemmas 2.3 and 2.4, the desired inequality is obtained by estimating the identity

$$\begin{aligned}&(I-hA(w))^{-1}-(I-hA(\hat{w}))^{-1}\\ &\quad = h(I-hA(w))^{-1}(A(w)-A(\hat{w}))(I-hA(\hat{w}))^{-1}\end{aligned}$$

for $r>0$, $h\in(0,h_X(r))$ and $w\in B_Y(r)\cap D$. ∎

The next proposition shows that the so-called range condition holds under an additional assumption.

Proposition 2.6. *Suppose that* (A1) *through* (A4) *hold. Suppose further that*

(2.1) $(I-hA(w))^{-1}(D)\subset D$ *for* $r>0$*,* $h\in(0,h_X(r)]$ *and* $w\in B_Y(r)\cap D$*.*

Then for any $u\in D$ *with* $A(u)u\in\overline{Y}$ *and* $\varepsilon>0$ *there exists* $h_0>0$ *such that to each* $h\in(0,h_0]$ *there corresponds* $u_h\in D$ *satisfying* $\|u_h-u\|_Y\le\varepsilon$ *and*

$$(u_h-u)/h=A(u_h)u_h.$$

Proof. Let $\varepsilon>0$ and let $u\in D$ be such that $A(u)u\in\overline{Y}$. Then, we set $r_0=\|u\|_Y+\varepsilon$ and $\rho_0=M_X(r_0)c_A(r_0)r_0$. Since $A(u)u\in\overline{Y}$, there exists $y\in Y$ such that $4c_A(r_0)M_X(r_0)\|A(u)u-y\|_X\le\varepsilon$. Choose $h_0>0$ so small that

$$h_0<h_X(r_0),\quad 2h_0M_X(r_0)M_{dA}(r_0)r_0<1,$$

$$2h_0c_A(r_0)M_X(r_0)(c_A(r_0)(r_0+\|y\|_Y)+2M_{dA}(r_0)\rho_0r_0)\le\varepsilon,$$

and let $h\in(0,h_0]$. Then, we want to find $u_h\in D$ satisfying the desired conditions. To do this, we define a subset E of Y by

$$E=\{v\in D;\|v-u\|_Y\le\varepsilon,\ \|v-u\|_X\le h\rho_0\}$$

and a mapping $\Phi:E\to Y$ by

$$\Phi v=(I-hA(v))^{-1}u\quad\text{for } v\in E.$$

To show that $\Phi(E)\subset E$, let $v\in E$. Since $E\subset B_Y(r_0)\cap D$, it follows from (2.1) that $\Phi v\in D$. By conditions (A4) and (A1) we see that

$$\|\Phi v-u\|_X=h\|(I-hA(v))^{-1}A(v)u\|_X\le hM_X(r_0)c_A(r_0)\|u\|_Y; \tag{2.2}$$

hence $\|\Phi v-u\|_X\le h\rho_0$ by the definition of ρ_0. Since

$$\begin{aligned}&\|A(v)(\Phi v-u)\|_X\\ &\quad\le\|(I-hA(v))^{-1}y-y\|_X+\|((I-hA(v))^{-1}-I)(A(v)-A(u))u\|_X\\ &\qquad+\|((I-hA(v))^{-1}-I)(A(u)u-y)\|_X\end{aligned}$$

and $(I-hA(v))^{-1}y-y=h(I-hA(v))^{-1}A(v)y$, we have

$$\begin{aligned}\|A(v)(\Phi v-u)\|_X \leq\ & hM_X(r_0)c_A(r_0)\|y\|_Y \\ & +2M_X(r_0)M_{dA}(r_0)\|v-u\|_X\|u\|_Y \\ & +2M_X(r_0)\|A(u)u-y\|_X\end{aligned} \tag{2.3}$$

by (A1), (A4) and Lemma 2.3. Adding (2.2) and (2.3) and using the fact that $\|u\|_Y \leq r_0$ and $\|v-u\|_X \leq h\rho_0$, we find that $\|\Phi v-u\|_Y \leq \varepsilon$ by (A1). It is thus proved that $\Phi(E) \subset E$, by the definition of E.

Consider the sequence $\{v_i\}_{i=0}^{\infty}$ in E defined inductively by $v_0=u$ and $v_i=\Phi v_{i-1}$ for $i=1,2,\ldots$ Since $E \subset B_Y(r_0)\cap D$, we have by Lemma 2.3

$$\begin{aligned}\|\Phi v-\Phi\hat{v}\|_X &= h\|(I-hA(v))^{-1}(A(v)-A(\hat{v}))\Phi\hat{v}\|_X \\ &\leq hM_X(r_0)M_{dA}(r_0)r_0\|v-\hat{v}\|_X.\end{aligned}$$

Since $hM_X(r_0)M_{dA}(r_0)r_0<1$, the above inequality implies that $\{v_i\}_{i=0}^{\infty}$ is a Cauchy sequence in X, and hence there exists $v\in X$ such that $v_i\to v$ in X as $i\to\infty$. It should be noticed by the definition of $\{v_i\}$ that $A(v_{i-1})v_i=(v_i-u)/h\to(v-u)/h$ in X as $i\to\infty$. Since

$$\|A(v_{i-1})(v_i-v_j)\|_X \leq \|A(v_{i-1})v_i-A(v_{j-1})v_j\|_X+M_{dA}(r_0)r_0\|v_{i-1}-v_{j-1}\|_X$$

and $\|v_i-v_j\|_Y \leq c_A(r_0)(\|v_i-v_j\|_X+\|A(v_{i-1})(v_i-v_j)\|_X)$, we see that $v_i\to v$ in Y as $i\to\infty$. Since E is a closed subset of Y, it follows that $v\in E$ and $(v-u)/h=A(v)v$. The proof is thus complete. ■

In addition to conditions (A1) through (A4), we introduce a notion of 'local quasi-dissipativity' of $\{A(w); w\in W\}$ in the following sense.

(D1) There exists a family $\{\|\cdot\|_w; w\in D\}$ of norms in X such that for each $r>0$ there exists $c_X(r)\geq 1$ satisfying

$$c_X(r)^{-1}\|x\|_X \leq \|x\|_w \leq c_X(r)\|x\|_X \quad \text{for } x\in X \text{ and } w\in B_Y(r)\cap D.$$

(D2) For each $r>0$ there exists $\omega(r)\geq 0$ such that

$$\|(I-hA(u_h))^{-1}x\|_{u_h} \leq \exp(\omega(r)(1+\eta)h)\|x\|_u$$

for $x\in X$, $h\in(0,h_X(r)]$, $\eta>0$ and u_h, $u\in B_Y(r)\cap D$ with $\|u_h-u-hA(u_h)u_h\|\leq h\eta$.

A condition similar to (D1)-(D2) was proposed by Hughes et al. [8], but certain smoothness assumption of norms in w was imposed there.

Proposition 2.7. *Suppose that conditions* (D1), (D2) *and the assumptions of Proposition* 2.6 *are satisfied. Suppose further that*

(2.4) *for each* $\alpha \geq 0$ *there exists* $r > 0$ *such that* $u \in D$ *and* $\|u\|_u + \|A(u)u\|_u \leq \alpha$ *imply that* $\|u\|_Y \leq r$.

Then there exists $G \in C(\mathbb{R}_+; \mathbb{R}_+)$ *such that*

$$\limsup_{h \downarrow 0} (\phi(u_h) - \phi(u))/h \leq G(\phi(u)) \quad \text{for } u \in D \text{ with } A(u)u \in \overline{Y}, \tag{2.5}$$

where $\phi(u) = \|u\|_u + \|A(u)u\|_u$ *for* $u \in D$, *and* $\{u_h\}$ *is the sequence in* D *specified in Proposition* 2.6.

Proof. Let $u \in D$ be such that $A(u)u \in \overline{Y}$. Let $\varepsilon > 0$ and set $r_\varepsilon = \|u\|_Y + \varepsilon$. Then, by Proposition 2.6 there exists $h_0 > 0$ such that to each $h \in (0, h_0]$ there corresponds $u_h \in D$ satisfying $\|u_h - u\|_Y \leq \varepsilon$ and $(u_h - u)/h = A(u_h)u_h$. Let $h \in (0, h_0] \cap (0, h_X(r_\varepsilon)]$. Then we have $u_h = (I - hA(u_h))^{-1}u$ by condition (A4). Since $u, u_h \in B_Y(r_\varepsilon) \cap D$, we see by (D2) that

$$\|u_h\|_{u_h} + \|A(u_h)u_h\|_{u_h} \leq \exp(\omega(r_\varepsilon)h)(\|u\|_u + \|A(u_h)u\|_u). \tag{2.6}$$

Since $\|A(u_h)u - A(u)u\|_X \leq hM_{dA}(r_\varepsilon)\|A(u_h)u_h\|_X\|u\|_Y$ by Lemma 2.3, we have by (D1) and (A1)

$$\|A(u_h)u - A(u)u\|_u \leq hc_X(r_\varepsilon)c_A(r_\varepsilon)M_{dA}(r_\varepsilon)r_\varepsilon^2. \tag{2.7}$$

Combining (2.6) and (2.7), and taking the limsup as $h \downarrow 0$, we find

$$\limsup_{h \downarrow 0} (\phi(u_h) - \phi(u))/h \leq \overline{\omega}(r)\phi(u) + \overline{c}_X(r)\overline{c}_A(r)\overline{M}_{dA}(r)r^2, \tag{2.8}$$

where $r = \|u\|_Y$. Here $\overline{\omega}$, $\overline{c}_X$, $\overline{c}_A$ and $\overline{M}_{dA}$ are nondecreasing, continuous functions dominating ω, c_X, c_A and M_{dA} respectively. These functions can be constructed in a way similar to the construction of $\overline{F}$ below.

To obtain the desired function G, we employ the two functions F and $\overline{F}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ defined by $F(\xi) = \sup\{\|u\|_Y; u \in D,\ \phi(u) \leq \xi\}$ and $\overline{F}(\xi) = \int_\xi^{\xi+1} F(\sigma)\, d\sigma$, respectively. Since F and $\overline{F}$ are nondecreasing, $\overline{F} \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\|u\|_Y \leq F(\phi(u))$ for $u \in D$ and $F(\xi) \leq \overline{F}(\xi)$ for $\xi \geq 0$, we see by (2.8) that the function G, defined by $G(\xi) = \overline{\omega}(\overline{F}(\xi))\xi + \overline{c}_X(\overline{F}(\xi))\overline{c}_A(\overline{F}(\xi))\overline{M}_{dA}(\overline{F}(\xi))\overline{F}(\xi)^2$, is the desired one satisfying (2.5). ∎

As a special case of our main result (Theorem 2.9) stated later, we can show the local existence of C^1-solution to (QE; u_0).

Theorem 2.8. *Suppose that conditions* (A1) *through* (A4), (D1), (D2), (2.1) *and* (2.4) *are satisfied. Then for each* $u_0 \in D$ *with* $A(u_0)u_0 \in \overline{Y}$*, there exists a unique function* u *in* $C([0,\tau_0);Y) \cap C^1([0,\tau_0);X)$ *satisfying* $u(0) = u_0$*,*

$$u'(t) = A(u(t))u(t) \quad \textit{for } t \in [0,\tau_0),$$

$$\phi(u(t)) \le m(t;\phi(u_0)) \quad \textit{for } t \in [0,\tau_0),$$

where ϕ *is specified by Proposition* 2.7 *and* τ_0 *is the maximal existence time of the maximal solution* $m(t;\phi(u_0))$ *to*

$$p'(t) = G(p(t)), \quad p(0) = \phi(u_0).$$

Because of localized conditions, the Cauchy problem (QE; u_0) may have only local C^1-solutions by Theorem 2.8. The purpose of this paper is to discuss the unique global existence of C^1-solution to (QE; u_0). To do this, it is necessary to consider the growth of C^1-solutions. Here the growth of a C^1-solution is specified by using a vector-valued functional $\varphi = (\varphi_i)_{i=1}^n : D \to \mathbb{R}_+^n$ such that each φ_i is lower semicontinuous on D, and a *comparison function* $g = (g_i)_{i=1}^n \in C(\mathbb{R}_+^n;\mathbb{R}^n)$ satisfying the following conditions:

(g1) $g_i(0) \ge 0$ for $i = 1,2,\ldots,n$.

(g2) For each $i = 1,2,\ldots,n$, $g_i(r)$ is nondecreasing in r_j with $j \ne i$.

In order to consider global C^1-solutions to the Cauchy problem (QE; u_0) satisfying the growth condition

$$\varphi(u(t)) \le m(t;\varphi(u_0))),$$

where the order '$\le$' in $\mathbb{R}^n$ is defined in the way that $\alpha = (\alpha_i)_{i=1}^n \le \beta = (\beta_i)_{i=1}^n$ if and only if $\alpha_i \le \beta_i$ for all $i = 1,2,\ldots,n$, we employ the following range condition with growth condition (φ)-(R):

(φ) For each $\alpha \in \mathbb{R}_+^n$ there exists $r > 0$ such that $\varphi(u) \le \alpha$ implies $\|u\|_Y \le r$.

(R) For each $\varepsilon > 0$ and $u \in D$ with $A(u)u \in \overline{Y}$ there exist $h \in (0,\varepsilon]$ and $u_h \in D$ such that

$$(u_h - u)/h = A(u_h)u_h,$$

$$\|u_h - u\|_Y \le \varepsilon,$$

$$(\varphi(u_h) - \varphi(u))/h \le g^\varepsilon(\varphi(u)),$$

where the i-th component g_i^ε of g^ε is defined by $g_i^\varepsilon(p) = g_i(p) + \varepsilon$ for $p \in \mathbb{R}_+^n$.

In the following, the number r, defined by $r = \sup\{\|u\|_Y; u \in D, \varphi(u) \le \alpha\}$, is called *the number specified in condition* (φ) *by* α.

For $\alpha \in \mathbb{R}^n_+$, we denote by $\tau(\alpha)$ the maximal existence time of the maximal solution $m(t;\alpha) = (m_i(t;\alpha))_{i=1}^n$ to the Cauchy problem for g

$$p'(t) = g(p(t)), \quad p(0) = \alpha.$$

The main theorem in this paper is stated as follows:

Theorem 2.9. *Suppose that conditions* (A1) *through* (A4), (D1), (D2), (φ) *and* (R) *are satisfied. Then for each* $u_0 \in D$ *with* $A(u_0)u_0 \in \overline{Y}$, *there exists a unique function* $u \in C([0,\tau_0); Y) \cap C^1([0,\tau_0); X)$ *satisfying* $u(0) = u_0$,

$$u'(t) = A(u(t))u(t) \quad \textit{for } t \in [0,\tau_0),$$

$$\varphi(u(t)) \le m(t;\varphi(u_0)) \quad \textit{for } t \in [0,\tau_0),$$

where $\tau_0 = \tau(\varphi(u_0))$.

3. Basic Lemmas for the Construction of Approximate Solutions

For $\varepsilon > 0$ and $\alpha \in \mathbb{R}^n_+$, we denote by $\tau^\varepsilon(\alpha)$ the maximal existence time of the maximal solution $m^\varepsilon(t;\alpha) = (m_i^\varepsilon(t;\alpha))_{i=1}^n$ to the Cauchy problem for g^ε

$$p'(t) = g^\varepsilon(p(t)), \quad p(0) = \alpha.$$

We start with the following lemma.

Lemma 3.1. *Let* $\eta > 0$, $\tau > 0$, $F \in C^1([0,\tau]; X)$ *and* $t \in [0,\tau)$. *Let* $u \in D$, $w \in Y$ *and assume that* $A(u)u \in \overline{Y}$ *and* $A(u)w + F(t) \in \overline{Y}$. *Then there exist* $\delta \in (0,\eta]$, $u_\delta \in D$ *and* $w_\delta \in Y$ *satisfying the following ten conditions:*

(i) $t + \delta < \tau$.

(ii) $\|u_\delta - u - \delta A(u_\delta)u_\delta\|_X \le \delta\eta$.

(iii) $\|u_\delta - u\|_Y \le \eta$.

(iv) $\varphi(u_\delta) \le m^\eta(\delta;\varphi(u))$.

(v) $\|A(u_\delta)u_\delta - (I - \delta A(u_\delta))^{-1}(A(u)u + \delta(dA(u)A(u)u)u)\|_X \le \delta\eta$.

(vi) $\|w_\delta - w - \delta(A(u_\delta)w_\delta + F(t+\delta))\|_X \le \delta\eta$.

(vii) $\|w_\delta - w\|_Y \le \eta$.

(viii) $\|A(u_\delta)w_\delta + F(t+\delta) - (I - \delta A(u_\delta))^{-1}(A(u)w + F(t)$

$$+\delta((dA(u)A(u)u)w + F'(t)))\|_X \le \delta\eta.$$

(ix) $A(u_\delta)u_\delta \in \overline{Y}$.

(x) $A(u_\delta)w_\delta + F(t+\delta) \in \overline{Y}$.

