TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1603-1614, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

BOUNDEDNESS OF STABLE DOMAINS OF TRANSCENDENTAL FUNCTIONS

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Abstract. Boundedness of components of the Fatou sets of iteration of transcendental entire or meromorphic functions are investigated in this paper.

1. Introduction and Main Results

For an integer $m \geq 1$, Σ_m denotes the one-sided word space, i.e.

$$\Sigma_m = \prod_1^{\infty} \{1, 2, \cdots, m\} = \{1, 2, \cdots, m\} \times \{1, 2, \cdots, m\} \times \cdots .$$

Let $f_j (j = 1, 2, \cdots, m)$ be a transcendental meromorphic function in $\mathbb{C}$. For

$$w = (w_1, w_2, \cdots, w_n, \cdots) \in \Sigma_m,$$

$f_{w_n} \circ \cdots \circ f_{w_1}$ is defined in $\mathbb{C}$ except for at most a countably infinite set:

$$\cup_{j=1}^{n-1} \{z \in \mathbb{C} : f_{w_j} \circ \cdots \circ f_{w_1}(z) = \infty\},$$

where $f_{w_j} \in \{f_1, f_2, \cdots, f_m\}, j = 1, 2, \cdots$. The Fatou set F_w on w is defined by

$$F_w = \{z \in \mathbb{C} : \{f_{w_n} \circ \cdots \circ f_{w_1}(z)\}_n \text{ is defined and normal in a neighborhood of } z\}.$$

The Julia set on w $J_w = \overline{\mathbb{C}} \backslash F_w$. J_w is closed and perfect, F_w is open. Let U be a component of F_w. For any $n \geq 1$, there is a component U_n of F_w such that $f_{w_n} \circ \cdots \circ f_{w_1}(U) \subset U_n$. U is said to be wandering if for any $n \neq k$,

$$f_{w_n} \circ \cdots \circ f_{w_1}(U) \cap f_{w_k} \circ \cdots \circ f_{w_1}(U) = \emptyset.$$

Received June 25, 2004; revised August 31, 2005.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: 30D05, 37F50.
Key words and phrases: Entire function; Fatou set; Julia set; Meromorphic function; Random iteration.

If

$$f = f_{w_1} = f_{w_2} = \cdots = f_{w_n} = \cdots ,$$

the component U of the Fatou set $F(f)$ is called pre-periodic domain if there exist $n > k \geq 0$ such that $U_n = U_k$, where U_n and U_k are the components of the Fatou set $F(f)$ and $f^n(U) \subseteq U_n$, $f^k(U) \subseteq U_k$. U is called invariant under f if $f(U) \subset U$. U is called completely invariant under f, $z \in U$ if and only if $f(z) \in U$. For more details, we refer to [6]. For $r > 0$, we define

$$L(r, f) = \min_{|z|=r}\{|f(z)|\}.$$

If $f(z)$ is entire, we define

$$M(r, f) = \max_{|z|=r}\{|f(z)|\}.$$

The first result is stated below.

Theorem 1. *Let $f_j(j = 1, 2, \cdots, m)$ be transcendental entire functions with the properties: for some constant $d > 1$ and all sufficiently large $r > 0$, there is $r_j \in (r, r^d)$ such that*

$$L(r_j, f_j) > M(r, f_j)^d, j = 1, 2, \cdots, m.$$

Then for $w = (w_1, w_2, \cdots) \in \Sigma_m$, all components of F_w are bounded.

Remark. It was proved in [19] that if set $g(z) = f_m \circ \cdots \circ f_1(z)$, $1 \leq m < \infty$, then the Fatou set $F(g)$ has no unbounded components, this is a special case of Theorem 1. There are some research on the bounded components of the Fatou set. Let $f(z)$ be a transcendental entire function in $\mathbb{C}$. There is a problem based on [4]:

Problem. Does $F(f)$ have only bounded components if the growth order of $f(z)$ is less than $\frac{1}{2}$?

There are some papers on this problem, see [2, 4, 11, 13, 16-18]. But, the problem still remains open. Wang [16] proved that the answer to this problem is affirmative if the growth order and lower order of $f(z)$ both lie in $(0, \frac{1}{2})$. Zheng and Wang [19] extended Wang's result to the case of the composition of finitely many entire functions under the same conditions. Here, we give a generalization of the result below.

Corollary 1. *Let $f_j(z)(j = 1, 2, \cdots, m)$ be entire functions of growth order and lower order lie in $(0, \frac{1}{2})$. Then for $w = (w_1, w_2, \cdots) \in \Sigma_m$, all components of F_w are bounded.*

By using Theorem in [5], Corollary 1 immediately follows from the following result.

Corollary 2. *Let $f_j(z)(j = 1, 2, \cdots, m)$ be entire functions of finite order and $\sup\{\rho(f_i), i = 1, \cdots, m\} < \infty$, where $\rho(f_i)$ is the order of f_i. Suppose that for some $\alpha \in (0, 1)$,*

$$L(r, f_j) > M(r, f_j)^\alpha, r \in E_j,$$

where E_j is a set of values r with nonzero lower logarithmic density, and for some $\epsilon > 0$,

$$D_j = \{r : \log M(r, f_j) > r^\epsilon\}$$

has positive lower logarithmic density, $j = 1, 2, \cdots, m$. Then for $w = (w_1, w_2, \cdots) \in \Sigma_m$, all components of F_w are bounded.

Corollary 1 and Corollary 2 were proved by Zheng and Wang [19] for the case $g(z) = f_m \circ \cdots \circ f_1(z)$, $1 \leq m < \infty$. See also [15].

