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EXISTENCE AND ASYMPTOTIC STABILITY OF SOLUTIONS TO A FUNCTIONAL-INTEGRAL EQUATION

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Abstract. Using the Darbo's fixed-point theorem associated with the measure of noncompactness due to Banaś, we establish the existence and asymptotic stability of solutions for a functional-integral equation. An example which shows the importance of our result is also included.

1. Introduction

It is well known that integral equations have many useful applications in describing numerous events and problems of the real world, and the theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory. For details, we refer to [1-18] and the references therein.

The purpose of this paper is to consider the existence of solutions for the following functional-integral equation

$$x(t) = f(t, x(t)) + g(t, x(t)) \int_0^t u(t, s, x(s))ds, \quad \forall t \in [0, \infty), \tag{1.1}$$

where f, g and u are given continuous functions while x is an unknown function. Recently, Banaś and Rzepka [11, 13] studied the behavior of solutions for equation (1.1) with either $f \equiv 0$ or $g \equiv 1$.

By applying Darbo's fixed-point theorem associated with the measure of noncompactness due to Banaś, we obtain a sufficient condition for the existence and asymptotic stability of solutions for equation (1.1). The result presented in this paper extends proper corresponding results of Banaś' and Rzepka [11, 13].

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2. Preliminaries

In this section, we give a collection of auxiliary facts which will be needed further on. Let $R = (-\infty, \infty)$ and $R_+ = [0, +\infty)$. Assume that $(E, ||\cdot||)$ is an infinite dimensional Banach space with zero element θ and $B(\theta, r)$ stands for the closed ball centered at θ and with radius r. Let $B(E)$ denote the family of all nonempty bounded subsets of E and μ be a measure of noncompactness on $B(E)$. The famous fixed-point theorem due to Darbo [16] states as follows:

Theorem 2.1. *Let D be a nonempty bounded closed convex subset of the space E and let $f : D \to D$ be a continuous mapping such that $\mu(fA) \le k\mu(A)$ for each nonempty subset A of D, where $k \in [0, 1)$ is a constant and μ is a measure of noncompactness on $B(E)$. Then f has at least one fixed point in D.*

Assume that $BC(R_+)$ denotes the Banach space of all bounded and continuous functions $x : R_+ \to R$ equipped with the standard norm

$$||x|| = \sup\{|x(t)| : t \in R_+\}, \quad x \in BC(R_+).$$

For any nonempty bounded subset X of $BC(R_+)$, $x \in X, T > 0$ and $\epsilon \ge 0$, define

$$w^T(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T] \text{ with } |t - s| \le \epsilon\},$$
$$w^T(X, \epsilon) = \sup\{w^T(x, \epsilon) : x \in X\}, \quad w_0^T(X) = \lim_{\epsilon\to 0} w^T(X, \epsilon),$$
$$w_0(X) = \lim_{T\to\infty} w_0^T(X), \quad X(t) = \{x(t) : x \in X\},$$
$$diam\, X(T) = \sup\{|x(t) - y(t)| : x, y \in X\} \text{ and}$$
$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diam\, X(t).$$

It follows from Banaś [3] that the function $\mu(X)$ is a sublinear measure of noncompactness in the space $BC(R_+)$. A solution $x = x(t)$ of equation (1.1) is said to be *asymptotically stable* on the interval R_+ of for any $\epsilon > 0$ there exist $T > 0$ and $r > 0$ such that $x \in B(\theta, r)$ and

$$|x(t) - y(t)| \le \epsilon, \quad \forall t > T,$$

where $y = y(t) \in B(\theta, r)$ is a arbitrary solution of equation (1.1).

3. Main Result

Now we make the following assumptions.

(a) $f, g : R_+ \times R \to R$ are continuous and $f(\cdot, 0), g(\cdot, 0) \in BC(R_+)$;

(b) there exist a constant $k \in [0, 1)$ and a continuous function $m : R_+ \to R_+$ satisfying

$$|f(t,x) - f(t,y)| \leq k|x - y| \quad \text{and} \quad |g(t,x) - g(t,y)| \leq m(t)|x - y|,$$
$$\forall t \geq 0 \text{ and } x, y \in R;$$

(c) $u : R_+ \times R_+ \times R \to R$ is a continuous function and there exist continuous functions $a, b : R_+ \to R_+$ satisfying

$$|u(t,s,x)| \leq a(t)b(s), \quad \forall t, s \in R_+ \text{ and } x \in R,$$
$$\lim_{t\to\infty} a(t) \int_0^t b(s)ds = \lim_{t\to\infty} m(t)a(t) \int_0^t b(s)ds = 0;$$

(d) there exists a constant $q \in [0, 1)$ such that

$$m(t)a(t) \int_0^t b(s)ds \leq q, \quad \forall t \in R_+ \text{ and } k + q < 1.$$

Theorem 3.1. *Under assumptions* (a)-(d)*, equation* (1.1) *has at least one solution* $x = x(t)$ *which belongs to the space* $BC(R_+)$ *and is asymptotically stable on the interval* R_+.