Proof. Let $\tau > 0$, $F \in C^1([0,\tau];X)$ and $t \in [0,\tau)$. Let $u \in D$, $w \in Y$ and assume that $A(u)u \in \overline{Y}$ and $A(u)w + F(t) \in \overline{Y}$. Let $\eta > 0$. Then, by range condition (R) there exists a null sequence $\{h_k\}$ of positive numbers and a sequence $\{u_k\}$ in D such that

$$(u_k - u)/h_k = A(u_k)u_k, \tag{3.1}$$

$$\|u_k - u\|_Y \le 1/k, \tag{3.2}$$

$$(\varphi(u_k) - \varphi(u))/h_k \le g^{1/k}(\varphi(u)), \tag{3.3}$$

for $k = 1, 2, \ldots$ It follows from (3.1) that

$$A(u_k)u_k \in \overline{Y} \quad \text{for } k = 1, 2, \ldots \tag{3.4}$$

We prove that for sufficiently large k,

$$\varphi(u_k) \le m^\eta(h_k; \varphi(u)). \tag{3.5}$$

To do this, choose a positive integer k_0 such that $k_0\eta > 1$ and set $p(t) = \varphi(u) + tg^{1/k_0}(\varphi(u))$ for $t \ge 0$. Then we have $(d/dt)(m_i^\eta(t;\varphi(u)) - p_i(t))|_{t=0} = g_i^\eta(\varphi(u)) - g_i^{1/k_0}(\varphi(u)) > 0$ for $1 \le i \le n$. By the continuity of m^η and p, one finds $\delta_0 > 0$ such that $\delta_0 < \tau^\eta(\varphi(u))$ and $(d/dt)(m^\eta(t;\varphi(u)) - p(t)) \ge 0$ for $t \in [0,\delta_0]$. Since $m^\eta(0;\varphi(u)) - p(0) = 0$, we have $m^\eta(t;\varphi(u)) \ge p(t)$ for $t \in [0,\delta_0]$. Substituting $t = h_k$ into this inequality, we have, by (3.3), (3.5) for k large enough to satisfy $k \ge k_0$ and $h_k \le \delta_0$. Since the sequence $\{u_k\}$ is convergent in Y as $k \to \infty$ (by (3.2)), there exists $r_0 > 0$ such that $u_k \in B_Y(r_0) \cap D$ for $k \ge 1$. By condition (A4) we notice that $(I - h_kA(u_k))^{-1} \in B(X)$ for sufficiently large k and $\|(I - h_kA(u_k))^{-1}\|_X$ is bounded as $k \to \infty$. Since $u_k = (I - h_kA(u_k))^{-1}u$ for sufficiently large k, it follows from (A2) that

$$\begin{aligned} A(u_k)u_k = \quad & (I - h_kA(u_k))^{-1}\Big(A(u)u \\ & + h_k \textstyle\int_0^1 (dA(\theta u_k + (1-\theta)u)A(u_k)u_k)u\, d\theta\Big). \end{aligned}$$

Since $\|u_k - u\|_Y \to 0$ and $\|(I - h_kA(u_k))^{-1}\|_X$ is bounded as $k \to \infty$, we have

$$\begin{aligned} & \lim_{k\to\infty} \|h_k^{-1}(A(u_k)u_k \\ & \quad -(I - h_kA(u_k))^{-1}(A(u)u + h_k(dA(u)A(u)u)u))\|_X = 0. \end{aligned} \tag{3.6}$$

Next, consider the sequence $\{w_k\}$ in Y defined by

$$w_k = (I - h_k A(u_k))^{-1}(w + h_k F(t + h_k)) \quad \text{for } k = 1, 2, \dots \tag{3.7}$$

Then, we find the relation

$$A(u_k)w_k + F(t + h_k) = (I - h_k A(u_k))^{-1}(A(u_k)w + F(t + h_k)); \tag{3.8}$$

hence

$$A(u_k)w_k + F(t + h_k) \in \overline{Y} \quad \text{for } k = 1, 2, \dots \tag{3.9}$$

By (3.7) we have

$$w_k - w = h_k(I - h_k A(u_k))^{-1}(A(u_k)w + F(t + h_k)),$$

which vanishes in X as $k \to \infty$, since the sequence $\{u_k\}$ converges to u in Y and $\|(I - h_k A(u_k))^{-1}\|_X$ is bounded as $k \to \infty$. By (3.8) we have

$$A(u_k)(w_k - w) = (I - h_k A(u_k))^{-1}(A(u_k)w + F(t + h_k)) - (A(u_k)w + F(t + h_k)).$$

An application of Lemma 2.1 to the right-hand side implies that $A(u_k)(w_k - w)$ tends to zero in X as $k \to \infty$. It follows from (A1) that

$$\|w_k - w\|_Y \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

Using (3.8) combined with the identity

$$A(u_k)w - A(u)w = h_k \int_0^1 (dA(\theta u_k + (1 - \theta)u)A(u_k)u_k)w \, d\theta,$$

we see by Lemma 2.3, (A3) and (3.2) that

$$\begin{aligned} &\|h_k^{-1}(A(u_k)w_k + F(t + h_k) - (I - h_k A(u_k))^{-1}(A(u)w + F(t) \\ &\qquad + h_k((dA(u)A(u)u)w + F'(t))))\|_X \text{ vanishes as } k \to \infty. \end{aligned} \tag{3.11}$$

The desired element $(\delta, u_\delta, w_\delta) \in (0, \eta] \times D \times Y$ can be found by (3.1), (3.2), (3.4) through (3.7), and (3.9) through (3.11). ■

The following lemma will be used for the construction of approximate solutions with nice properties.

Lemma 3.2. *Let $v_0 \in D$, $z_0 \in Y$, $\tau \in (0, \tau(\varphi(v_0)))$, $F \in C^1([0, \tau]; X)$ and assume that $A(v_0)v_0 \in \overline{Y}$ and $A(v_0)z_0 + F(0) \in \overline{Y}$. Let η be a positive number satisfying $\eta \leq 1$ and $\tau < \tau^\eta(\varphi(v_0))$. Then there exists a sequence $\{(s_j, v_j, z_j)\}_{j=0}^\infty$ in $[0, \tau) \times D \times Y$ such that the following conditions are satisfied:*

(i) $0 = s_0 < s_1 < \cdots < s_j < \cdots < \tau$ *for* $j = 1, 2, \ldots$

(ii) $s_j - s_{j-1} \leq \eta$ *for* $j = 1, 2, \ldots$

(iii) $\|v_j - v_{j-1} - (s_j - s_{j-1})A(v_j)v_j\|_X \leq (s_j - s_{j-1})\eta$ *for* $j = 1, 2, \ldots$

(iv) $\|v_j - v_{j-1}\|_Y \leq \eta$ *for* $j = 1, 2, \ldots$.

(v) $\|A(v_j)v_j - (I - (s_j - s_{j-1})A(v_j))^{-1}(A(v_{j-1})v_{j-1}$
$+(s_j - s_{j-1})(dA(v_{j-1})A(v_{j-1})v_{j-1})v_{j-1})\|_X \leq (s_j - s_{j-1})\eta$ *for* $j = 1, 2, \ldots$

(vi) $\varphi(v_j) \leq m^\eta(s_j - s_{j-1}; \varphi(v_{j-1}))$ *for* $j = 1, 2, \ldots$

(vii) $A(v_j)v_j \in \overline{Y}$ *for* $j = 1, 2, \ldots$.

(viii) $\|z_j - z_{j-1} - (s_j - s_{j-1})(A(v_j)z_j + F(s_j))\|_X \leq (s_j - s_{j-1})\eta$ *for* $j = 1, 2, \ldots$.

(ix) $\|z_j - z_{j-1}\|_Y \leq \eta$ *for* $j = 1, 2, \ldots$.

(x) $\|A(v_j)z_j + F(s_j) - (I - (s_j - s_{j-1})A(v_j))^{-1}(A(v_{j-1})z_{j-1} + F(s_{j-1}) + (s_j - s_{j-1})((dA(v_{j-1})A(v_{j-1})v_{j-1})z_{j-1} + F'(s_{j-1})))\|_X \leq (s_j - s_{j-1})\eta$ *for* $j = 1, 2, \ldots$.

(xi) $A(v_j)z_j + F(s_j) \in \overline{Y}$ *for* $j = 1, 2, \ldots$

(xii) $\lim_{j\to\infty} s_j = \tau$.

To prove Lemma 3.2 we need the following basic estimates.

Lemma 3.3. *Let* $r > 0$, $\tau > 0$, $\eta \in (0, 1]$ *and* $F \in C^1([0, \tau]; X)$. *Let* $\{\delta_j\}_{j=1}^K$ *be a sequence in* $(0, h_X(r)]$, $\{v_j\}_{j=0}^K$ *a sequence in* $B_Y(r) \cap D$, $\{z_j\}_{j=0}^K$ *a sequence in* Y *such that they satisfy that* $\sum_{j=1}^K \delta_j \leq \tau$ *and that*

$$\|v_j - v_{j-1} - \delta_j A(v_j)v_j\|_X \leq \delta_j \eta, \tag{3.12}$$

$$\|z_j - z_{j-1} - \delta_j(A(v_j)z_j + F(s_j))\|_X \leq \delta_j \eta, \tag{3.13}$$

$$\begin{aligned} &\|A(v_j)z_j + F(s_j) - (I - \delta_j A(v_j))^{-1}(A(v_{j-1})z_{j-1} + F(s_{j-1}))\|_X \\ &\quad \leq \delta_j M_0(r, \|F\|_{C^1})(\|z_{j-1}\|_Y + 1) \end{aligned} \tag{3.14}$$

for $1 \leq j \leq K$, *where* $s_0 = 0$, $s_j = \sum_{k=1}^j \delta_k$ *for* $1 \leq j \leq K$, *and* M_0 *is a nonnegative function defined on* $\mathbb{R}_+^2$ *and nondecreasing in each variable. Then there exist nonnegative functions* M_i, $i = 1, 2, 3$, *defined on* $\mathbb{R}_+^4$ *which are nondecreasing with respect to each variable and satisfy the following conditions:*

(a) $\|z_j\|_Y \leq M_1(\tau, r, \|z_0\|_Y, \|F\|_{C^1})$ *for* $0 \leq j \leq K$.

(b) $\|z_j - z_k\|_X \leq M_2(\tau, r, \|z_0\|_Y, \|F\|_{C^1})(s_j - s_k)$ *for* $0 \leq k \leq j \leq K$.

(c) $$\left\| A(v_j)z_j + F(s_j) - \prod_{l=p+1}^{j} (I - \delta_l A(v_l))^{-1} (A(v_p)z_p + F(s_p)) \right\|_X \le M_3(\tau, r, \|z_0\|_Y, \|F\|_{C^1})(s_j - s_p) \qquad \textit{for } 0 \le p \le j \le K.$$

Proof. To prove (a) we use the sequence $\{a_j\}_{j=0}^K$ in $\mathbb{R}_+$ defined by

$$a_j = \|z_j\|_{v_j} + \|A(v_j)z_j + F(s_j)\|_{v_j} \quad \text{for } 0 \le j \le K.$$

If we set $\varepsilon_j = (z_j - z_{j-1})/\delta_j - (A(v_j)z_j + F(s_j))$ for $1 \le j \le K$, then we have $\|\varepsilon_j\|_X \le \eta$ (by (3.13)) and $z_j = (I - \delta_j A(v_j))^{-1}(z_{j-1} + \delta_j(F(s_j) + \varepsilon_j))$ for $1 \le j \le K$. By (3.12) we use conditions (D1) and (D2) to find

$$\|z_j\|_{v_j} \le \exp(\omega(r)(1+\eta)\delta_j)(\|z_{j-1}\|_{v_{j-1}} + c_X(r)\delta_j(\|F\|_\infty + \eta)) \tag{3.15}$$

for $1 \le j \le K$. By (3.14) we have

$$\begin{aligned}\|A(v_j)z_j + F(s_j)\|_{v_j} \quad &\le \exp(\omega(r)(1+\eta)\delta_j)(\|A(v_{j-1})z_{j-1} + F(s_{j-1})\|_{v_{j-1}} \\ &\quad + \delta_j c_X(r) M_0(r, \|F\|_{C^1})(\|z_{j-1}\|_Y + 1))\end{aligned}$$

for $1 \le j \le K$. Since

$$\|z_{j-1}\|_Y \le c_A(r)(\|z_{j-1}\|_X + \|A(v_{j-1})z_{j-1}\|_X) \le c_A(r)(c_X(r)a_{j-1} + \|F\|_\infty)$$

for $1 \le j \le K$ (by (A1) and (D1)), it follows that

$$\begin{aligned}&\|A(v_j)z_j + F(s_j)\|_{v_j} \le \exp(\omega(r)(1+\eta)\delta_j) \\ &\quad (\|A(v_{j-1})z_{j-1} + F(s_{j-1})\|_{v_{j-1}} \\ &\quad + \delta_j c_X(r) M_0(r, \|F\|_{C^1})(c_A(r)(c_X(r)a_{j-1} + \|F\|_\infty) + 1))\end{aligned} \tag{3.16}$$

for $1 \le j \le K$. Adding (3.15) and (3.16), we have

$$a_j \le \exp((\omega(r)(1+\eta) + M(r, \|F\|_{C^1}))\delta_j)(a_{j-1} + \delta_j M(r, \|F\|_{C^1}))$$

for $1 \le j \le K$, where M is a function on $\mathbb{R}_+^2$ satisfying the same properties as M_0. This implies that

$$a_j \le \exp((\omega(r)(1+\eta) + M(r, \|F\|_{C^1}))s_j)(a_0 + s_j M(r, \|F\|_{C^1})) \tag{3.17}$$

for $0 \le j \le K$. Since $\|z_j\|_Y \le c_A(r)(c_X(r)a_j + \|F\|_\infty)$ for $0 \le j \le K$ and $a_0 \le c_X(r)(c_A(r)\|z_0\|_Y + \|F\|_\infty)$ by (A1) and (D1), the desired inequality (a) follows from (3.17).

Let $0 \le k < j \le K$. By (3.13) and (a), we have

$$\|z_l - z_{l-1}\|_X \le \delta_l(\eta + c_A(r)M_1(\tau, r, \|z_0\|_Y, \|F\|_{C^1}) + \|F\|_\infty)$$

for $k+1 \le l \le j$. Adding the above inequality from $l = k+1$ to j, we obtain the desired inequality (b) with $M_2(t, r, \lambda, \mu) = 1 + \mu + \overline{c}_A(r)M_1(t, r, \lambda, \mu)$.

To prove (c), let $0 \le p < j \le K$, and set $P_{j,k} = \prod_{l=k+1}^{j}(I - \delta_l A(v_l))^{-1}$ for k with $p \le k \le j$. By (3.12) we use condition (D2) to find that $\|P_{j,k}\|_X \le c_X(r)^2 \exp(\omega(r)(1+\eta)\tau)$ for $p \le k \le j$. This inequality together with (a) and (3.14) implies that

$$\begin{aligned}&\|P_{j,k}(A(v_k)z_k + F(s_k)) - P_{j,k-1}(A(v_{k-1})z_{k-1} + F(s_{k-1}))\|_X \\ &\quad \le c_X(r)^2 \exp(\omega(r)(1+\eta)\tau)\delta_k M_0(r, \|F\|_{C^1})(M_1(\tau, r, \|z_0\|_Y, \|F\|_{C^1}) + 1)\end{aligned}$$

for $p+1 \le k \le j$. The desired inequality (c) is obtained by adding the above inequality from $k = p+1$ to j. ■

Corollary 3.4. *Let $r > 0$, $\tau > 0$ and $\eta \in (0, 1]$. Let $\{\delta_j\}_{j=1}^K$ be a sequence in $(0, h_X(r)]$ and $\{v_j\}_{j=0}^K$ a sequence in $B_Y(r) \cap D$ such that they satisfy that $\sum_{j=1}^K \delta_j \le \tau$ and*

(3.18) $$\|v_j - v_{j-1} - \delta_j A(v_j)v_j\|_X \le \delta_j \eta \quad \textit{for } 1 \le j \le K.$$

Then there exists a nonnegative function M_4 defined on $\mathbb{R}_+^3$ that is nondecreasing with respect to each variable and that satisfies

$$\left\|\prod_{l=p+1}^{j}(I - \delta_l A(v_l))^{-1}y - \prod_{l=p+1}^{k}(I - \delta_l A(v_l))^{-1}y\right\|_X \le M_4(\tau, r, \|y\|_Y)(s_j - s_k)$$

for $0 \le p \le k \le j \le K$ and $y \in Y$.

Proof. Let $y \in Y$ and $0 \le p < K$. Then, the sequence $\{y_j\}_{j=p}^K$ in Y, defined by $y_p = y$ and $y_j = (I - \delta_j A(v_j))^{-1}y_{j-1}$ for $p+1 \le j \le K$, satisfies the identities that $y_j - y_{j-1} - \delta_j A(v_j)y_j = 0$ and

$$A(v_j)y_j = (I - \delta_j A(v_j))^{-1}(A(v_{j-1})y_{j-1} + (A(v_j) - A(v_{j-1}))y_{j-1})$$

for $p+1 \le j \le K$. By condition (A4), Lemma 2.3 and (3.18) we find that

$$\begin{aligned}&\|(I - \delta_j A(v_j))^{-1}(A(v_j) - A(v_{j-1}))y_{j-1}\|_X \\ &\quad \le M_X(r)M_{dA}(r)\delta_j(\eta + c_A(r)r)\|y_{j-1}\|_Y\end{aligned}$$

for $p+1 \le j \le K$. We thus see that the sequence $\{y_j\}_{j=p}^K$ satisfies (3.13) and (3.14) with $z_j = y_{j+p}$, $F = 0$ and $M_0(r, \mu) = \overline{M}_X(r)\overline{M}_{dA}(r)(1 + \overline{c}_A(r)r)$. The desired result is a direct consequence of Lemma 3.3 (b). ■

Proof of Lemma 3.2. For each $(u, w, t) \in D \times Y \times [0, \tau)$ satisfying $A(u)u$, $A(u)w + F(t) \in \overline{Y}$, we define $\delta(u, w, t)$ by the supremum of $\delta \in (0, \eta]$ such that there exists $u_\delta \in D$ and $w_\delta \in Y$ satisfying (i) through (x) in Lemma 3.1. It should be noticed that $\delta(u, w, t) > 0$ by Lemma 3.1. Let $k \geq 1$ and assume that a sequence $\{(s_j, v_j, z_j)\}_{j=0}^{k-1}$ in $[0, \tau) \times D \times Y$ has been chosen such that conditions (i) through (xi) in Lemma 3.2 are satisfied. Since $v_{k-1} \in D$ and $z_{k-1} \in Y$ satisfy (vii) and (xi) with $j = k - 1$, we have $\delta(v_{k-1}, z_{k-1}, s_{k-1}) > 0$ by the notice mentioned above. By the definition of $\delta(v_{k-1}, z_{k-1}, s_{k-1})$, there exist $\delta_k \in (0, \eta]$, $v_k \in D$ and $z_k \in Y$ such that $\delta_k > \delta(v_{k-1}, z_{k-1}, s_{k-1})/2$ and such that (s_k, v_k, z_k) satisfies (i) through (xi) with $j = k$, where $s_k = s_{k-1} + \delta_k$.

It remains to show that (xii) holds. To do this, suppose to the contrary that $\bar{s} := \lim_{j\to\infty} s_j < \tau$. Then we first show that $\{v_j\}$ and $\{z_j\}$ are convergent sequences in Y. To check the assumptions of Lemma 3.3, set $\alpha_0 = (\alpha_{0,i})_{i=0}^n$ where $\alpha_{0,i} = \sup\{m_i^\eta(t; \varphi(v_0)); t \in [0, \tau]\} < \infty$ for $i = 1, 2, \ldots, n$, and let r_0 be the number specified in condition (φ) by α_0. Then, we have $v_j \in B_Y(r_0) \cap D$ for $j = 1, 2, \ldots$ (by (vi)). Since the sequence $\{s_j\}$ is convergent as $j \to \infty$, we choose a positive integer p_0 such that $s_j - s_{j-1} \leq h_X(r_0)$ for $j \geq p_0$. We see by Lemma 3.2 (iii), (viii) and (x) that (3.12), (3.13) and (3.14) are satisfied with $M_0(r, \mu) = \overline{M}_X(r)(\overline{M}_{dA}(r)\overline{c}_A(r)r + \mu) + 1$. We therefore apply Lemma 3.3 and Corollary 3.4 to find that the sequence $\{z_j\}$ is bounded in Y as $j \to \infty$ and that $\lim_{j,k\to\infty} \|z_j - z_k\|_X = 0$ and

$$\begin{aligned}\limsup_{j,k\to\infty} \quad & \|A(v_j)z_j - A(v_k)z_k\|_X \\ & \leq 2c_X(r_0)^2 \exp(\omega(r_0)(1+\eta)\tau)\|A(v_p)z_p + F(s_p) - y\|_X \\ & \qquad + 2M_3(\tau, r_0, \|z_0\|_Y, \|F\|_{C^1})(\bar{s} - s_p)\end{aligned}$$

for $p \geq p_0$ and $y \in Y$. Since $A(v_p)z_p + F(s_p) \in \overline{Y}$, we see that $\{A(v_j)z_j\}$ is a Cauchy sequence in X. By condition (iii) we have

$$\|v_j - v_k\|_X \leq (c_A(r_0)r_0 + \eta)(s_j - s_k),$$

which tends to zero as $j, k \to \infty$. Since

$$\begin{aligned}\|z_j - z_k\|_Y \leq c_A(r_0)(\|z_j - z_k\|_X \quad & + \|A(v_j)z_j - A(v_k)z_k\|_X \\ & + \|A(v_k) - A(v_j)\|_{Y,X}\|z_k\|_Y)\end{aligned}$$

and since the facts shown above imply the right-hand side vanishes as $k, j \to \infty$, the completeness of Y ensures that the sequence $\{z_j\}$ converges in Y as $j \to \infty$. By $\bar{z}$ we denote the limit $\{z_j\}$ in Y. By the above argument with $F = 0$, we see that the sequence $\{v_j\}$ converges in Y to some $\bar{v}$ as $j \to \infty$.

Now, the closedness of D in Y implies that $\bar{v} \in D$. Since $A(v_k)v_k$, $A(v_k)z_k + F(s_k) \in \overline{Y}$ and since $A(v_k)v_k \to A(\bar{v})\bar{v}$ and $A(v_k)z_k + F(s_k) \to A(\bar{v})\bar{z} + F(\bar{s})$ as $k \to \infty$, we have $A(\bar{v})\bar{v}$, $A(\bar{v})\bar{z} + F(\bar{s}) \in \overline{Y}$. We therefore apply Lemma 3.1 to find $\delta \in (0, \eta/2] \cap (0, h_X(r_0)/2]$, $v_\delta \in D$ and $z_\delta \in Y$ satisfying the following ten conditions:

$$\bar{s} + \delta < \tau.$$

$$\|v_\delta - \bar{v} - \delta A(v_\delta)v_\delta\|_X \leq \delta(\eta/2).$$

$$\|v_\delta - \bar{v}\|_Y \leq \eta/2.$$

$$\varphi(v_\delta) \leq m^{\eta/2}(\delta; \varphi(\bar{v})). \tag{3.19}$$

$$\|A(v_\delta)v_\delta - (I - \delta A(v_\delta))^{-1}(A(\bar{v})\bar{v} + \delta(dA(\bar{v})A(\bar{v})\bar{v})\bar{v})\|_X \leq \delta(\eta/2). \tag{3.20}$$

$$\|z_\delta - \bar{z} - \delta(A(v_\delta)z_\delta + F(\bar{s} + \delta))\|_X \leq \delta(\eta/2).$$

$$\|z_\delta - \bar{z}\|_Y \leq \eta/2.$$

$$\begin{aligned}\|A(v_\delta)z_\delta + F(\bar{s} + \delta) - (I - \delta A(v_\delta))^{-1}(A(\bar{v})\bar{z} + F(\bar{s})\\ + \delta((dA(\bar{v})A(\bar{v})\bar{v})\bar{z} + F'(\bar{s})))\|_X \leq \delta(\eta/2).\end{aligned}$$

$$A(v_\delta)v_\delta \in \overline{Y}.$$

$$A(v_\delta)z_\delta + F(\bar{s} + \delta) \in \overline{Y}.$$

Set $\gamma_k = \bar{s} + \delta - s_{k-1}$ for $k \geq 1$. Then we want to show that (i) through (x) in Lemma 3.1 with $(t, \delta, u_\delta, w_\delta, u, w)$ replaced by $(s_{k-1}, \gamma_k, v_\delta, z_\delta, v_{k-1}, z_{k-1})$ are satisfied. It is obvious that $\gamma_k \to \delta$ as $k \to \infty$, $s_{k-1} + \gamma_k = \bar{s} + \delta < \tau$ and that $\gamma_k \in (0, \eta) \cap (0, h_X(r_0))$ for k sufficiently large. By Lemma 3.2 (vi) we have $\varphi(v_j) \leq m^\eta(s_j - s_{k-1}; \varphi(v_{k-1}))$ for $j \geq k$, so that $\varphi(\bar{v}) \leq m^\eta(\bar{s} - s_{k-1}; \varphi(v_{k-1}))$. This together with (3.19) implies $\varphi(v_\delta) \leq m^\eta(\gamma_k; \varphi(v_{k-1}))$. Since $v_\delta \in B_Y(r_0) \cap D$ we see by (A4) that $(I - hA(v_\delta))^{-1}$ is analytic in $h \in (0, h_X(r_0))$; hence $\lim_{k\to\infty}(I - \gamma_k A(v_\delta))^{-1} = (I - \delta A(v_\delta))^{-1}$ in $B(X)$. By (3.20) we then have

$$\begin{aligned}\|A(v_\delta)v_\delta - (I - \gamma_k A(v_\delta))^{-1}(A(v_{k-1})v_{k-1}\\ + \gamma_k(dA(v_{k-1})A(v_{k-1})v_{k-1})v_{k-1})\|_X \leq \gamma_k \eta\end{aligned}$$

for sufficiently large k. This means that condition (v) in Lemma 3.1 with (δ, u_δ, u) replaced by $(\gamma_k, v_\delta, v_{k-1})$ is satisfied. All the other conditions are checked similarly. By the definition of $\delta(v_{k-1}, z_{k-1}, s_{k-1})$, it is thus shown that $\gamma_k \leq \delta(v_{k-1}, z_{k-1}, s_{k-1})$ for sufficiently large k. Since $\delta(v_{k-1}, z_{k-1}, s_{k-1}) < 2\delta_k \to 0$ as $k \to \infty$, we have $\delta \leq 0$, which contradicts to the fact that δ is positive. ∎

4. Approximate Solutions and Mild Solutions

Our purpose is to find a unique global classical solution to (QE; u_0). By *a classical solution to* (QE; u_0) *on* $[0,T]$ we mean a function $u \in C^1([0,T];X) \cap C([0,T];Y)$ satisfying (QE; u_0) for $t \in [0,T]$. A classical solution on $[0,\infty)$ is said to be *global*. For our purpose we need the construction of approximate solutions with 'nice' properties, although we employ the so-called *method of discretization in time* in the following sense:

Definition 4.1. Let $\varepsilon > 0$, $u_0 \in D$ and $T > 0$. Assume that $\{(t_i,u_i)\}_{i=0}^N$ is a sequence in $[0,\infty) \times D$ satisfying the following three conditions:

(εi) $0 = t_0 < t_1 < \cdots < t_{N-1} < T \le t_N$.

(εii) $t_i - t_{i-1} \le \varepsilon$ for $i = 1,2,\ldots,N$.

(εiii) $\|u_i - u_{i-1} - (t_i - t_{i-1})A(u_i)u_i\|_X \le \varepsilon(t_i - t_{i-1})$ for $i = 1,2,\ldots,N$.

Then, the function $u : [0,T] \to X$ defined by

$$u(t) = \begin{cases} u_0 & \text{for } t = 0, \\ u_i & \text{for } t \in (t_{i-1},t_i] \cap [0,T] \text{ and } i = 1,2\ldots,N \end{cases}$$

is called *an ε-approximate solution to* (QE; u_0) *on* $[0,T]$. If there exists $\alpha \in \mathbb{R}_+^n$ such that an ε-approximate solution u^ε satisfies $u^\varepsilon(t) \in D(\alpha)$ for $t \in [0,T]$, then u^ε is called *an ε-approximate solution constrained in* $D(\alpha)$. Here $D(\alpha) = \{u \in D; \varphi(u) \le \alpha\}$.

Definition 4.2. Let $\varepsilon > 0$, $u_0 \in D$ and $T > 0$. A function $u \in C([0,T];X)$ is called *a mild solution to* (QE; u_0) *on* $[0,T]$ if for each sufficiently small $\varepsilon > 0$ there exists an ε-approximate solution u^ε to (QE; u_0) on $[0,T]$ such that

$$\|u^\varepsilon(t) - u(t)\|_X \le \varepsilon \qquad \text{for } t \in [0,T].$$

It is easy to show the following fundamental result.

Proposition 4.3. *Let $u_0 \in D$ and $T > 0$. Then a classical solution to* (QE; u_0) *on* $[0,T]$ *is a mild solution.*

Conversely, if there exists a mild solution u to (QE; u_0) on $[0,T]$ such that it is the limit function of a sequence $\{u^\varepsilon\}$ of ε-approximate solutions to (QE; u_0) on $[0,T]$ in Y, then u is a classical solution to (QE; u_0) on $[0,T]$. In this case, the limit function $w(t)$ of the sequence $\{A(u^\varepsilon(t))u^\varepsilon(t)\}$ coincides with $A(u(t))u(t)$ and formally satisfies the equation $w'(t) = A(u(t))w(t) + (dA(u(t))w(t))u(t)$. For this reason, it is necessary to consider the problem of whether the inhomogeneous Cauchy problem involved with a mild solution u

$$\text{(CP; } x, f)^u \qquad \begin{cases} w'(t) = A(u(t))w(t) + f(t) & \text{for } t \in [0,T] \\ w(0) = x \end{cases}$$

is solvable, in discussing the convergence in Y of a sequence of approximate solutions to (QE; u_0) on $[0,T]$.

Definition 4.4. Let $u_0 \in D$, $T > 0$ and $\alpha \in \mathbb{R}^n_+$. Let $u \in C([0,T];X)$ be a mild solution to (QE; u_0) such that u is a uniform limit of ε-approximate solutions constrained in $D(\alpha)$. Let $\varepsilon > 0$ be sufficiently small. Then, by Definitions 4.1 and 4.2, there exists a sequence $\{(t_i, u_i)\}_{i=0}^N$ in $[0,\infty) \times D$ satisfying (εi) through (εiii) and the following conditions:

(εiv) $\quad u_i \in D(\alpha)$ for $i = 0, 1, \ldots, N$.

(εv) $\quad \|u(t_i) - u_i\|_X \le \varepsilon$ for $i = 0, 1, \ldots, N$.

Let $x \in X$ and $f \in C([0,T];X)$. Assume that $\{(w_i, f_i)\}_{i=0}^N$ is a sequence in $Y \times X$ satisfying the three conditions below:

(εvi) $\quad (w_i - w_{i-1})/(t_i - t_{i-1}) = A(u_i)w_i + f_i$ for $i = 1, 2, \ldots, N$.

(εvii) $\quad \|w_0 - x\|_X \le \varepsilon$.

(εviii) $\quad \|f(t) - f_i\|_X \le \varepsilon$ for $t \in [t_{i-1}, t_i] \cap [0,T]$ for $i = 1, 2, \ldots, N$.

Then the function $w : [0,T] \to X$, defined by

$$w(t) = \begin{cases} w_0 & \text{for } t = 0, \\ w_i & \text{for } t \in (t_{i-1}, t_i] \cap [0,T] \text{ and } i = 1, 2, \ldots, N, \end{cases}$$

is called *an ε-approximate solution to* (CP; x, f)^u *on* $[0,T]$.

Definition 4.5. Let $w \in C([0,T];X)$. The function w is called *a mild solution to* (CP; x, f)^u on $[0,T]$ if for each sufficiently small $\varepsilon > 0$ there exists an ε-approximate solution w^ε to (CP; x, f)^u on $[0,T]$ such that

$$\|w^\varepsilon(t) - w(t)\|_X \le \varepsilon \qquad \text{for } t \in [0,T].$$

The following is a key lemma to prove the convergence of ε-approximate solutions.

Lemma 4.6. *Let $0 < \bar\tau < \tau$, $r > 0$, $F \in C([0,\tau];X)$ and $f \in C([0,\bar\tau];X)$. Let $\{(t_i, u_i, w_i, \varepsilon_i, f_i)\}_{i=0}^N$ be a sequence in $[0,\tau) \times (B_Y(r) \cap D) \times Y \times X \times X$ satisfying the following three conditions:*

(i) $0 = t_0 < t_1 < \cdots < t_{N-1} < \bar{\tau} \le t_N < \tau$.

(ii) $(u_i - u_{i-1})/(t_i - t_{i-1}) = A(u_i)u_i + \varepsilon_i$ *for* $i = 1, 2, \ldots, N$.

(iii) $(w_i - w_{i-1})/(t_i - t_{i-1}) = A(u_i)w_i + f_i$ *for* $i = 1, 2, \ldots, N$.

Using this sequence, we define two functions u, $w : [0, \bar{\tau}] \to Y$ *by*

$$u(t) = \begin{cases} u_0 & \textit{for } t = 0, \\ u_i & \textit{for } t \in (t_{i-1}, t_i] \cap [0, \bar{\tau}] \textit{ and } i = 1, 2, \ldots, N, \end{cases}$$

$$w(t) = \begin{cases} w_0 & \textit{for } t = 0, \\ w_i & \textit{for } t \in (t_{i-1}, t_i] \cap [0, \bar{\tau}] \textit{ and } i = 1, 2, \ldots, N. \end{cases}$$

Let $\{(s_j, v_j, z_j, \eta_j, F_j)\}_{j=0}^{\infty}$ *be a sequence in* $[0, \tau) \times (B_Y(r) \cap D) \times Y \times X \times X$ *satisfying the following three conditions:*

(iv) $0 = s_0 < s_1 < \cdots < s_j < \cdots < \tau$ *and* $\lim_{j\to\infty} s_j = \tau$.

(v) $(v_j - v_{j-1})/(s_j - s_{j-1}) = A(v_j)v_j + \eta_j$ *for* $j = 1, 2, \ldots$

(vi) $(z_j - z_{j-1})/(s_j - s_{j-1}) = A(v_j)z_j + F_j$ *for* $j = 1, 2, \ldots$

Using this sequence, we define two functions v, $z : [0, \tau) \to Y$ *by*

$$v(t) = \begin{cases} v_0 & \textit{for } t = 0, \\ v_j & \textit{for } t \in (s_{j-1}, s_j] \textit{ and } j = 1, 2, \ldots, \end{cases}$$

$$z(t) = \begin{cases} z_0 & \textit{for } t = 0, \\ z_j & \textit{for } t \in (s_{j-1}, s_j] \textit{ and } j = 1, 2, \ldots \end{cases}$$

Let K *be a positive integer satisfying* $s_{K-1} < \bar{\tau} \le s_K$ *and assume that*

$$|\Delta| := \max_{1\le i\le N}(t_i - t_{i-1}) \le \min_{1\le j\le K+1}(s_j - s_{j-1}), \tag{4.1}$$

$$|\Delta| \le h_X(r). \tag{4.2}$$

Then we have

$$\begin{aligned} \|u(t) - v(t)\|_X &\le M(\bar{\tau}, r, \varepsilon)(\|u_0 - v_0\|_X + |\Delta| \\ &\quad + \delta + \eta + \varepsilon) \quad \textit{for } t \in [0, \bar{\tau}], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \|w(t) - z(t)\|_X &\le M(\bar{\tau}, r, \varepsilon)\Big(\|w_0 - z_0\|_X + \widetilde{M}(\|u_0 - v_0\|_X \\ &\quad + |\Delta| + \delta + \eta + \varepsilon) + \widetilde{\delta} + \widetilde{\eta} + \widetilde{\varepsilon} + \rho_F(|P|) \\ &\quad + |\Delta|(\|F\|_\infty + \|f\|_\infty) + \|F - f\|_{L'(0,t;x)}\Big) \text{ for } t \in [0, \widetilde{\tau}], \end{aligned} \tag{4.4}$$

where $|P| = \max_{1\le j\le K+1}(s_j - s_{j-1})$, $\widetilde{M} = \max_{1\le j\le K}\|z_j\|_Y$,

$$\begin{aligned}
&\varepsilon = \max_{1\le i\le N}\|\varepsilon_i\|_X, \quad \tilde{\varepsilon} = \max_{1\le i\le N}\sup\{\|f(t)-f_i\|_X; t\in[t_{i-1},t_i]\cap[0,\bar{\tau}]\},\\
&\delta = \max_{1\le j\le K+1}\|v_j - v_{j-1}\|_Y, \quad \tilde{\delta} = \max_{1\le j\le K+1}\|z_j - z_{j-1}\|_Y,\\
&\eta = \max_{1\le j\le K+1}\|\eta_j\|_X, \quad \tilde{\eta} = \max_{1\le j\le K+1}\sup\{\|F(t)-F_j\|_X; t\in[s_{j-1},s_j]\},\\
&\rho_F(r) = \max\{\|F(s)-F(t)\|_X; s,t\in[0,\tau],\ |s-t|\le r\}
\end{aligned}$$

and $M(\bar{\tau}, r, \varepsilon)$ *is a nonnegative function on* $\mathbb{R}^3_+$ *and nondecreasing with respect to each variables.*