Proof. First of all let us fix a function $x \in BC(R_+)$ and define

$$(Fx)(t) = f(t, x(t)) + g(t, x(t)) \int_0^t u(t, s, x(s))ds, \quad \forall t \in I. \tag{3.1}$$

In light of (3.1) and assumptions (a)-(d), we infer that Fx is continuous on R_+ and that

$$\begin{aligned}
&|(Fx)(t)| \\
&\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + |g(t,x(t)) \\
&\quad -g(t,0)| \int_0^t |u(t,s,x(s))|ds + |g(t,0)| \int_0^t |u(t,s,x(s))|ds \\
&\leq k|x(t)| + |f(t,0)| + m(t)|x(t)|a(t) \int_0^t b(s)ds + |g(t,0)|a(t) \int_0^t b(s)ds \\
&\leq (k+q)|x(t)| + |f(t,0)| + |g(t,0)|a(t) \int_0^t b(s)ds \\
&\leq (k+q)|x(t)| + A
\end{aligned} \tag{3.2}$$

for any $t \in R_+$, where $A = \sup\{|f(t,0)|+|g(t,0)|a(t)\int_0^t b(s)ds : t \in R_+\} < \infty$. Thus Fx is bounded on R_+. It follows that the operator F transforms the space $BC(R_+)$ into itself. Notice that (3.2) implies that

$$||Fx|| \le (k+q)||x|| + A.$$

This means that the operator F maps the ball $B(\theta, r)$ into itself, where

$$r = \frac{A}{1-k-q}. \tag{3.3}$$

Now we claim that F is continuous on the ball $B(\theta, r)$. For any $\epsilon > 0$ and $x, y \in B(\theta, r)$ with $||x-y|| \le \epsilon$ and $t \in R_+$, in terms of (3.1) and assumptions (b)-(d), we infer that

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\le |f(t,x(t)) - f(t,y(t))| \\
&\quad + \left| g(t,x(t)) \int_0^t u(t,s,x(s))ds - g(t,x(t)) \int_0^t u(t,s,y(s))ds \right| \\
&\quad + \left| g(t,x(t)) \int_0^t u(t,s,y(s))ds - g(t,y(t)) \int_0^t u(t,s,y(s))ds \right| \\
&\le k|x(t)-y(t)| + |g(t,x(t))| \int_0^t |u(t,s,x(s)) - u(t,s,y(s))|ds \\
&\quad + |g(t,x(t)) - g(t,y(t))| \int_0^t |u(t,s,u(s))|ds \\
&\le k||x-y|| + [m(t)|x(t)| + |g(t,0)|] \int_0^t |u(t,s,x(s)) - u(t,s,y(s))|ds \\
&\quad + m(t)|x(t)-y(t)|a(t) \int_0^t b(s)ds \\
&\le (k+q)\epsilon + [rm(t) + |g(t,0)|] \int_0^t |u(t,s,x(s)) - u(t,s,y(s))|ds.
\end{aligned} \tag{3.4}$$

On the other hand, assumptions (a)-(c) ensure that there exists a positive number T satisfying

$$\{rm(t) + \sup\{|g(t,0)| : t \in R_+\}\}a(t) \int_0^t b(s)ds < (1-k-q)\frac{\epsilon}{2}, \quad \forall t \ge T. \tag{3.5}$$

Suppose that $t \ge T$. It follows from (3.4) and (3.5) that

$$|(Fx)t - (Fy)(t)| \le (k+q)\epsilon + 2[rm(t) + |g(t,0)|]a(t) \int_0^t b(s)ds < \epsilon. \tag{3.6}$$

Suppose that $t < T$. Put

$w(\epsilon) = \sup\{|u(t,s,x) - u(t,s,y)| : t,s \in [0,T]$ and $x,y \in [t,r]$ with $|x-y| \leq \epsilon\}$.