Proof. Consider the function $\tilde{z}:[0,\tau)\to Y$ defined by

$$\tilde{z}(t) = z_{j-1} + (t-s_{j-1})(z_j - z_{j-1})/(s_j - s_{j-1}) \quad \text{for } t\in[s_{j-1},s_j] \text{ and } j=1,2,\dots$$

Then, by (4.2) we use condition (A4) to find that

$$\tilde{z}(t_i) = (I-(t_i-t_{i-1})A(u_i))^{-1}(\tilde{z}(t_{i-1}) + (t_i-t_{i-1})\tilde{f}_i)$$

for $1\le i\le N$, where $\tilde{f}_i := (\tilde{z}(t_i)-\tilde{z}(t_{i-1}))/(t_i-t_{i-1}) - A(u_i)\tilde{z}(t_i)$ for $1\le i\le N$. By (iii) we have $w_i = (I-(t_i-t_{i-1})A(u_i))^{-1}(w_{i-1}+(t_i-t_{i-1})f_i)$ for $1\le i\le N$. Since $\|\varepsilon_i\|_X\le\varepsilon$ for $1\le i\le N$, it follows from (D1) and (D2) that

$$\begin{aligned}
\|\tilde{z}(t_i)-w_i\|_{u_i} \quad &\le \exp(\omega(r)(1+\varepsilon)(t_i-t_{i-1}))\Big(\|\tilde{z}(t_{i-1})-w_{i-1}\|_{u_{i-1}}\\
&\qquad + (t_i-t_{i-1})c_X(r)\|\tilde{f}_i - f_i\|_X\Big)
\end{aligned}$$

for $1\le i\le N$; hence

$$\begin{aligned}
\|\tilde{z}(t_i)-w_i\|_X \quad &\le c_X(r)^2\exp(\omega(r)(1+\varepsilon)(\bar{\tau}+\varepsilon))\\
&\qquad \times\Big(\|z_0-w_0\|_X + \textstyle\sum_{k=1}^i (t_k-t_{k-1})\|\tilde{f}_k - f_k\|_X\Big)
\end{aligned}\tag{4.5}$$

for $1\le i\le N$. By the definition of $\tilde{z}$ we see that $1\le l\le N$, $1\le p\le K$ and $s_{p-1}\le t_l\le s_{p+1}$ imply that

$$\|z_p - \tilde{z}(t_l)\|_Y \le \max(\|z_p - z_{p-1}\|_Y,\ \|z_{p+1}-z_p\|_Y) \le \tilde{\delta}. \tag{4.6}$$

By the preceding arguments, the estimate of the last term on the right-hand side of (4.5) necessary to obtain the desired inequality (4.4). To do this, let $1\le k\le N$ and $\sigma\in(t_{k-1},t_k)$. Since $t_{k-1}\le t_{N-1}<\bar{\tau}\le s_K$, there exists an integer q with $1\le q\le K$ such that $t_{k-1}\in[s_{q-1},s_q)$. By condition (4.1), one of the following cases happens:

$$\text{(I)}\quad t_k \in [s_{q-1}, s_q), \qquad\qquad \text{(II)}\quad t_k \in [s_q, s_{q+1}).$$

In both cases, we have $s_{q-1} \le t_{k-1} < \sigma < t_k < s_{q+1}$ and $v(\sigma) = v_q$ or v_{q+1}. Since $u(\sigma) = u_k$ we have

$$\tag{4.7}\begin{aligned}
&\|A(v_q)z_q + F_q - A(u_k)\tilde{z}(t_k) - f_k\|_X \\
&\le \|(A(v(\sigma)) - A(u(\sigma)))z_q\|_X + \|(A(v_q) - A(v_{q+1}))z_q\|_X \\
&\quad + \|A(u_k)(\tilde{z}(t_k) - z_q)\|_X + \|F(\sigma) - \bar{f}(\sigma)\|_X + \|\bar{f}(\sigma) - f_k\|_X \\
&\quad + \|F_q - F(s_q)\|_X + \|F(s_q) - F(\sigma)\|_X \\
&\le M_{dA}(r)\widetilde{M}\|v(\sigma) - u(\sigma)\|_X + M_{dA}(r)c_A(r)\widetilde{M}\delta + c_A(r)\tilde{\delta} \\
&\quad + \|F(\sigma) - \bar{f}(\sigma)\|_X + \tilde{\varepsilon} + \tilde{\eta} + \rho_F(|P|),
\end{aligned}$$

where $\bar{f}(t) = f(t \wedge \bar{\tau})$ for $t \in [0, \tau]$. Here we have used Lemma 2.3 and (4.6) with $(l, p) = (k, q)$ to obtain the last inequality.

Now, we begin by considering the case (II). By the definition of $\tilde{f}_k$, we find

$$\tag{4.8}\begin{aligned}
\tilde{f}_k - f_k = {}& ((t_k - s_q)/(t_k - t_{k-1}))((z_{q+1} - z_q)/(s_{q+1} - s_q) \\
&- A(u_k)\tilde{z}(t_k) - f_k) + ((s_q - t_{k-1})/(t_k - t_{k-1})) \\
&((z_q - z_{q-1})/(s_q - s_{q-1}) - A(u_k)\tilde{z}(t_k) - f_k).
\end{aligned}$$

Since $A(v_{q+1})z_{q+1} + F_{q+1} - (A(v_q)z_q + F_q)$ is written as

$$A(v_{q+1})(z_{q+1} - z_q) + (A(v_{q+1}) - A(v_q))z_q + (F_{q+1} - F(s_q)) + (F(s_q) - F_q),$$

we have

$$\|A(v_{q+1})z_{q+1} + F_{q+1} - (A(v_q)z_q + F_q)\|_X \le c_A(r)\tilde{\delta} + M_{dA}(r)c_A(r)\delta\widetilde{M} + 2\tilde{\eta}.$$

This inequality together with (4.7) implies that

$$\tag{4.9}\begin{aligned}
&\|(z_{q+1} - z_q)/(s_{q+1} - s_q) - A(u_k)\tilde{z}(t_k) - f_k\|_X \\
&\le M_{dA}(r)\widetilde{M}\|u(\sigma) - v(\sigma)\|_X + 2M_{dA}(r)c_A(r)\widetilde{M}\delta + 2c_A(r)\tilde{\delta} + 3\tilde{\eta} \\
&\quad + \rho_F(|P|) + \|F(\sigma) - \bar{f}(\sigma)\|_X + \tilde{\varepsilon}.
\end{aligned}$$

Applying (4.7) and (4.9) to (4.8), we have

$$\tag{4.10}\begin{aligned}
\|f_k - \tilde{f}_k\|_X \le {}& M_{dA}(r)\widetilde{M}\|u(\sigma) - v(\sigma)\|_X \\
&+ 2M_{dA}(r)c_A(r)\widetilde{M}\delta + 2c_A(r)\tilde{\delta} + 3\tilde{\eta} \\
&+ \rho_F(|P|) + \|F(\sigma) - \bar{f}(\sigma)\|_X + \tilde{\varepsilon}.
\end{aligned}$$

In the case (I), (4.10) is also valid by (4.7) because $\tilde{f}_k - f_k = A(v_q)z_q + F_q - A(u_k)\tilde{z}(t_k) - f_k$ by the definition of $\tilde{f}_k$. The term $\sum_{k=1}^{i}(t_k - t_{k-1})\|\tilde{f}_k - f_k\|_X$ in (4.5) is thus estimated by integrating (4.10) over (t_{k-1}, t_k) and adding the resulting inequality from $k = 1$ to i.

Now, let $t \in [0, \bar{\tau}]$. Then there exist $1 \le i \le N$ and $1 \le j \le K$ such that $t \in (t_{i-1}, t_i] \cap (s_{j-1}, s_j]$. Since (4.1) implies $s_{j-1} < t_i \le s_{j+1}$, it follows from (4.6) that $\|z_j - \tilde{z}(t_i)\|_X \le c_A(r)\tilde{\delta}$. We substitute (4.10) into (4.5) to estimate $\|\tilde{z}(t_i) - w_i\|_X$. This yields that

$$\begin{aligned} \|w(t)-z(t)\|_X &\le M(\bar{\tau}, r, \varepsilon)\Big(\|w_0 - z_0\|_X + \widetilde{M}\textstyle\int_0^t \|u(\sigma)-v(\sigma)\|_X\, d\sigma \\ &\qquad + \displaystyle\int_0^t \|F(\sigma) - f(\sigma)\|_X\, d\sigma + \tilde{\varepsilon} + \widetilde{M}(|\Delta| + \delta) \\ &\qquad + |\Delta|(\|F\|_\infty + \|f\|_\infty) + \tilde{\delta} + \tilde{\eta} + \rho_F(|P|)\Big). \end{aligned} \tag{4.11}$$

If $f_i = \varepsilon_i$, $F_j = \eta_j$ and $F = f = 0$, then $w_i = u_i, z_j = v_j$, $\tilde{\varepsilon} = \varepsilon$, $\tilde{\delta} = \delta$, $\tilde{\eta} = \eta$, $\widetilde{M} \le r$ and $\rho_F = 0$. In this special case we have by (4.11),

$$\|u(t) - v(t)\|_X \le M(\bar{\tau}, r, \varepsilon)\Big(\|u_0 - v_0\|_X + \textstyle\int_0^t \|u(\sigma) - v(\sigma)\|_X\, d\sigma + |\Delta| + \delta + \eta + \varepsilon\Big)$$

for $t \in [0, \bar{\tau}]$. An application of Gronwall's inequality gives the desired inequality (4.3). Substituting (4.3) into (4.11), we obtain the desired inequality (4.4). ■

The continuous dependence of mild solutions on initial data is given by

Proposition 4.7. *Let $\beta \ge \alpha \ge 0$ and let u_0, $\hat{u}_0 \in D(\alpha)$ be such that $A(u_0)u_0$, $A(\hat{u}_0)\hat{u}_0 \in \overline{Y}$. Let $\bar{\tau} \in (0, \tau(\alpha))$. Let u and $\hat{u}$ be mild solutions to* (QE; u_0) *and* (QE; $\hat{u}_0$) *on $[0, \bar{\tau}]$ respectively such that they are uniform limit of ε-approximate solutions constrained in $D(\beta)$. Then we have*

$$\|u(t) - \hat{u}(t)\|_X \le C(\bar{\tau}, \alpha, \beta)\|u_0 - \hat{u}_0\|_X \qquad \textit{for } t \in [0, \bar{\tau}],$$

where $C(\bar{\tau}, \alpha, \beta)$ denotes a constant depending on $\bar{\tau}, \alpha$ and β.

Proof. By assumption and the definition of mild solutions to (QE; u_0) on $[0, \bar{\tau}]$, for each sufficiently small $\varepsilon > 0$ there exists an ε-approximate solution u^ε to (QE; u_0) on $[0, \bar{\tau}]$ constrained in $D(\beta)$ such that $\|u(t) - u^\varepsilon(t)\|_X \le \varepsilon$ for $t \in [0, \bar{\tau}]$. We use Lemma 4.6 to estimate the difference between u^ε and an ε-approximate solution $\hat{u}^\varepsilon$ to (QE; $\hat{u}_0$) on $[0, \bar{\tau}]$. To do this, let $\tau \in (\bar{\tau}, \tau(\alpha))$ and let $\gamma = (\gamma_i)_{i=1}^n$ where $\gamma_i = \sup\{m_i(t; \alpha) + 1; t \in [0, \tau]\} \vee \beta_i$ for $i = 1, 2, \ldots, n$. Denote by r the number specified in condition (φ) by the vector γ. The number r depends only on α, β. Then there exists $\eta_0 \in (0, 1] \cap (0, h_X(r)]$ such that $\eta \in (0, \eta_0]$ implies that

$\tau < \tau^\eta(\alpha)$ and $m^\eta(t;\alpha) \le m(t;\alpha)+1$ for $t \in [0,\tau]$. Let $\eta \in (0,\eta_0]$. Then we apply Lemma 3.2 with $z_0 = v_0 = \hat{u}_0$ and $F = 0$ to find a sequence $\{(s_j, v_j)\}_{j=0}^\infty$ in $[0,\tau)\times D$ satisfying (i) through (vii) and (xii) in Lemma 3.2. Condition (vi) implies that $v_j \in D(\gamma)$ for $j \ge 1$; hence $v_j \in B_Y(r)\cap D$ for $j \ge 1$ and $u^\varepsilon(t) \in B_Y(r)\cap D$ for $t \in [0,\bar\tau]$. Let K be the positive integer satisfying $s_{K-1} < \bar\tau \le s_K$, and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \le \min_{1\le j\le K+1}(s_j - s_{j-1})$ and $\varepsilon_0 \le h_X(r)$. Let $\varepsilon \in (0,\varepsilon_0]$. Then we have by Lemma 4.6.

$$\|u^\varepsilon(t) - v(t)\|_X \le M(\bar\tau, r, \varepsilon)(\|u_0 - \hat{u}_0\|_X + 2\varepsilon + 2\eta) \quad \text{for } t \in [0,\bar\tau], \tag{4.12}$$

where $v : [0,\tau) \to X$ is the function defined by $v(t) = \hat{u}_0$ for $t = 0$, and v_j for $t \in (s_{j-1}, s_j]$ and $j = 1, 2, \ldots$ Since $\hat{u}^\varepsilon$ satisfies an estimate similar to (4.12), it follows that

$$\|u^\varepsilon(t) - \hat{u}^\varepsilon(t)\|_X \le 2M(\bar\tau, r, \varepsilon)(\|u_0 - \hat{u}_0\|_X + 2(\varepsilon+\eta))$$

for $t \in [0,\bar\tau]$. The desired result is obtained by letting $\varepsilon \to 0$ in the above inequality. ∎

Proposition 4.8. *Let $\beta \ge \alpha \ge 0$, and let $u_0 \in D(\alpha)$ be such that $A(u_0)u_0 \in \overline{Y}$. Let $\bar\tau \in (0,\tau(\alpha))$. Let u be a mild solution to* (QE;u_0) *on $[0,\bar\tau]$ which is a uniform limit of ε-approximate solutions constrained in $D(\beta)$. Let x, $\hat{x} \in \overline{Y}$ and f, $\hat{f} \in C([0,\bar\tau];X)$. Let w and $\hat{w}$ be mild solutions to* (CP;x,f)^u *and* (CP;$\hat{x}$,$\hat{f}$)^u *on $[0,\bar\tau]$ respectively. Then we have*

$$\|w(t) - \hat{w}(t)\|_X \le C(\bar\tau,\alpha,\beta)(\|x - \hat{x}\|_X + \|f - \hat{f}\|_{L^1(0,t;X)}) \quad \text{for } t \in [0,\bar\tau].$$

Proof. By the definition of mild solutions of (CP;x, f)^u on $[0,\bar\tau]$, for each sufficiently small $\varepsilon > 0$ there exists an ε-approximate solution w^ε to (CP;x,f)^u on $[0,\bar\tau]$ such that $\|w^\varepsilon(t) - w(t)\|_X \le \varepsilon$ for $t \in [0,\bar\tau]$. We use Lemma 4.6 to estimate the difference between w^ε and an ε-approximate solution $\hat{w}^\varepsilon$ to (CP; $\hat{x}$, $\hat{f}$)^u on $[0,\bar\tau]$. To do this, let $\tau \in (\bar\tau, \tau(\alpha))$ and let $\gamma = (\gamma_i)_{i=1}^n$ where $\gamma_i = \sup\{m_i(t;\alpha)+1; t \in [0,\tau]\} \vee \beta_i$ for $i = 1, 2, \ldots, n$. Denote by r the number specified in condition (φ) by the vector γ. The number r depends only on α, β. Then there exists $\eta_0 \in (0,1]\cap(0,h_X(r)]$ such that $\eta \in (0,\eta_0]$ implies that $\tau < \tau^\eta(\alpha)$ and $m^\eta(t;\alpha) \le m(t;\alpha)+1$ for $t \in [0,\tau]$. Let $\eta \in (0,\eta_0]$ and let (z_0, F) in $Y \times C^1([0,\tau];X)$ be fixed arbitrarily such that $A(u_0)z_0 + F(0) \in \overline{Y}$. Then there exists a sequence $\{(s_j, v_j, z_j)\}_{j=0}^\infty$ in $[0,\tau)\times D\times Y$ satisfying conditions (i) through (xii) with v_0 replaced by u_0 in Lemma 3.2. It should be noticed by Lemma 3.3 that

$$\|z_j\|_Y \le M_1(\tau, r, \|z_0\|_Y, \|F\|_{C^1})$$

for $j \geq 1$. Similarly to the argument in the proof of Proposition 4.7, we find by Lemma 4.6

$$\begin{aligned} \|w^\varepsilon(t) - z(t)\|_X \leq\ & M(\bar{\tau}, r, \varepsilon)(2M_1(\tau, r, \|z_0\|_Y, \|F\|_{C^1})(\varepsilon + \eta) \\ & + \|w^\varepsilon(0) - z_0\|_X + 2\eta + 2\rho_F(\eta) + \varepsilon \\ & + \varepsilon(\|F\|_\infty + \|f\|_\infty) + \|F - f\|_{L^1(0,t;X)}) \end{aligned} \tag{4.13}$$

for $t \in [0, \bar{\tau}]$, where $z : [0, \tau) \to Y$ is the function defined by $z(0) = z_0$ and $z(t) = z_j$ for $t \in (s_{j-1}, s_j]$ and $j = 1, 2, \ldots$ Since an estimate similar to (4.13) holds for $\hat{w}^\varepsilon$, we find

$$\begin{aligned} \|w(t) - \hat{w}(t)\|_X \leq\ & C(\bar{\tau}, \alpha, \beta)(\|x - z_0\|_X + \|\hat{x} - z_0\|_X \\ & + \|F - f\|_{L^1(0,t;X)} + \|F - \hat{f}\|_{L^1(0,t;X)}) \end{aligned}$$

for $t \in [0, \bar{\tau}]$. By Lemma 4.9 below, the desired result is obtained by letting $z_0 \to x$ in X and $F \to f$ in $C([0, \bar{\tau}]; X)$ in the above inequality. ■

Lemma 4.9. *Let $T > 0$, $u_0 \in D$ and $A(u_0)u_0 \in \overline{Y}$. Then the set*

$$E = \{(z_0, F); z_0 \in Y,\ F \in C^1([0, T]; X),\ A(u_0)z_0 + F(0) \in \overline{Y}\}$$

is dense in $\overline{Y} \times C([0, T]; X)$.

Proof. Let $x \in \overline{Y}$ and $f \in C([0, T]; X)$. Then there exists a sequence $\{(x_k, f_k)\}$ in $Y \times C^1([0, T]; X)$ such that $\|x_k - x\|_X \to 0$ and $\|f_k - f\|_\infty \to 0$ as $k \to \infty$. Choose a null sequence $\{h_k\}$ of positive numbers such that $h_k \in (0, h_X(\|u_0\|_Y)]$ for $k \geq 1$, and put $z_k = (I - h_k A(u_0))^{-1}(x_k + h_k f_k(0))$ for $k = 1, 2, \ldots$ Then it is easily seen that $z_k \in Y$ and $A(u_0)z_k + f_k(0) = (I - h_k A(u_0))^{-1}(A(u_0)x_k + f_k(0)) \in Y$; hence $(z_k, f_k) \in E$ for $k = 1, 2, \ldots$ Since

$$\begin{aligned} \|z_k - x\|_X \leq\ & \|(I - h_k A(u_0))^{-1}\|_X(\|x_k - x\|_X + h_k\|f_k(0)\|_X) \\ & + \|(I - h_k A(u_0))^{-1}x - x\|_X, \end{aligned}$$

we have $\lim_{k\to\infty} z_k = x$ in X, by Lemma 2.1 and condition (A4). ■

Lemma 4.10. *Let $\beta \geq \alpha \geq 0$ and let $u_0 \in D(\alpha)$ be such that $A(u_0)u_0 \in \overline{Y}$. Let $\bar{\tau} \in (0, \tau(\alpha))$ and let u be a mild solution to* (QE;u_0) *on $[0, \bar{\tau}]$ such that u is a uniform limit of ε-approximate solutions constrained in $D(\beta)$. Let $x \in \overline{Y}$, $f \in C([0, \bar{\tau}]; X)$ and let w be a mild solution to* (CP; x, f)^u *on $[0, \bar{\tau}]$. Then we have*

$$\|w(t) - y\|_X \leq C(\bar{\tau}, \alpha, \beta)(\|x - y\|_X + \|f\|_{L^1(0,t;X)} + tc_A(r)\|y\|_Y)$$

for $t \in [0, \bar{\tau}]$ and $y \in Y$, where r is the number specified in condition (φ) *by β.*

Proof. By assumption, for each sufficiently small $\varepsilon > 0$ there exists an ε-approximate solution u^ε to (QE; u_0) on $[0, \bar{\tau}]$ constrained in $D(\beta)$ such that $\|u^\varepsilon(t) - u(t)\|_X \leq \varepsilon$ for $t \in [0, \bar{\tau}]$. It follows from Lemma 2.3 that $A(u^\varepsilon(t))$ converges in $B(Y, X)$ uniformly on $[0, \bar{\tau}]$ as $\varepsilon \downarrow 0$. Let $y \in Y$, and set $\hat{f}(t) = -\lim_{\varepsilon \downarrow 0} A(u^\varepsilon(t))y$ and $\hat{w}^\varepsilon(t) = y$ for $t \in [0, \bar{\tau}]$. Then we have

$$(\hat{w}^\varepsilon(t_i^\varepsilon) - \hat{w}^\varepsilon(t_{i-1}^\varepsilon))/(t_i^\varepsilon - t_{i-1}^\varepsilon) = A(u_i^\varepsilon)\hat{w}^\varepsilon(t_i^\varepsilon) - A(u_i^\varepsilon)y$$

for $i = 1, 2, \ldots, N^\varepsilon$, where $\{(t_i^\varepsilon, u_i^\varepsilon)\}_{i=0}^{N^\varepsilon}$ is the sequence in $[0, \infty) \times D$ by which the ε-approximate solution u^ε is defined as in Definition 4.1. The continuity of $\hat{f}$ in X on $[0, \bar{\tau}]$ follows from that of u, by Lemma 2.3. Since $\max_{1 \leq i \leq N^\varepsilon} \sup\{\|\hat{f}(t) + A(u_i^\varepsilon)y\|_X; t \in [t_{i-1}^\varepsilon, t_i^\varepsilon] \cap [0, \bar{\tau}]\} \to 0$ as $\varepsilon \downarrow 0$, we see that $\hat{w}^\varepsilon$ is an ε-approximate solution to (CP; y, $\hat{f}$)^u on $[0, \bar{\tau}]$. This implies that the function $w(t) = y$ for $t \in [0, \bar{\tau}]$ is a mild solution to (CP; y,$\hat{f}$)^u on $[0, \bar{\tau}]$. Since $\|\hat{f}(t)\|_X = \lim_{\varepsilon \downarrow 0} \| A(u^\varepsilon(t))y\|_X \leq c_A(r)\|y\|_Y$ for $t \in [0, \bar{\tau}]$, the desired result is obtained by Proposition 4.8. ∎

5. Existence of Classical Solutions

In this section we discuss the convergence of ε-approximate solutions of (QE; u_0) and give the proof of our main theorem (Theorem 2.9).

Lemma 5.1. *Let $u_0 \in D$ and $A(u_0)u_0 \in \overline{Y}$. Let $\bar{\tau} \in (0, \tau(\varphi(u_0)))$ and $\alpha \in \mathbb{R}_+^n$. Assume that for each sufficiently small $\varepsilon > 0$, there exists an ε-approximate solution u^ε to* (QE; u_0) *on $[0, \bar{\tau}]$ constrained in $D(\alpha)$. Then there exists a mild solution u to* (QE; u_0) *on $[0, \bar{\tau}]$ such that*

$$\sup\{\|u^\varepsilon(t) - u(t)\|_X; t \in [0, \bar{\tau}]\} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Arguments similar to those in the proof of Proposition 4.7 imply that there exists a function $u : [0, \bar{\tau}] \to X$ such that $\sup\{\|u^\varepsilon(t) - u(t)\|_X; t \in [0, \bar{\tau}]\} \to 0$ as $\varepsilon \to 0$. We have only to show the continuity of u in X on $[0, \bar{\tau}]$. Since u^ε is constrained in $D(\alpha)$, there exists $r > 0$ such that $u^\varepsilon(t) \in B_Y(r)$ for $t \in [0, \bar{\tau}]$ and $\varepsilon > 0$ (by condition (φ)). It follows from (εiii) that $\|u^\varepsilon(t) - u^\varepsilon(s)\|_X \leq (c_A(r)r + \varepsilon)(|t - s| + \varepsilon)$ for t, $s \in [0, \bar{\tau}]$. This implies that $u \in C([0, \bar{\tau}]; X)$. ∎

Lemma 5.2. *Let $u_0 \in D$ be such that $A(u_0)u_0 \in \overline{Y}$, and let $0 < \bar{\tau} < \tau < \tau(\varphi(u_0))$. Let $\alpha_i = \sup\{m_i(t; \varphi(u_0)) + 1; t \in [0, \tau]\}$ for $i = 1, 2, \ldots, n$ and put $\alpha = (\alpha_i)_{i=1}^n$. Then there exists a mild solution to* (QE; u_0) *on $[0, \bar{\tau}]$ which is a uniform limit of ε-approximate solutions constrained in $D(\alpha)$.*

Proof. We choose $\varepsilon_0 \in (0,1]$ such that $\varepsilon \in (0,\varepsilon_0]$ implies that $\tau < \tau^\varepsilon(\varphi(u_0))$ and $m^\varepsilon(t;\varphi(u_0)) \le m(t;\varphi(u_0)) + 1$ for $t \in [0,\tau]$. By Lemma 3.2 with $v_0 = z_0 = u_0$ and $F = 0$, there exists an ε-approximate solution constrained in $D(\alpha)$ for each $\varepsilon \in (0,\varepsilon_0]$. The desired result follows from Lemma 5.1. ■

Lemma 5.3. *Let $u_0 \in D$ be such that $A(u_0)u_0 \in \overline{Y}$, and let $\alpha \in \mathbb{R}^n_+$ and $T \in (0,\tau(\varphi(u_0)))$. Let u be a mild solution to* (QE; u_0) *on $[0,T]$ such that u is a uniform limit of ε-approximate solutions constrained in $D(\alpha)$. Then, for any $x \in \overline{Y}$ and $f \in C([0,T];X)$, there exists a mild solution to* (CP; x, f)^u *on $[0,T]$.*

Proof. We begin by showing the lemma under the assumption that there exists an ε-approximate solution w^ε to (CP; x, f)^u on $[0,T]$ for sufficiently small $\varepsilon > 0$. In a way similar to the proof of Proposition 4.8, we see that $\sup\{\|w^\lambda(t) - w^\mu(t)\|_X; t \in [0,T]\} \to 0$ as λ, $\mu \to 0$. This implies the existence of a function $w : [0,T] \to X$ such that $\sup\{\|w^\varepsilon(t) - w(t)\|_X; t \in [0,T]\} \to 0$ as $\varepsilon \to 0$. To prove the continuity of w in X on $[0,T]$, let $\tau \in (T,\tau(\varphi(u_0)))$ and $(z_0,F) \in Y \times C^1([0,\tau];X)$ be fixed arbitrarily such that $A(u_0)z_0 + F(0) \in \overline{Y}$. Let $z : [0,\tau) \to Y$ be the function constructed in the proof of Proposition 4.8, by using a sequence $\{(s_j,v_j,z_j)\}_{j=0}^\infty$ in $[0,\tau) \times D \times Y$ satisfying conditions (i) through (xii) with v_0 replaced by u_0 in Lemma 3.2. Then we have, by Lemma 3.2 (viii),

$$\|z(t) - z(s)\|_X \le (|t-s| + \eta)(c_A(r)M_1(\tau,r,\|z_0\|_Y,\|F\|_{C^1}) + \|F\|_\infty + \eta)$$

for t, $s \in [0,T]$, where r is a constant depending only on α and $\varphi(u_0)$. After combining this inequality with (4.13) and letting $\varepsilon \to 0$, we take the limit as $\eta \downarrow 0$ to find

$$\begin{aligned}\|w(t) - w(s)\|_X \quad &\le C(\|x - z_0\|_X + \|F - f\|_{L^1(0,T;X)}) \\ &\qquad + |t-s|(c_A(r)M_1(\tau,r,\|z_0\|_Y,\|F\|_{C^1}) + \|F\|_\infty)\end{aligned}$$

for $t,s \in [0,T]$, where C is a positive constant independent of t,s,z_0,F. By Lemma 4.9, this inequality implies that w is continuous in X on $[0,T]$.

By the above argument we have only to show the existence of ε-approximate solution to (CP; x, f)^u on $[0,T]$ for sufficiently small $\varepsilon > 0$. Let $\varepsilon \in (0,h_X(r))$. By assumption, there exists a sequence $\{(t_i,u_i)\}_{i=0}^N$ in $[0,\infty) \times D$ satisfying conditions (εi) through (εv). Since $x \in \overline{Y}$, we choose $w_0 \in Y$ such that $\|w_0 - x\|_X \le \varepsilon$, and define a sequence $\{w_i\}_{i=1}^N$ in Y inductively by

$$w_i = (I - (t_i - t_{i-1})A(u_i))^{-1}(w_{i-1} + (t_i - t_{i-1})f(t_{i-1}))$$

for $i = 1,2,\ldots,N$. Then, the function $w : [0,T] \to X$, defined by $w(t) = w_0$ for $t = 0$ and w_i for $t \in (t_{i-1},t_i] \cap [0,T]$ and $i = 1,2,\ldots,N$, is an ε-approximate solution to (CP; x, f)^u on $[0,T]$, since $\|f(t) - f(t_{i-1})\|_X \le \rho_f(\varepsilon)$

for $t \in [t_{i-1}, t_i] \cap [0, T]$ and $i = 1, 2, \ldots, N$, where ρ_f stands for the modulus of continuity of f in X on $[0, T]$. ∎

To prove our main theorem (Theorem 2.9), assume that $u_0 \in D$, $A(u_0)u_0 \in \overline{Y}$ and set $\tau_0 = \tau(\varphi(u_0))$ in the rest of this section. Take arbitrarily $\bar{\tau} \in (0, \tau_0)$ and $\tau \in (\bar{\tau}, \tau_0)$. Let $\alpha_i = \sup\{m_i(t; \varphi(u_0)) + 1; t \in [0, \tau]\}$ for $i = 1, 2, \ldots, n$ and put $\alpha = (\alpha_i)_{i=1}^n$. Denote by r the number specified in condition (φ) by the vector α, and let $\lambda_0 \in (0, h_X(r))$. Then, we define an operator B by

$$B(w, z) = (dA(w)z)(I - \lambda_0 A(w))^{-1}(w - \lambda_0 z)$$

for $(w, z) \in (B_Y(r) \cap D) \times X$, and investigate some properties of B, which will be used to show the convergence in Y of approximate solutions of (QE; u_0).

Lemma 5.4. (i) *For each $R > 0$ there exist $L_B(R) > 0$ and a nondecreasing function $\rho_B(R; \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_B(R; \sigma) \downarrow 0$ as $\sigma \downarrow 0$ and that*

$$\|B(w, z) - B(\hat{w}, \hat{z})\|_X \le \rho_B(R; \|w - \hat{w}\|_X) + L_B(R)\|z - \hat{z}\|_X$$

for (w, z), $(\hat{w}, \hat{z}) \in (B_Y(r) \cap D) \times B_X(R)$.
(ii) *For each $R > 0$ there exists $M_B(R) > 0$ such that*

$$\|B(w, z)\|_X \le M_B(R) \quad \text{for } (w, z) \in (B_Y(r) \cap D) \times B_X(R).$$

Proof. Since assertion (ii) is a direct consequence of assertion (i), we have only to show assertion (i). To do this, let (w, z), $(\hat{w}, \hat{z}) \in (B_Y(r) \cap D) \times B_X(R)$. Since $B(w, z) - B(\hat{w}, \hat{z})$ is written as

$$\begin{aligned} &((dA(w) - dA(\hat{w}))z)(I - \lambda_0 A(w))^{-1}(w - \lambda_0 z) \\ &\quad + (dA(\hat{w})(z - \hat{z}))(I - \lambda_0 A(w))^{-1}(w - \lambda_0 z) \\ &\quad + (dA(\hat{w})\hat{z})((I - \lambda_0 A(w))^{-1} - (I - \lambda_0 A(\hat{w}))^{-1})(w - \lambda_0 z) \\ &\quad + (dA(\hat{w})\hat{z})(I - \lambda_0 A(\hat{w}))^{-1}(w - \hat{w} - \lambda_0(z - \hat{z})), \end{aligned}$$

assertion (i) follows from condition (A3), Lemmas 2.2, 2.4 and 2.5. ∎

The above lemma implies that the operator $B : (B_Y(r) \cap D) \times X \to X$ is uniquely extensible to the operator $\widetilde{B} : \overline{B_Y(r) \cap D} \times X \to X$.

Lemma 5.5. *Let u be a mild solution to* (QE; u_0) *on $[0, \bar{\tau}]$ obtained by Lemma 5.2. Let $T \in (0, \bar{\tau}]$ and assume that there exists $w \in C([0, T]; X)$ such that w is* *a mild solution to* (CP; $A(u_0)u_0,\ f^w)^u$ *on* $[0,T]$, *where* $f^w(t) = \widetilde{B}(u(t), w(t))$ *for* $t \in [0,T]$. *Then* u *is a classical solution to* (QE; u_0) *on* $[0,T]$ *and satisfies*

$$w(t) = A(u(t))u(t) \quad \text{for } t \in [0,T],$$
$$\varphi(u(t)) \le m(t;\varphi(u_0)) \quad \text{for } t \in [0,T].$$

Proof. Let $\eta \in (0,1]$ be such that $\tau < \tau^\eta(\varphi(u_0))$ and $m^\eta(t;\varphi(u_0)) \le m(t;\varphi(u_0)) + 1$ for $t \in [0,\tau]$. By Lemma 3.2 with $v_0 = z_0 = u_0$ and $F = 0$, we then find a sequence $\{(s_j, v_j)\}_{j=0}^K$ in $[0,\tau) \times D$ satisfying the following conditions:

(i) $0 = s_0 < s_1 < \cdots < s_{K-1} < T \le s_K$.

(ii) $s_j - s_{j-1} \le \eta$ for $j = 1, 2, \dots, K$.

(iii) $\|v_j - v_{j-1} - (s_j - s_{j-1})A(v_j)v_j\|_X \le (s_j - s_{j-1})\eta$ for $j = 1, 2, \dots, K$, where $v_0 = u_0$.

(iv) $\|v_j - v_{j-1}\|_Y \le \eta$ for $j = 1, 2, \dots, K$.

(v) $\varphi(v_j) \le m^\eta(s_j;\varphi(u_0))$ for $j = 0, 1, \dots, K$.

(vi) $\|A(v_j)v_j - (I - (s_j - s_{j-1})A(v_j))^{-1}(A(v_{j-1})v_{j-1}$

$$+(s_j - s_{j-1})\widetilde{B}(v_{j-1}, A(v_{j-1})v_{j-1}))\|_X \le (s_j - s_{j-1})\eta \text{ for } j = 1, 2, \dots, K.$$

Here it should be noticed that condition (v) implies that $v_j \in D(\alpha)$ for $0 \le j \le K$; hence there exists $r > 0$ such that $v_j \in B_Y(r) \cap D$ for $0 \le j \le K$.