It follows from the uniform continuity of the function $u = t(t,s,x)$ on the set $[0,T] \times [0,T] \times [-r,r]$ that $\lim_{\epsilon \to 0} w(\epsilon) = 0$. In view of (3.4), we deduce that

$$(3.7)\quad |(Fx)(t) - (Fy)(t)| \leq (k+q)\epsilon + Tw(\epsilon)\sup\{m(t) + |g(t,v)| : t \in [0,T]\}.$$

From (3.6) and (3.7), we get immediately that the operator F is continuous on the ball $B(\theta, r)$.

Now let X be a nonempty subset of $B(\theta, r)$. We assert that

$$\mu(FX) \leq (k+q)\mu(X). \tag{3.8}$$

Indeed, by virtue of assumptions (b)-(d), we conclude that for any $x, y \in X$ and $t \in R_+$,

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\leq k|x(t) - y(t)| + [|g(t,x(t)) - g(t,0)| + |g(t,0)|]\int_0^t |u(t,s,x(s)) \\
&\quad -u(t,s,y(s))|ds + |g(t,x(t)) - g(t,y(t))|\int_0^t |u(t,s,y(s))|ds \\
&\leq (k+q)|x(t) - y(t)| + 2(rm(t) + |g(t,0)|)a(t)\int_0^t b(s)ds,
\end{aligned}$$

which gives that

$$\begin{aligned}
diam(FX)(t) &\leq (k+q)diam\, X(t) + 2(rm(t) \\
&\quad +|g(t,0)|)a(t)\int_0^t b(s)ds, \quad \forall t \in R_+,
\end{aligned} \tag{3.9}$$

It follow from (3.9) and assumptions (a) and (c) that

$$\limsup_{t\to\infty} diam(FX)(t) \leq (k+q)\limsup_{t\to\infty} diam\, X(t). \tag{3.10}$$

For any $T > 0$, $\epsilon > 0$, $x \in X$ and $t, p \in [0,T]$ with $|t-p| \leq \epsilon$, by assumptions (b)-(d) and (3.1), we arrive at

$$\begin{aligned}
&|(Fx)(t)-(Fx)(p)| \\
&\leq |f(t,x(t))-f(p,x(p))|+|g(t,x(t))-g(p,x(p))|\int_0^t |u(t,s,x(s))|ds \\
&\quad +|g(p,x(p))|\left|\int_0^t u(t,s,x(s))ds-\int_0^p u(p,s,x(s))ds\right| \\
&\leq |f(t,x(t))-f(t,x(p))|+|f(t,x(p))-f(p,x(p))| \\
&\quad +[|g(t,x(t))-g(t,x(p))|+|g(t,x(p))-g(p,x(p))|]a(t)\int_0^t b(s)ds \\
&\quad +[|g(p,x(p))-g(p,0)|+|g(p,v)|]\left[\left|\int_p^t u(t,s,x(s))ds\right|\right. \\
&\quad \left.+\int_0^p |u(t,s,x(s))-u(p,s,x(s))|ds\right] \\
&\leq k|x(t)-x(p)|+w_r^T(f,\epsilon)+[m(t)|x(t)-x(p)|+w_r^T(g,\epsilon)]a(t)\int_0^t b(s)ds \\
&\quad +[m(p)|x(p)|+|g(p,0)|][a(t)|t-p|\sup\{b(s):s\in[0,T]\}+Tw_r^T(u,\epsilon)] \\
&\leq (k+q)w^T(x,\epsilon)+w_r^T(f,\epsilon)+w_r^T(g,\epsilon)a(t)\int_0^t b(s)ds \\
&\quad +[rm(p)+|g(p,0)|][\epsilon a(t)\sup\{b(s):s\in[0,T]\}+Tw_r^T(u,\epsilon)],
\end{aligned}$$

where

$$\begin{aligned}
&w_r^T(f,\epsilon)=\sup\{|f(t,x)-f(p,x)|:t,p\in[0,T] \text{ with } |t-p|\leq\epsilon \text{ and } x\in[-r,r]\}, \\
&w_r^T(g,\epsilon)=\sup\{|g(t,x)-g(p,x)|:t,p\in[0,T] \text{ with } |t-p|\leq\epsilon \text{ and } x\in[-r,r]\}, \\
&w_r^T(u,\epsilon)=\sup\{|u(t,s,x)-u(p,s,x)|:t,p,s\in[0,T] \text{ with } |t-p| \\
&\qquad\qquad \leq\epsilon \text{ and } x\in[-r,r]\}.
\end{aligned}$$