Since the function $v^\eta : [0,T] \to X$, defined by $v^\eta(t) = u_0$ for $t = 0$ and v_j for $t \in (s_{j-1}, s_j] \cap [0,T]$ and $j = 1, 2, \dots, K$, is an η-approximate solution to (QE; u_0) on $[0,T]$ constrained in $D(\alpha)$, it follows from Lemma 5.1 and Proposition 4.7 that

$$\sup\{\|v^\eta(t) - u(t)\|_X; t \in [0,T]\} \to 0 \quad \text{as } \eta \to 0.$$

Let $z_0 \in Y$ be such that $\|z_0 - A(u_0)u_0\|_X \le \eta$ and define a sequence $\{z_j\}_{j=0}^K$ in Y inductively by

$$z_j = (I - (s_j - s_{j-1})A(v_j))^{-1}(z_{j-1} + (s_j - s_{j-1})f^w(s_{j-1})) \tag{5.1}$$

for $j = 1, 2, \dots, K$. Since $f^w \in C([0,T];X)$ by Lemma 5.4, we see that the function $z^\eta : [0,T] \to X$, defined by $z^\eta(t) = z_0$ for $t = 0$ and z_j for $t \in (s_{j-1}, s_j] \cap [0,T]$ and $j = 1, 2, \dots, K$, is an η-approximate solution to (CP; $A(u_0)u_0,\ f^w)^u$. By the first part in the proof of Lemma 5.3 and by Proposition 4.8, we have

$$\sup\{\|z^\eta(t) - w(t)\|_X; t \in [0,T]\} \to 0 \quad \text{for } \eta \to 0.$$

By (vi) and (5.1) we have

$$\begin{aligned}\|z_j - A(v_j)v_j\|_{v_j} &\le \exp(\omega(r)(1+\eta)(s_j - s_{j-1}))(\|z_{j-1} - A(v_{j-1})v_{j-1}\|_{v_{j-1}} \\ &\quad + c_X(r)(s_j - s_{j-1})\|\widetilde{B}(v_{j-1}, A(v_{j-1})v_{j-1}) \\ &\quad - \widetilde{B}(u(s_{j-1}), w(s_{j-1}))\|_X + c_X(r)(s_j - s_{j-1})\eta) \end{aligned} \tag{5.2}$$

for $j = 1, 2, \ldots, K$. Lemma 5.4 implies that

$$\begin{aligned} &\|\widetilde{B}(v_{j-1}, A(v_{j-1})v_{j-1}) - \widetilde{B}(u(s_{j-1}), w(s_{j-1}))\|_X \\ &\le \rho_B(R; \sup\{\|v^\eta(t) - u(t)\|_X; t \in [0,T]\}) \\ &\quad + L_B(R)c_X(r)\|A(v_{j-1})v_{j-1} - z_{j-1}\|_{v_{j-1}} \\ &\quad + L_B(R)\sup\{\|z^\eta(t) - w(t)\|_X; t \in [0,T]\} \end{aligned} \tag{5.3}$$

for $j = 1, 2, \ldots, K$, where $R = \max(c_A(r)r,\ \sup\{\|w(t)\|_X; t \in [0,T]\})$. Combining (5.2) and (5.3), we have

$$\begin{aligned}\|z_j - A(v_j)v_j\|_{v_j} &\le \exp((\omega(r)(1+\eta) + c_X(r)^2 L_B(R))(s_j - s_{j-1})) \\ &\quad \times(\|z_{j-1} - A(v_{j-1})v_{j-1}\|_{v_{j-1}} + c_X(r)C(R,\eta)(s_j - s_{j-1})) \end{aligned} \tag{5.4}$$

for $j = 1, 2, \ldots, K$, where

$$\begin{aligned} C(R,\eta) &= \rho_B(R; \sup\{\|v^\eta(t) - u(t)\|_X; t \in [0,T]\}) \\ &\quad + L_B(R)\sup\{\|z^\eta(t) - w(t)\|_X; t \in [0,T]\} + \eta. \end{aligned}$$

Solving (5.4), we obtain the inequality

$$\|z_j - A(v_j)v_j\|_{v_j} \le \exp((\omega(r)(1+\eta) + c_X(r)^2 L_B(R))s_j)c_X(r)(C(R,\eta)s_j + \eta)$$

for $j = 0, 1, \ldots, K$. This implies that

$$\sup\{\|z^\eta(t) - A(v^\eta(t))v^\eta(t)\|_X; t \in [0,T]\} \to 0 \quad \text{as } \eta \to 0.$$

Since $v^\eta(t) \to u(t)$ and $A(v^\eta(t))v^\eta(t) \to w(t)$ in X uniformly on $[0,T]$, we see that $v^\eta(t)$ converges in Y to $u(t)$ uniformly on $[0,T]$ and that $A(u(t))u(t) = w(t)$ for $t \in [0,T]$. It follows from (iii) and (v) that u is a classical solution to (QE; u_0) on $[0,T]$ satisfying $\varphi(u(t)) \le m(t; \varphi(u_0))$ for $t \in [0,T]$. ■

Proof of Theorem 2.9. Let u be the unique mild solution to (QE; u_0) on $[0,\bar{\tau}]$ such that u is a uniform limit of ε-approximate solutions constrained in $D(\alpha)$. The existence of u is ensured by Lemma 5.2.

We first show that there exist $T \in (0, \bar{\tau}]$ and $v \in C([0,T]; X)$ such that v is a mild solution to (CP; $A(u_0)u_0, f^v)^u$ on $[0,T]$, where $f^v(t) = \widetilde{B}(u(t), v(t))$ for $t \in [0,T]$. To do this, we introduce a subset $\mathbb{X}$ of $C([0,T]; X)$ defined by

$$\mathbb{X} = \{v \in C([0,T]; X); v(0) = A(u_0)u_0,\ \|v(t) - A(u_0)u_0\|_X \leq 1 \ \text{ for } t \in [0,T]\},$$

where $T \in (0, \bar{\tau}]$ is yet to be determined. It is obvious that $\mathbb{X}$ is a nonempty, closed subset of $C([0,T]; X)$. Let $v \in \mathbb{X}$. Since $A(u_0)u_0 \in \overline{Y}$ and $f^v \in C([0,T]; X)$, it follows from Lemma 5.3 and Proposition 4.8 that there exists a unique mild solution $z^v \in C([0,T]; X)$ to (CP; $A(u_0)u_0$, $f^v)^u$ on $[0,T]$. We define a mapping $\Phi : \mathbb{X} \to C([0,T]; X)$ by $\Phi v = z^v$ for $v \in \mathbb{X}$. By Proposition 4.8, Lemmas 4.10 and 5.4, we have

$$\|(\Phi v)(t) - y\|_X \leq M(\bar{\tau}, \alpha)(\|A(u_0)u_0 - y\|_X + T(M_B(R) + c_A(r)\|y\|_Y)),$$

$$\|(\Phi v)(t) - (\Phi \hat{v})(t)\|_X \leq M(\bar{\tau}, \alpha) L_B(R) T \|v - \hat{v}\|_\infty,$$

for $t \in [0,T]$ and $y \in Y$, where $R = \|A(u_0)u_0\|_X + 1$ and $y \in Y$. Since $A(u_0)u_0 \in \overline{Y}$, an element $y \in Y$ can be chosen such that $(M(\bar{\tau}, \alpha)+1)\|A(u_0)u_0 - y\|_X \leq 1/2$. It follows that for sufficiently small $T \in (0, \bar{\tau}]$, Φ maps $\mathbb{X}$ into itself and Φ is strictly contractive on $\mathbb{X}$. By Banach's fixed point theorem, the mapping Φ has a unique fixed point v in $\mathbb{X}$. This implies that v is a mild solution to (CP; $A(u_0)u_0$, $f^v)^u$ on $[0,T]$.

Now, we define $t_{\max}$ by the supremum of $T \in [0, \bar{\tau}]$ such that there exists $v \in C([0,T]; X)$ being a mild solution to (CP; $A(u_0)u_0$, $f^v)^u$ on $[0,T]$. Then, we have $0 < t_{\max} \leq \bar{\tau}$ by the above argument. By Proposition 4.8 and the definition of $t_{\max}$ there exists $\bar{v} \in C([0, t_{\max}); X)$ such that for each $T \in (0, t_{\max})$ the restriction of $\bar{v}$ on $[0,T]$ is a mild solution to (CP; $A(u_0)u_0$, $f^{\bar{v}})^u$ on $[0,T]$. By Lemma 5.5 we see that u is a unique classical solution to (QE; u_0) on $[0, t_{\max})$ satisfying the following conditions:

$$\varphi(u(t)) \leq m(t; \varphi(u_0)) \quad \text{ for } t \in [0, t_{\max}). \tag{5.5}$$

$$\bar{v}(t) = A(u(t))u(t) \quad \text{ for } t \in [0, t_{\max}). \tag{5.6}$$

Once the fact that $t_{\max} = \bar{\tau}$ is shown, the proof of Theorem 2.9 is complete because $\bar{\tau}$ is an arbitrary number in $(0, \tau_0)$.

Now, assume to the contrary that $\bar{\tau} > t_{\max}$. Then, we see that the limit $\lim_{t \uparrow t_{\max}} u(t)$ exists in Y. Indeed, take $F \in C([0, t_{\max}]; X)$ arbitrarily. Then, by Lemma 5.3 with $T = t_{\max}$, we see that there exists a mild solution $z \in C([0, t_{\max}]; X)$ to (CP; $A(u_0)u_0$, $F)^u$. By Proposition 4.8, we have

$$\|A(u(t))u(t) - z(t)\|_X \leq M(\bar{\tau}, \alpha) \int_0^t \|f^{\bar{v}}(\sigma) - F(\sigma)\|_X \, d\sigma \tag{5.7}$$

for $t \in [0, t_{\max})$; hence

$$\limsup_{t,\hat{t}\uparrow t_{\max}} \|A(u(t))u(t) - A(\hat{u}(t))\hat{u}(t)\|_X \le 2M(\bar{\tau},\alpha)\int_0^{t_{\max}} \|f^{\bar{v}}(\sigma) - F(\sigma)\|_X \, d\sigma.$$

By (5.5) and (5.6) we have $\|\bar{v}(t)\|_X \le c_A(r)r$ for $t \in [0, t_{\max})$; hence $f^{\bar{v}} \in L^\infty(0, t_{\max}; X) \subset L^1(0, t_{\max}; X)$ by Lemma 5.4. It follows by density argument that the limit $\lim_{t\uparrow t_{\max}} A(u(t))u(t)$ exists in X. Since u is a classical solution to (QE; u_0) on $[0, t_{\max})$ satisfying $u(t) \in D(\alpha)$ for $t \in [0, t_{\max})$, we have

$$\|u(t) - u(\hat{t})\|_X \le \int_t^{\hat{t}} \|A(u(\sigma))u(\sigma)\|_X \, d\sigma \le c_A(r)r|\hat{t} - t| \to 0 \quad \text{as } t, \hat{t} \uparrow t_{\max}.$$

These facts imply that the limit $\bar{u} := \lim_{t\uparrow t_{\max}} u(t)$ exists in Y.

Now, we introduce the space $\widetilde{\mathbb{X}}$ of all elements $z \in C([0,T]; X)$ such that $z(t) = A(u(t))u(t)$ $(= \bar{v}(t))$ for $t \in [0, t_{\max})$ and $\|z(t) - A(\bar{u})\bar{u}\|_X \le 1$ for $t \in [t_{\max}, T]$, and the mapping $\Psi : \widetilde{\mathbb{X}} \to C([0,T]; X)$ defined by $\Psi z = w^z$ where w^z is a unique mild solution to (CP; $A(u_0)u_0$, $f^z)^u$ on $[0,T]$, where $T \in (t_{\max}, \bar{\tau}]$ will be determined in the later arguments. It is easily seen that $\widetilde{\mathbb{X}}$ is a nonempty, closed subset of $C([0,T]; X)$ and that the mapping Ψ is unambiguous by Proposition 4.8 and Lemma 5.3. We want to show that Ψ has a unique fixed point in $\widetilde{\mathbb{X}}$ for some $T \in (t_{\max}, \bar{\tau}]$. To do this, let $z \in \widetilde{\mathbb{X}}$. To demonstrate that Ψ maps $\widetilde{\mathbb{X}}$ into itself, we notice that $(\Psi z)(t) = \bar{v}(t)$ for $t \in [0, t_{\max})$, because $f^z(t) = \widetilde{B}(u(t), \bar{v}(t))$ for $t \in [0, t_{\max})$. Consider the function $g \in C([0, \bar{\tau}]; X)$ defined by

$$g(t) = \begin{cases} \widetilde{B}(u(t), \bar{v}(t)) & \text{for } t \in [0, t_{\max}), \\ \widetilde{B}(\bar{u}, A(\bar{u})\bar{u}) & \text{for } t \in [t_{\max}, \bar{\tau}], \end{cases}$$

and let $\tilde{w}$ be a mild solution to (CP; $A(u_0)u_0$, $g)^u$ on $[0, \bar{\tau}]$. The existence of $\tilde{w}$ is guaranteed by Lemma 5.3. Since $f^z(t) = g(t)$ for $t \in [0, t_{\max})$ we find by Proposition 4.8

$$\|(\Psi z)(t) - \tilde{w}(t)\|_X \le M(\bar{\tau},\alpha)\int_{t_{\max}}^t \|f^z(\sigma) - \widetilde{B}(\bar{u}, A(\bar{u})\bar{u})\|_X \, d\sigma \tag{5.8}$$

for $t \in [t_{\max}, T]$. Since $\tilde{w}(t) = \bar{v}(t)$ for $t \in [0, t_{\max})$ (by uniqueness of mild solutions), we see by (5.6) that $\tilde{w}(t_{\max}) = A(\bar{u})\bar{u}$. Since $\|u(t)\|_Y \le r$ for $t \in [0, t_{\max})$ (by (5.5)), we have $\|f^z(t)\|_X = \|\widetilde{B}(u(t), z(t))\|_X \le M_B(c_A(r)r + 1)$ for $t \in [0, T]$ (by Lemma 5.4). It follows from (5.8) that

$$\|(\Psi z)(t) - A(\bar{u})\bar{u}\|_X \le \|\tilde{w}(t) - \tilde{w}(t_{\max})\|_X + 2M(\bar{\tau},\alpha)(t - t_{\max})M_B(c_A(r)r + 1)$$

for $t \in [t_{\max}, T]$. This inequality implies that Ψ maps $\widetilde{\mathbb{X}}$ into itself for T sufficiently close to $t_{\max}$. Let z, $\hat{z} \in \widetilde{\mathbb{X}}$. By Proposition 4.8 we have

$$\|(\Psi z)(t) - (\Psi \hat{z})(t)\|_X \leq M(\bar{\tau}, \alpha)(T - t_{\max}) L_B(c_A(r)r + 1)\|z - \hat{z}\|_\infty$$

for $t \in [t_{\max}, T]$. This means that the mapping Ψ is strictly contractive on $\widetilde{\mathbb{X}}$ if T is chosen sufficiently close to $t_{\max}$. We thus see that for some $T \in (t_{\max}, \bar{\tau}]$, the mapping Ψ has a unique fixed point w in $\widetilde{\mathbb{X}}$, and so w is a mild solution to (CP; $A(u_0)u_0$, $f^w)^u$ on $[0, T]$. This contradicts the maximality of $t_{\max}$; hence $t_{\max} = \bar{\tau}$. ∎

6. Quasi-linear Wave Equations of Kirchhoff Type with Acoustic Boundary Conditions

We consider the mixed problem for the quasi-linear wave equation of Kirchhoff type with acoustic boundary condition

$$\begin{cases} u_{tt}(x,t) - \beta(\|\nabla u(\cdot,t)\|^2)\Delta u(x,t) + \nu u_t(x,t) = 0 \quad \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\ m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) + \rho u_t(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u(x,t)}{\partial n} = \delta_t(x,t) \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad \delta(x,0) = \delta_0(x), \quad \delta_t(x,0) = \sigma_0(x), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here $\Gamma_0 \neq \emptyset$, Γ_0 and Γ_1 are closed and disjoint, and n represents the unit outward normal to Γ. The symbol $\|w\|$ is defined by $\|w\| = (\int_\Omega |w(x)|^2\,dx)^{1/2}$ for $w \in L^2(\Omega)$. The function $\beta \in C^1(\mathbb{R}_+;\mathbb{R})$ is assumed to satisfy that $\beta(s) \geq \beta_0 > 0$ for $s \in \mathbb{R}_+$, and $\nu, \rho > 0$. It is also assumed that d is a nonnegative, continuous function on Γ_1 and that m, k are positive, continuous functions on Γ_1.

Matsuyama and Ikehara [12] studied the global existence and decay property of solutions in the case where $\Gamma_1 = \emptyset$. Frota and Goldstein [6] discussed the global well-posedness for the Carrier equations with acoustic boundary conditions by using Galerkin's and energy method.

Theorem 6.1. (I) *There exists $r > 0$ such that for each initial data $u_0 \in H^1(\Omega)$ with $\gamma(u_0) = 0$ on Γ_0 and $\Delta u_0 \in L^2(\Omega)$, $v_0 \in H^1(\Omega)$ with $\gamma(v_0) = 0$ on Γ_0, $\delta_0, \sigma_0 \in L^2(\Gamma_1)$ satisfying $\|\nabla u_0\| + \|\Delta u_0\| + \|\nabla v_0\| + |\delta_0| + |\sigma_0| \leq r$ and $\int_\Omega (\Delta u_0)\phi + \nabla u_0 \cdot \nabla\phi\,dx = \int_{\Gamma_1} \sigma_0\gamma(\phi)\,dS$ for any $\phi \in H^1(\Omega)$ with $\gamma(\phi) = 0$ on Γ_0, the mixed problem mentioned above has a unique solution $u \in C^1([0,\infty); H^1(\Omega)) \cap$*

$C^2([0,\infty); L^2(\Omega))$ *and* $\delta \in C^2([0,\infty); L^2(\Gamma_1))$ *such that* $\gamma(u) = 0$ *on* Γ_0*,* $\Delta u \in C([0,\infty); L^2(\Omega))$ *and* $\int_\Omega (\Delta u)\phi + \nabla u \cdot \nabla \phi \, dx = \int_{\Gamma_1} \delta_t \gamma(\phi)\, dS$ *for any* $\phi \in H^1(\Omega)$ *with* $\gamma(\phi) = 0$ *on* Γ_0*, where* $\gamma : H^1(\Omega) \to L^2(\Gamma)$ *is the trace map.*

(II) *Assume in addition that* d *is positive on* Γ_1*. Then, the following exponential decay of the energy holds:*

$$\|\nabla u(t)\| + \|u_t(t)\| + |\delta(t)| + |\delta_t(t)| \le M \exp(-\omega t) \qquad \text{for } t \ge 0.$$

Proof. Let $V = \{v \in H^1(\Omega);\ \gamma(v) = 0 \text{ on } \Gamma_0\}$ and $H(\Delta,\Omega) = \{v \in V; \Delta v \in L^2(\Omega)\}$. Notice that

$$\|v\| \le C\|\nabla v\| \qquad \text{for } v \in V, \tag{6.1}$$

$$|\gamma(v)| \le C\|\nabla v\| \quad \text{for } v \in V, \tag{6.2}$$

where $|w| := (\int_{\Gamma_1} |w(x)|^2\, dS)^{1/2}$ for $w \in L^2(\Gamma_1)$. For simplicity in notation, we write u for $\gamma(u)$ in the following arguments.