That is,

$$\begin{aligned}
w^T(Fx,\epsilon) &\leq (k+q)w^T(x,\epsilon)+w_r^T(f,\epsilon)+w_r^T(g,\epsilon)a(t)\int_0^t b(s)ds \\
&\quad +[rm(p)+|g(p,0)|][\epsilon a(t)\sup\{b(s):s\in[0,T]\} \\
&\quad +Tw_r^T(u,\epsilon)]
\end{aligned} \tag{3.11}$$

Note that assumptions (a) and (c) guarantee that the functions $f = f(t,x)$, $g = g(t,x)$ are uniformly continuous on the set $[0,T] \times [-r,r]$, and the function $u = u(t,s,x)$ is uniformly continuous on the set $[0,T] \times [0,T] \times [-r,r]$. It follows that

$$\lim_{\epsilon\to 0} w_r^T(f,\epsilon) = \lim_{\epsilon\to 0} w_r^T(g,\epsilon) = \lim_{\epsilon\to 0} w_r^T(u,\epsilon) = 0. \tag{3.12}$$

Using assumptions (a)-(d), (3.11) and (3.12), we conclude that

$$w_0^T(FX) \le (k+q)w_0^T(X) \quad \text{and} \quad w_0(FX) \le (k+q)w_0(X). \tag{3.13}$$

It is clear that (3.10) and (3.13) yield that (3.8) holds. It follows from Theorem 2.1 that F has at least one fixed point $x = x(t) \in B(\theta, r)$. That is, equation (1.1) has at least one solution $x = x(t) \in B(\theta, r)$.

Finally, we show that the solution $x = x(t)$ of equation (1.1) is asymptotically stable on the interval R_+. Let ϵ be an arbitrary positive number and let r be defined by (3.3). It follows assumptions (a)-(c) that there exists $T > 0$ such that

$$[rm(t) + |g(t,0)|]a(t)\int_0^t b(s)ds \le \frac{1-k-q}{2}\epsilon, \quad \forall t \ge T. \tag{3.14}$$

For solutions $x = x(t)$ and $y = y(t)$ of equation (1.1) in $B(\theta, r)$, by (3.1) and assumptions (b)-(d), we deduce that

$$\begin{aligned}
&|x(t) - y(t)| \\
&= |(Fx)(t) - (Fy)(t)| \\
&\le |f(t,x(t)) - f(t,y(t))| \\
&\quad + [|g(t,x(t)) - g(t,v)| + |g(t,v)|]\int_0^t |u(t,s,x(s)) - u(t,s,y(s))|ds \\
&\quad + |g(t,x(t)) - g(t,y(t))|\int_0^t |u(t,s,y(s))|ds \\
&\le k|x(t) - y(t)| + 2[m(t)|x(t)| + |g(t,0)|]a(t)\int_0^t b(s)ds \\
&\quad + m(t)|x(t) - y(t)|a(t)\int_0^t b(s)ds \\
&\le (k+q)|x(t) - y(t)| + 2[rm(t) + |g(t,0)|]a(t)\int_0^t b(s)ds
\end{aligned} \tag{3.15}$$

for any $t \in R_+$. Using (3.14) and (3.15), we derive that

$$|x(t) - y(t)| \le \frac{2}{1-k-q}[rm(t) + |g(t,0)|]a(t)\int_0^t b(x)ds \le \epsilon, \quad \forall t \ge T.$$

That is, the solution $x = x(t)$ of equation (1.1) is asymptotically stable on the interval R_+. This completes the proof.

Remark 3.1. If $f(t, x) = 0$ for any $t \in R_+$ and $x \in R$, then Theorem 3.1 reduces to Theorem 2 of Banaś' and Rzepka [11]. If $g(t, x) = 1$ for any $t \in R_+$ and $x \in R$, then Theorem 3.1 reduces to Theorem 2 of Banaś and Rzepka [13].

Remark 3.2. The asymptotic stability of the equation $x = x(t)$ of the equation (1.1) can be obtained as a consequence of the fact that all solutions of that equation belong to the so-called kernel of the measure of noncompactnrss μ (cf. [4, 11] and [13]).