Let $X = V \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)$ and let Y be the space of all elements $(u, v, \delta, \sigma) \in H(\Delta,\Omega) \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1)$ such that

$$\int_\Omega (\Delta u)\phi + \nabla u \cdot \nabla \phi \, dx = \int_{\Gamma_1} \sigma\phi \, dS$$

for any $\phi \in V$. For simplicity in notation, the above identity is written as

$$\langle \Delta u, \phi\rangle + \langle \nabla u, \nabla\phi\rangle = \langle \sigma, \phi\rangle_{\Gamma_1}.$$

The spaces X and Y are real Banach spaces under the norms defined by

$$\|(u, v, \delta, \sigma)\|_X = (\|\nabla u\|^2 + \|v\|^2 + |\delta|^2 + |\sigma|^2)^{1/2} \quad \text{for } (u, v, \delta, \sigma) \in X$$

and

$$\|(u, v, \delta, \sigma)\|_Y = (\|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla v\|^2 + |\delta|^2 + |\sigma|^2)^{1/2} \quad \text{for } (u, v, \delta, \sigma) \in Y$$

respectively.

We use Theorem 2.9 with $D = W = Y$ to solve the above-mentioned mixed problem. For each $(w, z, \xi, \eta) \in Y$, define a linear operator $A((w, z, \xi, \eta))$ in X by

$$A((w, z, \xi, \eta))(u, v, \delta, \sigma) = (v, \beta(\|\nabla w\|^2)\Delta u - \nu v, \sigma, -(1/m)(d\sigma + k\delta + \rho v))$$

for $(u, v, \delta, \sigma) \in D(A((w, z, \xi, \eta))) = Y$. Then, we have $A((w, z, \xi, \eta)) \in B(Y, X)$ by (6.2). By using the identity that

$$\Delta u = \beta(\|\nabla w\|^2)^{-1}(\beta(\|\nabla w\|^2)\Delta u - \nu v) + \nu\beta(\|\nabla w\|^2)^{-1} v$$

and the property that $\beta(s) \geq \beta_0 > 0$ for $s \in \mathbb{R}_+$, condition (A1) is easily checked. Condition (A2) is satisfied with the operator $dA((w,z,\xi,\eta)) \in B(X, B(Y,X))$ defined by

$$(dA((w,z,\xi,\eta))(\tilde{w},\tilde{z},\tilde{\xi},\tilde{\eta}))(u,v,\delta,\sigma) = (0, 2\beta'(\|\nabla w\|^2)\langle \nabla w, \nabla \tilde{w}\rangle \Delta u, 0, 0)$$

for $(w,z,\xi,\eta), (u,v,\delta,\sigma) \in Y$ and $(\tilde{w},\tilde{z},\tilde{\xi},\tilde{\eta}) \in X$. We easily see that the operator $dA((w,z,\xi,\eta))$ defined above satisfies condition (A3), using the continuity of β' on $\mathbb{R}_+$.

To check condition (A4), let $r > 0$ and $(w,z,\xi,\eta) \in B_Y(r)$. By a routine argument with the help of the Riesz representation theorem, we see that for each $\lambda > 0$ and $(u,v,\delta,\sigma) \in X$, there exists a unique $(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) \in Y$ such that $(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \lambda A((w,z,\xi,\eta))(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) = (u,v,\delta,\sigma)$; namely

$$(u_\lambda - u)/\lambda = v_\lambda \tag{6.3}$$

$$(v_\lambda - v)/\lambda = \beta(\|\nabla w\|^2)\Delta u_\lambda - \nu v_\lambda \tag{6.4}$$

$$(\delta_\lambda - \delta)/\lambda = \sigma_\lambda \tag{6.5}$$

$$(\sigma_\lambda - \sigma)/\lambda = -(1/m)(d\sigma_\lambda + k\delta_\lambda + \rho v_\lambda) \tag{6.6}$$

$$\langle \Delta u_\lambda, \phi\rangle + \langle \nabla u_\lambda, \nabla \phi\rangle = \langle \sigma_\lambda, \phi\rangle_{\Gamma_1} \qquad \text{for } \phi \in V. \tag{6.7}$$

To check the remained part of condition (A4), we employ the family of norms in X defined by

$$\|(u,v,\delta,\sigma)\|_{(w,z,\xi,\eta)} = (\rho\|\nabla u\|^2 + \rho\beta(\|\nabla w\|^2)^{-1}\|v\|^2 + |k^{1/2}\delta|^2 + |m^{1/2}\sigma|^2)^{1/2}$$

for $(w,z,\xi,\eta) \in Y$. Clearly, condition (D1) is satisfied, since $\rho > 0$ and k, m are positive continuous functions on Γ_1. Substituting $\phi = v_\lambda$ $(= (u_\lambda - u)/\lambda) \in V$ into (6.7) we find, by (6.4) and (6.6),

$$\begin{aligned}\beta(\|\nabla w\|^2)^{-1}\langle (v_\lambda - v)/\lambda + \nu v_\lambda, v_\lambda\rangle + \langle \nabla u_\lambda, \nabla(u_\lambda - u)/\lambda\rangle \\ + \langle \sigma_\lambda, (1/\rho)(m(\sigma_\lambda - \sigma)/\lambda + d\sigma_\lambda + k\delta_\lambda)\rangle_{\Gamma_1} = 0.\end{aligned}$$

By convexity of norms, we have

$$\begin{aligned}&(\|(I - \lambda A((w,z,\xi,\eta)))^{-1}(u,v,\delta,\sigma)\|^2_{(w,z,\xi,\eta)} \\ &- \|(u,v,\delta,\sigma)\|^2_{(w,z,\xi,\eta)})/\lambda + 2\nu\rho\beta(\|\nabla w\|^2)^{-1}\|v_\lambda\|^2 + 2|d^{1/2}\sigma_\lambda|^2 \leq 0,\end{aligned} \tag{6.8}$$

where v_λ, σ_λ are defined by $(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) = (I - \lambda A((w,z,\xi,\eta)))^{-1}(u,v,\delta,\sigma)$. This inequality, together with condition (D1), implies that the remained part of (A4) is satisfied.

To check (D2), let $(u, v, \delta, \sigma) \in X$, $(w, z, \xi, \eta), (w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda) \in B_Y(r)$ and $\|(w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda) - \lambda A((w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda))(w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda) - (w, z, \xi, \eta)\|_X \le \lambda\eta$. Then, we notice that $\|\nabla w\|, \|\nabla w_\lambda\|, \|\nabla z_\lambda\| \le r$ and

(6.9) $$\|\nabla(w_\lambda - w - \lambda z_\lambda)\| \le \lambda\eta.$$

By the inequality (6.8) we find that

$$\|(I - \lambda A((w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda)))^{-1}(u, v, \delta, \sigma)\|^2_{(w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda)} \le \|(u, v, \delta, \sigma)\|^2_{(w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda)}.$$

Condition (D2) follows from this inequality combined with (6.9) and the inequalities that

$$\frac{\|(u, v, \delta, \sigma)\|^2_{(w_\lambda, z_\lambda, \xi_\lambda, \eta_\lambda)} - \|(u, v, \delta, \sigma)\|^2_{(w,z,\xi,\eta)}}{\|(u, v, \delta, \sigma)\|^2_{(w,z,\xi,\eta)}} \le \frac{|\beta(\|\nabla w\|^2) - \beta(\|\nabla w_\lambda\|^2)|}{\beta(\|\nabla w_\lambda\|^2)}$$

and $\beta(\|\nabla w_\lambda\|^2)^{-1}|\beta(\|\nabla w\|^2) - \beta(\|\nabla w_\lambda\|^2)| \le 2\beta_0^{-1} M_{\beta'}(r^2) r \|\nabla(w - w_\lambda)\|$, where $M_{\beta'}(R) = \sup\{|\beta'(s)|; s \in [0, R]\}$ for $R \in \mathbb{R}_+$.

To check (φ)-(R) we employ the three functionals on X defined by

$$\phi_1(u, v, \delta, \sigma) = \|(u, v, \delta, \sigma)\|^2_{(u,v,\delta,\sigma)},$$
$$\phi_2(u, v, \delta, \sigma) = \beta(\|\nabla u\|^2)^{-1}\rho\|\nu u + v\|^2 + \rho\|\nabla u\|^2 + |m^{1/2}\sigma|^2 + |k^{1/2}\delta|^2,$$
$$\phi_3(u, v, \delta, \sigma) = |\rho u + m\sigma + d\delta|^2 + |(km)^{1/2}\delta|^2$$

and the functional on Y defined by

$$\phi_4(u, v, \delta, \sigma) = \|A((u, v, \delta, \sigma))(u, v, \delta, \sigma)\|^2_{(u,v,\delta,\sigma)}.$$

Let $(u, v, \delta, \sigma) \in Y$. Then, we see by Proposition 2.6 that there exists a sequence $\{(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)\}$ in Y such that

(6.10) $$(u_\lambda - u)/\lambda = v_\lambda,$$

(6.11) $$(v_\lambda - v)/\lambda = \beta(\|\nabla u_\lambda\|^2)\Delta u_\lambda - \nu v_\lambda,$$

(6.12) $$(\delta_\lambda - \delta)/\lambda = \sigma_\lambda,$$

(6.13) $$(\sigma_\lambda - \sigma)/\lambda = -(1/m)(d\sigma_\lambda + k\delta_\lambda + \rho v_\lambda),$$

(6.14) $$\langle \Delta u_\lambda, \phi\rangle + \langle \nabla u_\lambda, \nabla\phi\rangle = \langle \sigma_\lambda, \phi\rangle_{\Gamma_1} \quad \text{for any } \phi \in V$$

and such that $u_\lambda \to u$ in V, $\Delta u_\lambda \to \Delta u$ in $L^2(\Omega)$, $v_\lambda \to v$ in V, $\sigma_\lambda \to \sigma$ in $L^2(\Gamma_1)$

and $\delta_\lambda \to \delta$ in $L^2(\Gamma_1)$ as $\lambda \downarrow 0$. By (6.8) with $(w, z, \xi, \eta) = (u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)$, we find that

$$(\phi_1(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \phi_1(u, v, \delta, \sigma))/\lambda + 2\nu\rho\beta(\|\nabla u_\lambda\|^2)^{-1}\|v_\lambda\|^2 + 2|d^{1/2}\sigma_\lambda|^2 \\ \leq \rho(\beta(\|\nabla u_\lambda\|^2)^{-1} - \beta(\|\nabla u\|^2)^{-1})\|v\|^2/\lambda.$$

Since $\lambda^{-1}(\beta(\|\nabla u_\lambda\|^2) - \beta(\|\nabla u\|^2)) = \int_0^1 2\beta'(\|\nabla(\theta u_\lambda + (1-\theta)u)\|^2)\langle\nabla(\theta u_\lambda + (1-\theta)u), \nabla v_\lambda\rangle\, d\theta$ we have

$$\begin{aligned} &\limsup_{\lambda\downarrow 0}(\phi_1(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \phi_1(u, v, \delta, \sigma))/\lambda \\ &\quad +2\nu\rho\beta(\|\nabla u\|^2)^{-1}\|v\|^2 + 2|d^{1/2}\sigma|^2 \\ &\leq -2\rho\beta'(\|\nabla u\|^2)\beta(\|\nabla u\|^2)^{-2}\langle\nabla u, \nabla v\rangle\|v\|^2. \end{aligned} \tag{6.15}$$

Substituting $\phi = v_\lambda + \nu u_\lambda \in V$ into (6.14) and noting that $(v_\lambda - v)/\lambda + \nu v_\lambda = ((v_\lambda + \nu u_\lambda) - (v + \nu u))/\lambda$ we find, by a way similar to the derivation of (6.15),

$$\limsup_{\lambda\downarrow 0}(\phi_2(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \phi_2(u, v, \delta, \sigma))/\lambda + 2\nu\rho\|\nabla u\|^2 + 2|d^{1/2}\sigma|^2 \\ \leq -2\rho\beta'(\|\nabla u\|^2)\beta(\|\nabla u\|^2)^{-2}\langle\nabla u, \nabla v\rangle\|\nu u + v\|^2 + 2\nu\rho\langle\sigma, u\rangle_{\Gamma_1}.$$

Substituting (6.10) and (6.12) into (6.13) we have

$$((\rho u_\lambda + m\sigma_\lambda + d\delta_\lambda) - (\rho u + m\sigma + d\delta))/\lambda + k\delta_\lambda = 0. \tag{6.16}$$

Since $\langle k\delta_\lambda, \rho u_\lambda + m\sigma_\lambda + d\delta_\lambda\rangle_{\Gamma_1} = \langle k\delta_\lambda, \rho u_\lambda\rangle_{\Gamma_1} + \langle k\delta_\lambda, m(\delta_\lambda - \delta)/\lambda\rangle_{\Gamma_1} + |(kd)^{1/2}\delta_\lambda|^2$, we take the inner product of (6.16) and $\rho u_\lambda + m\sigma_\lambda + d\delta_\lambda$ to find that

$$\limsup_{\lambda\downarrow 0}(\phi_3(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \phi_3(u, v, \delta, \sigma))/\lambda + 2\langle k\delta, \rho u\rangle_{\Gamma_1} + 2|(kd)^{1/2}\delta|^2 \leq 0.$$

By (6.8) we have

$$\begin{aligned} &(\|(I - \lambda A((u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)))^{-1}A((u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda))(u, v, \delta, \sigma)\|^2_{(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)} \\ &\quad -\|A((u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda))(u, v, \delta, \sigma)\|^2_{(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)})/\lambda \\ &\leq -2\nu\rho\beta(\|\nabla u_\lambda\|^2)^{-1}\|\beta(\|\nabla u_\lambda\|^2)\Delta u_\lambda - \nu v_\lambda\|^2 \\ &\quad -2|d^{1/2}(1/m)(d\sigma_\lambda + k\delta_\lambda + \rho v_\lambda)|^2. \end{aligned} \tag{6.17}$$

By the definition of the norm $\|(u, v, \delta, \sigma)\|_{(w,z,\xi,\eta)}$ we have

$$\begin{aligned} &(\|A((u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda))(u, v, \delta, \sigma)\|^2_{(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda)} \\ &\quad -\|A((u, v, \delta, \sigma))(u, v, \delta, \sigma)\|^2_{(u, v, \delta, \sigma)})/\lambda \\ &= \rho(\beta(\|\nabla u_\lambda\|^2)^{-1}\|\beta(\|\nabla u_\lambda\|^2)\Delta u - \nu v\|^2 \\ &\quad -\beta(\|\nabla u\|^2)^{-1}\|\beta(\|\nabla u\|^2)\Delta u - \nu v\|^2)/\lambda. \end{aligned} \tag{6.18}$$

Adding (6.17) and (6.18), and passing to the limsup as $\lambda \downarrow 0$, we have

$$\begin{aligned}
&\limsup_{\lambda\downarrow 0}(\phi_4(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \phi_4(u,v,\delta,\sigma))/\lambda \\
&\qquad +2\rho\nu\beta(\|\nabla u\|^2)^{-1}\|\beta(\|\nabla u\|^2)\Delta u - \nu v\|^2 \\
&\qquad +2|d^{1/2}(1/m)(\rho v + d\sigma + k\delta)|^2 \qquad\qquad (6.19)\\
&\quad \le 2\rho\beta(\|\nabla u\|^2)^{-2}\beta'(\|\nabla u\|^2)\langle \nabla u, \nabla v\rangle \|\beta(\|\nabla u\|^2)\Delta u - \nu v\|^2 \\
&\qquad +4\rho\nu\beta(\|\nabla u\|^2)^{-2}\beta'(\|\nabla u\|^2)\langle \nabla u, \nabla v\rangle \langle \beta(\|\nabla u\|^2)\Delta u - \nu v, v\rangle.
\end{aligned}$$

Consider the functional φ_0 on Y defined by $\varphi_0 = \phi_1 + \phi_4$. Then we have, by (6.15) and (6.19),

$$\begin{aligned}
&\limsup_{\lambda\downarrow 0}(\varphi_0(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \varphi_0(u,v,\delta,\sigma))/\lambda \\
&\qquad +2\nu\rho\beta(\|\nabla u\|^2)^{-1}(\|v\|^2 + \|\beta(\|\nabla u\|^2)\Delta u - \nu v\|^2) \\
&\le 2(\nu+1)\rho\beta(\|\nabla u\|^2)^{-2}|\beta'(\|\nabla u\|^2)|\|\nabla u\|\|\nabla v\|(\|v\|^2 + \|\beta(\|\nabla u\|^2)\Delta u - \nu v\|^2).
\end{aligned}$$

Since

$$2\rho\|\nabla u\|\|\nabla v\| \le \rho(\|\nabla u\|^2 + \|\nabla v\|^2) \le \varphi_0(u,v,\delta,\sigma), \tag{6.20}$$

we have

$$\begin{aligned}
&\limsup_{\lambda\downarrow 0}(\varphi_0(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \varphi_0(u,v,\delta,\sigma))/\lambda \\
&\qquad \le (f_0(\varphi_0(u,v,\delta,\sigma)) - 2\nu)^+\varphi_0(u,v,\delta,\sigma),
\end{aligned} \tag{6.21}$$

where $s^+ = \max(s,0)$ for $s \in \mathbb{R}$ and f_0 is a nondecreasing, continuous function on $\mathbb{R}_+$ such that $f_0(0) = 0$. The function G, defined by

$$G(s) = (f_0(s) - 2\nu)^+ s$$

for $s \in \mathbb{R}_+$, is a comparison function such that for each $\alpha_0 > 0$ with $f_0(\alpha_0) < 2\nu$, the Cauchy problem for G with the initial condition $r(0) = \alpha_0$ has the global maximal solution $m(t;\alpha_0) = \alpha_0$. Since

$$\limsup_{\lambda\downarrow 0}(\varphi_0(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \varphi_0(u,v,\delta,\sigma))/\lambda \le G(\varphi_0(u,v,\delta,\sigma)),$$

condition (φ)-(R) is shown to be satisfied. Theorem 2.9 therefore asserts that for each $\alpha_0 > 0$ with $f_0(\alpha_0) < 2\nu$, the mixed problem has a unique solution (u,δ) in the class

$$C([0,\infty); H(\Delta,\Omega)) \cap C^1([0,\infty); V) \cap C^2([0,\infty); L^2(\Omega)) \times C^2([0,\infty); L^2(\Gamma_1))$$

if the initial data $(u_0, v_0, \delta_0, \sigma_0)$ satisfies $\varphi_0(u_0, v_0, \delta_0, \sigma_0) \leq \alpha_0$.

If d is positive on Γ_1 then the exponential decay of the energy is obtained. To do this, we use the functional φ_1 on X defined by $\varphi_1 = a\phi_1 + b\phi_2 + \phi_3$, where $a, b > 0$ are yet to be determined. We have

$$\begin{aligned}
&\limsup_{\lambda\downarrow 0}(\varphi_1(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \varphi_1(u, v, \delta, \sigma))/\lambda \\
&\qquad +2\nu\rho(a\beta(\|\nabla u\|^2)^{-1}\|v\|^2 + b\|\nabla u\|^2) + 2(a+b)d_0|\sigma|^2 + 2k_0 d_0|\delta|^2 \\
&\leq (2/\beta_0)|\beta'(\|\nabla u\|^2)|\|\nabla u\|\|\nabla v\|\varphi_1(u, v, \delta, \sigma) + 2b\nu\rho\langle\sigma, u\rangle_{\Gamma_1} - 2\rho\langle k\delta, u\rangle_{\Gamma_1}
\end{aligned}$$

where d_0, k_0 are positive constants such that $d(x) \geq d_0$ and $k(x) \geq k_0$ for $x \in \Gamma_1$. Since $|\langle k\delta, u\rangle_{\Gamma_1}| \leq \varepsilon_0|\delta|^2 + C_{\varepsilon_0}\|\nabla u\|^2$ for any $\varepsilon_0 > 0$ and $|\langle\sigma, u\rangle_{\Gamma_1}| \leq \varepsilon_1\|\nabla u\|^2 + C_{\varepsilon_1}|\sigma|^2$ for any $\varepsilon_1 > 0$, we find

$$\begin{aligned}
&\limsup_{\lambda\downarrow 0}(\varphi_1(u_\lambda, v_\lambda, \delta_\lambda, \sigma_\lambda) - \varphi_1(u, v, \delta, \sigma))/\lambda \\
&\qquad +(c_0 - f_1(\varphi_0(u, v, \delta, \sigma)))\varphi_1(u, v, \delta, \sigma) \leq 0,
\end{aligned}$$

by choosing $\varepsilon_0, b, \varepsilon_1$ and a in order so that $2\rho\varepsilon_0 \leq k_0 d_0$, $2\rho(C_{\varepsilon_0} + b\nu\varepsilon_1) \leq \nu\rho b$ and $2b\nu\rho C_{\varepsilon_1} \leq (a+b)d_0$ and noting that $\varphi_1(u, v, \delta, \sigma) \leq c(\beta(\|\nabla u\|^2)^{-1}\|v\|^2 + \|\nabla u\|^2 + |\sigma|^2 + |\delta|^2)$ for some constant $c > 0$. Here f_1 is a nondecreasing, continuous function on $\mathbb{R}_+$ such that $f_1(0) = 0$ and we have used (6.20). This combined with (6.21) implies that condition (φ)-(R) is satisfied with $\varphi = (\varphi_0, \varphi_1)$ and the comparison function $g = (g_0, g_1)$ defined by $g_0(r_0, r_1) = (f_0(r_0) - 2\nu)^+ r_0$ and $g_1(r_0, r_1) = (f_1(r_0) - c_0)r_1$ for $r = (r_0, r_1) \in \mathbb{R}_+^2$. If $\alpha_0 > 0$ is chosen such that $f_0(\alpha_0) < 2\nu$ and $f_1(\alpha_0) < c_0$, then the maximal solution $m(t;\alpha)$ of the Cauchy problem for g with initial condition $r(0) = \alpha = (\alpha_0, \alpha_1)$ is given by

$$\begin{cases} m_0(t;\alpha) = \alpha_0, \\ m_1(t;\alpha) = \exp((f_1(\alpha_0) - c_0)t)\alpha_1. \end{cases}$$

We therefore have $\varphi_0(u(t), v(t), \delta(t), \sigma(t)) \leq \alpha_0$ and $\varphi_1(u(t), v(t), \delta(t), \sigma(t)) \leq \exp((f_1(\alpha_0) - c_0)t)\alpha_1$ for $t \geq 0$ if the initial data $(u_0, v_0, \delta_0, \sigma_0)$ satisfies the two conditions that $\varphi_0(u_0, v_0, \delta_0, \sigma_0) \leq \alpha_0$ and $\varphi_1(u_0, v_0, \delta_0, \sigma_0) \leq \alpha_1$. Since $\|(u, v, \delta, \sigma)\|_X^2 \leq f_2(\varphi_0(u, v, \delta, \sigma))\varphi_1(u, v, \delta, \sigma)$ for $(u, v, \delta, \sigma) \in X$, where f_2 is a nondecreasing function on $\mathbb{R}_+$, we have $\|\nabla u\|^2 + \|u_t\|^2 + |\delta|^2 + |\delta_t|^2 \leq M\exp(-\omega t)$ for $t \geq 0$. ∎

7. Quasi-linear Wave Equations with Wentzell Boundary Conditions

The systematic study of the second order Cauchy problems for operators with Wentzell boundary conditions was initiated by Favini, Goldstein, Goldstein and

Romanelli [5]. In [7] a general framework was developed which allows to study the initial-boundary value problems for quasi-linear equations with Wentzell boundary conditions. This section is devoted to another approach to such problems. We apply Theorem 2.8 to the initial-boundary value problem for the quasi-linear wave equation with Wentzell boundary condition

$$\tag{7.1}\begin{cases} u_{tt}(x,t) = \phi(x,u_x(x,t))u_{xx}(x,t) + \psi(x,u(x,t),u_x(x,t),u_t(x,t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in [0,1], \\ \phi(j,u_x(j,t))u_{xx}(j,t) + \psi(j,u(j,t),u_x(j,t),u_t(j,t)) \\ \qquad\qquad\qquad\qquad = \beta_j(u_x(j,t)) + \gamma_j(u(j,t)) \quad \text{for } j = 0,1. \end{cases}$$

Theorem 7.1. *Assume that the following conditions are satisfied:*

(ϕ) $\phi \in C^1([0,1]\times\mathbb{R};\mathbb{R})$ *and there exists* $\phi_0 > 0$ *such that* $\phi(x,p) \geq \phi_0$ *for* $(x,p) \in [0,1]\times\mathbb{R}$.

(ψ) $\psi \in C^1([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R};\mathbb{R})$.

(β) *For* $j = 0,1$, $\beta_j \in C^2(\mathbb{R};\mathbb{R})$ *and* $\beta_j(0) = 0$.

(γ) *For* $j = 0,1$, $\gamma_j \in C^2(\mathbb{R};\mathbb{R})$ *and* $\gamma_j(0) = 0$.

Then for each $(u_0,v_0) \in C^2([0,1];\mathbb{R})\times C^1([0,1];\mathbb{R})$ *with*

$$\phi(j,u_0'(j))u_0''(j) + \psi(j,u_0(j),u_0'(j),v_0(j)) = \beta_j(u_0'(j)) + \gamma_j(u_0(j))$$

for $j = 0,1$, *there exist* $T > 0$ *and a unique* $u \in C([0,T];C^2([0,1];\mathbb{R})) \cap C^1([0,T];C^1([0,1];\mathbb{R})) \cap C^2([0,T];C([0,1];\mathbb{R}))$ *such that* u *satisfies equation (7.1) for* $t \in [0,T]$ *and the initial condition* $u(x,0) = u_0(x)$ *and* $u_t(x,0) = v_0(x)$ *for* $x \in [0,1]$.

Proof. We use the homogeneous reduction technique due to Kato [10]. Let X be the space of all $(u,v,k,\xi,\eta) \in C^1[0,1]\times C[0,1]\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ such that $u(j) = \xi_j$ for $j = 0,1$, and let Y be the space of all $(u,v,k,\xi,\eta) \in C^2[0,1]\times C^1[0,1]\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ such that $u(j) = \xi_j$ and $v(j) = \eta_j$ for $j = 0,1$. The spaces X and Y are real Banach spaces under the norms $\|(u,v,k,\xi,\eta)\|_X = \|u\|_{C^1} + \|v\|_C + |k| + \|\xi\|_{\mathbb{R}^2} + \|\eta\|_{\mathbb{R}^2}$ and $\|(u,v,k,\xi,\eta)\|_Y = \|u\|_{C^2} + \|v\|_{C^1} + |k| + \|\xi\|_{\mathbb{R}^2} + \|\eta\|_{\mathbb{R}^2}$, respectively.

Let $D = \{(u,v,k,\xi,\eta) \in Y;\ k = 1\}$. To solve the problem (7.1), we apply Theorem 2.8 to the family $\{A((w,z,\zeta,f,g)); (w,z,\zeta,f,g) \in Y\}$ in $B(Y,X)$ defined by

$$\begin{aligned} &A((w,z,\zeta,f,g))(u,v,k,\xi,\eta) \\ &\quad = (v,\ \phi(\cdot,w')u'' + k\psi(\cdot,w,w',z),\ 0,\ \eta, B(w)u + C(w)\xi), \end{aligned}$$

where

$$B(w)u = \left(\left(\int_0^1 \beta_j'(\theta w'(j))\,d\theta\right)u'(j)\right)_{j=0,1} \qquad \text{for } w \in C^2[0,1] \text{ and } u \in C^1[0,1]$$

and

$$C(w)\xi = \left(\left(\int_0^1 \gamma_j'(\theta w(j))\,d\theta\right)\xi_j\right)_{j=0,1} \qquad \text{for } w \in C^2[0,1] \text{ and } \xi = (\xi_0, \xi_1) \in \mathbb{R}^2.$$

Conditions (A1) through (A3) are easily checked. To check condition (A4) we need the following lemma.

Lemma 7.2. ([17, Proposition 2.1] and [7]). (I) *Let* $E = \{(p,q) \in C^1[0,1] \times C[0,1]; p(0) = p(1) = 0\}$ *and* $w \in C^2[0,1]$. *Define an operator* $A_0(w)$ *in* E *by*

$$\begin{cases} (A_0(w)(p,q))(x) = (q(x),\ \phi(x, w'(x))p''(x)) \\ D(A_0(w)) = \{(p,q) \in C^2[0,1] \times C^1[0,1];\ p(0) = p(1) = 0,\ q(0) = q(1) = 0\}. \end{cases}$$

Then the following assertions hold:

(i) *The space* E *is a real Banach space under the norm*

$$\begin{aligned} \|(p,q)\|_w = \max(\ & \sup\{|q(x) + \sqrt{\phi(x,w'(x))}p'(x)|; x \in [0,1]\}, \\ & \sup\{|q(x) - \sqrt{\phi(x,w'(x))}p'(x)|; x \in [0,1]\}) \end{aligned}$$

for $(p,q) \in E$. *By* E_w *we denote the space* E *equipped with the norm* $\|(p,q)\|_w$.

(ii) *For* $r > 0$ *with* $\|w\|_{C^2} \le r$, *there exists* $\omega(r) \ge 0$ *such that for any* $\lambda > 0$ *with* $\lambda\omega(r) < 1$, $R(I - \lambda A_0(w)) = E_w$ *and*

$$\|(I - \lambda A_0(w))^{-1}(p,q)\|_w \le (1 - \lambda\omega(r))^{-1}\|(p,q)\|_w \qquad \text{for } (p,q) \in E_w.$$

(II) *The family* $\{\|(p,q)\|_w; w \in C^2[0,1]\}$ *of the norms defined above satisfies the following conditions:*

(i) *For each* $r > 0$ *there exists* $M_E(r) > 0$ *such that*

$$M_E(r)^{-1}(\|p\|_{C^1} + \|q\|_C) \le \|(p,q)\|_w \le M_E(r)(\|p\|_{C^1} + \|q\|_C) \tag{7.2}$$

for $(p,q) \in E$ *and* $\|w\|_{C^2} \le r$.

(ii) *For each* $r > 0$ *there exists* $L_E(r) > 0$ *such that*

$$\|(p,q)\|_w \leq \exp(L_E(r)\|w-\hat{w}\|_{C^1})\|(p,q)\|_{\hat{w}} \tag{7.3}$$

for $(p,q) \in E$ *and* $\|w\|_{C^2}, \|\hat{w}\|_{C^2} \leq r$.

The operator L_0 from $\mathbb{R}^2$ into $C^\infty[0,1]$, defined by $(L_0\xi)(x) = (1-x)\xi_0 + x\xi_1$ for $x \in [0,1]$, plays an important role in verifying condition (A4). Such an operator is called a *Dirichlet operator*. Let $(u,v,k,\xi,\eta) \in X$ and $(w,z,\zeta,f,g) \in B_Y(r) \cap D$. Set $p = u - L_0\xi$ and $q = v - L_0\eta$. Then we apply Banach's fixed point theorem to the mapping Φ from $E_w \times \mathbb{R}^2 \times \mathbb{R}^2$ into itself, defined by

$$\begin{aligned}\Phi((\tilde{p},\tilde{q}),\, \tilde{\xi},\, \tilde{\eta}) = {} & ((I-\lambda A_0(w))^{-1}((p,q) + \lambda(0, k\psi(\cdot,w,w',z) - L_0\tilde{d}(w))),\\ & \xi + \lambda\tilde{\eta}, \eta + \lambda\tilde{d}(w))\end{aligned}$$

for $(\tilde{p},\tilde{q}) \in E_w$ and $\tilde{\xi},\tilde{\eta} \in \mathbb{R}^2$, where $\tilde{d}(w) := B(w)(\tilde{p} + L_0\tilde{\xi}) + C(w)\tilde{\xi}$ for $w \in C^2[0,1]$. By Lemma 7.2, this yields that for sufficiently small $\lambda > 0$ depending on r, the problem

$$\begin{cases} ((p_\lambda,q_\lambda) - (p,q))/\lambda = A_0(w)(p_\lambda,q_\lambda) + (0, k_\lambda\psi(\cdot,w,w',z) - L_0\tilde{d}(w)),\\ (k_\lambda - k)/\lambda = 0,\\ (\xi_\lambda - \xi)/\lambda = \eta_\lambda,\\ (\eta_\lambda - \eta)/\lambda = \tilde{d}(w) \end{cases}$$

has a solution $(p_\lambda,q_\lambda,k_\lambda,\xi_\lambda,\eta_\lambda) \in C^2[0,1] \times C^1[0,1] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ satisfying that $|k_\lambda| = |k|$ and

$$\begin{aligned}&\|(p_\lambda,q_\lambda)\|_w + \|\xi_\lambda\|_{\mathbb{R}^2} + \|\eta_\lambda\|_{\mathbb{R}^2}\\ &\quad\leq (1-\lambda\omega(r))^{-1}(\|(p,q)\|_w + \lambda C(r)(|k_\lambda| + \|p_\lambda\|_{C^1} + \|\xi_\lambda\|_{\mathbb{R}^2}))\\ &\qquad + \|\xi\|_{\mathbb{R}^2} + \lambda\|\eta_\lambda\|_{\mathbb{R}^2} + \|\eta\|_{\mathbb{R}^2} + \lambda C(r)(\|p_\lambda\|_{C^1} + \|\xi_\lambda\|_{\mathbb{R}^2}).\end{aligned}$$

A combination of the above two estimates shows that

$$\begin{aligned}&(1-\lambda\beta(r))(\|(p_\lambda,q_\lambda)\|_w + |k_\lambda| + \|\xi_\lambda\|_{\mathbb{R}^2} + \|\eta_\lambda\|_{\mathbb{R}^2})\\ &\quad\leq \|(p,q)\|_w + |k| + \|\xi\|_{\mathbb{R}^2} + \|\eta\|_{\mathbb{R}^2}.\end{aligned} \tag{7.4}$$

We employ the family $\{\|(u,v,k,\xi,\eta)\|_{(w,z,\zeta,f,g)}; (w,z,\zeta,f,g) \in Y\}$ of norms in X defined by

$$\|(u,v,k,\xi,\eta)\|_{(w,z,\zeta,f,g)} = \|(u-L_0\xi, v-L_0\eta)\|_w + |k| + \|\xi\|_{\mathbb{R}^2} + \|\eta\|_{\mathbb{R}^2}.$$

Condition (D1) follows from (7.2). The inequality (7.4) together with (7.3) implies condition (D2). If we set $u_\lambda := p_\lambda + L_0\xi_\lambda$ and $v_\lambda := q_\lambda + L_0\eta_\lambda$, then we see that $(u_\lambda, v_\lambda, k_\lambda, \xi_\lambda, \eta_\lambda)$ is a solution of the equation $(u_\lambda, v_\lambda, k_\lambda, \xi_\lambda, \eta_\lambda) - \lambda A((w, z, \zeta, f, g))(u_\lambda, v_\lambda, k_\lambda, \xi_\lambda, \eta_\lambda) = (u, v, k, \xi, \eta)$ and

$$(1 - \lambda\beta(r))\|(u_\lambda, v_\lambda, k_\lambda, \xi_\lambda, \eta_\lambda)\|_{(w,z,\zeta,f,g)} \le \|(u, v, k, \xi, \eta)\|_{(w,z,\zeta,f,g)}.$$

This means that condition (A4) is satisfied. If $k = 1$ then $k_\lambda(= k) = 1$; hence condition (2.1) is satisfied. By condition (ϕ), for each $\alpha > 0$ there exists $r > 0$ such that $w \in C^2[0, 1]$ and $\|(p, q)\|_w \le \alpha$ imply that $\|p'\|_C, \|q\|_C \le r$; from which condition (2.4) holds. ∎

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