Example 3.1. Consider the following nonlinear functional-integral equation:

$$\begin{aligned} x(t) &= \frac{t}{1+t^2}\sin[t^2 - x(t)] + \cos\left[\sqrt{3t^4+1} - 2(t+1) + x(t)\right] \\ &\quad \times \int_0^t \frac{x(s)}{1+x^2(s)} \ln\left[1 + \frac{2(1+2s)|x(s)|}{3(1+t)(1+2t)^2}\right] ds, \quad \forall t \in R_+. \end{aligned} \tag{3.16}$$

Put

$$f(t,x) = \frac{x}{1+t^2}\sin(t^2 - x), \quad g(t,x) = \frac{\cos\left[\sqrt{3t^4+1} - 2(t+1) + x\right]}{1+\sqrt{t}},$$
$$\forall t \in R_+,\ x \in R,$$
$$u(t,s,x) = (1+\sqrt{t})\frac{x}{1+x^2}\ln\left[1 + \frac{(1+2s)|x|}{3(1+t)(1+2t)^2}\right], \quad \forall t, s \in R_+,\ x \in R,$$
$$k = \frac{1}{2}, \quad q = \frac{1}{3}, \quad m(t) = \frac{1}{1+\sqrt{t}}, \quad \forall t \in R_+,$$
$$a(t) = \frac{1+\sqrt{t}}{3(1+t)(1+2t)^2}, \quad b(t) = 1 + 2t, \quad \forall t \in R_+.$$

It is easy to verify that assumptions (a)-(d) are fulfilled. Consequently Theorem 3.1 ensures that equation (3.16) has at least one solution $x = x(t)$ which belongs to the space $BC(R_+)$ and is asymptotically stable on the interval R_+. However Theorem 2 in [11] and [13] are not applicable.

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REFERENCES

1. R. P. Agarwal and D. O'Reaan, Global existence for nonlinear operator inclusion, *Comput. Math. Appl.*, **38(11-12)** (1999), 131-139.
2. R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive solutions of differential, difference and integral equations*, Kluwer Academic, Dordecht, 1999.
3. J. Banaś, Measures of noncompactness in the space of continuous tempered functions, *Demonstratio Math.*, **14** (1981), 127-133.
4. J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, in: Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, 1980.
5. J. Banaś and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math.*, **41** (2001), 13-23.
6. J. Banaś and A. Martinon, Monotonic solutions of a quadratic integral equation of Volterra type, *Comput. Math. Appl.*, **47** (2004), 271-279.
7. J. Banaś, M. Lecko and W. G. El-Sayed, Existence theorems for some quadratic integral equations, *J. Math. Anal. Appl.*, **222** (1998), 276-285.
8. J. Banaś and K. Sadarangani, Solvability of Volterra-Stieltjes operator-integral equations and their applications, *Comput. Math. Appl.*, **41** (2001), 1535-1544.
9. J. Banaś, J. R. Rodriguez and K. Sadrangani, On a class of Urysohn-Sieltjes quadratic integral equations and their applications, *J. Comput. Appl. Math.*, **113** (2000), 35-50.
10. J. Banaś, J. Rocha and K. B. Sadarangani, Solvability of a nonlinear integral equation of Volterra type, *J. Comput. Appl. Math.*, **157** (2003), 31-48.
11. J. Banaś and B. Rzepka, On existence and asymptotic stability of solutions of a nonlinear integral equation, *J. Math. Anal. Appl.*, **284** (2003), 165-173.
12. J. Banaś, J. R. Rodriquez and K. Sadarangani, On a nonlinear quadratic integral equation of Urysohn-Stieltjes type and its applications, *Nonlinear Anal.*, **47** (2001), 1175-1186.
13. J. Banaś and B. Rzepka, An applicaion of a measure of noncompactness in the study of asymptotic stability, *Appl. Math. Lett.*, **16** (2003), 1-6.
14. B. Cahlon and M. Eskin, Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation, *J. Math. Anal. Appl.*, **83**, (1981), 159-171.
15. C. Corduneanu, *Integral equations and applications*, Cambrige University Press, Cambridge, MA, 1991.
16. G Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend. Sem. Mat. Univ. Padova*, **24** (1955), 84-92.
17. S. Hu, M. Khavani and W. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.*, **34** (1989), 261-266.

18. D. O'Regan and M. Meehan, *Existence theory for nonlinear integral and integrodifferential equations*, Kluwer Academic, Dordrecht, 1998.

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