Let $f_1, f_2. \cdots, f_m$ be meromorphic in $\mathbb{C}$ and $G =< f_1, f_2, \cdots, f_m >$ the semigroup generated by the generators $f_1, f_2, \cdots, f_m$, where the semigroup operation is the composition of the functions. The Fatou set $F(G)$ of G is defined by

$$F(G) = \{z \in \mathbb{C} : G \text{ is defined and normal in a neighborhood of } z\}.$$

The Julia set $J(G)$ is the complement of $F(G)$ in $\overline{\mathbb{C}}$. Obviously

$$J(G) = \overline{\cup_{w \in \Sigma_m} J_w}.$$

A component U of $F(G)$ is said to be wandering if for any $w = (w_1, \cdots, w_i, \cdots, w_j, \cdots) \in \Sigma_m$, $f_{w_i} \circ \cdots \circ f_{w_1}(U) \cap f_{w_j} \circ \cdots \circ f_{w_1}(U) = \emptyset$, $i \neq j$. There are no complete classification for the components of $F(G)$ yet. It may be interesting to find a way to classify the components of $F(G)$. In this paper, we studied the bounded wandering components of $F(G)$ for some semigroups G.

For the case of meromorphic functions with poles, we have the following.

Theorem 2. *Let $f_j(z)(j = 1, 2, \cdots, m)$ be transcendental meromorphic in $\mathbb{C}$ and have the properties: for some $d > 1$, for any positive number $\rho > 0$ and all sufficiently large r, there exist $r_j \in (r, r^d)$ such that*

$$\log^+ L(r_j, f_j) > \rho \log r, j = 1, 2, \cdots, m.$$

If U is a wandering component of $F(G)$ and there exists a point $z_0 \in U$ such that

$$\log^+ \log^+ |f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| = O(n), n \to \infty \tag{1}$$

for some $w = (w_1, \cdots, w_n, \cdots) \in \Sigma_w$, then U is bounded.

Theorem 2 is a generalization of those results in [15, 17, 19]. A transcendental meromorphic function $f(z)$ satisfies the first hypothese of Theorem 2, if the order $\sigma(f) < \frac{1}{2}$ and $\delta(\infty, f) > 1 - \cos \pi\sigma(f)$, where $\delta(\infty, f)$ is the Nevanlinna deficient number, see [8].

2. Preliminaries for the Proof of Theorem 1

Let $f_j(z)(j = 1, 2, \cdots, m)$ be transcendental and entire. For

$$w = (w_1, w_2, \cdots, w_n, \cdots) \in \Sigma_m,$$

a point z_0 is called a repelling fixed point of $f_{w_n} \circ \cdots \circ f_{w_1}(z)$ with order n if

$$\begin{aligned} &f_{w_n} \circ \cdots \circ f_{w_1}(z_0) = z_0, \\ &f_{w_k} \circ \cdots \circ f_{w_1}(z_0) \neq z_0, k = 1, 2, \cdots, n-1, \\ &|(f_{w_n} \circ \cdots \circ f_{w_1}(z_0))'| > 1. \end{aligned}$$

By using Schwick's method, see [12], we easily obtain the Lemma 1 bellow.

Lemma 1. *Let $f_j(z)(j = 1, 2, \cdots, m)$ be transcendental and entire. Then for*

$$w = (w_1, w_2, \cdots, w_n, \cdots) \in \Sigma_m,$$

all repelling fixed points of $f_{w_n} \circ \cdots \circ f_{w_1}(z)$ are dense in J_w, $n = 1, 2, \cdots$.

Lemma 2. *Under the hypotheses of Lemma 1 and let U be a multiply connected component of F_w. Then*

(1) *$f_{w_n} \circ \cdots \circ f_{w_1}(z) \to \infty$ uniformly locally on U as $n \to \infty$;*
(2) *$f_{w_n} \circ \cdots \circ f_{w_1}(\gamma)$ winds the original point 0 at least once as n is sufficiently large, where γ is an un-contractible Jordan curve in U.*

Proof.

(1) If any uniformly locally convergent subsequence $\{f_{w_n} \circ \cdots \circ f_{w_1}(z)\}_n$ has a regularly finite limit in U, let γ be a Jordan curve in U and not contractible, then $\{f_{w_n} \circ \cdots \circ f_{w_1}\}_n$ is normal in the interior of γ. This is impossible. Because $J_w \cap int(\gamma) \neq \emptyset$, where $int(\gamma)$ denotes the interior of γ.
(2) Assume that for all sufficiently large n, $f_{w_n} \circ \cdots \circ f_{w_1}(\gamma)$ can not wind 0. Then $f_{w_n} \circ \cdots \circ f_{w_1}(z)$ have no zeros in $int(\gamma)$ for all sufficiently large n. By the minimum principle and (1), $f_{w_n} \circ \cdots \circ f_{w_1}(z) \to \infty$ in $int(\gamma)$ as $n \to \infty$. This contradicts the fact that $J_w \cap int(\gamma) \neq \emptyset$.

Lemma 2 was proved for the case of a single entire function, see [7].

Lemma 3. *Under the hypotheses of Lemma 1. If F_w has an unbounded component U, then all other components of F_w are simple connected. Furthermore, if U is multiply connected, then U is completely invariant component under $f_{w_n} \circ \cdots \circ f_{w_1}(z)$, $n = 1, 2, \cdots$, i.e., for any integer n*

$$U = f_{w_n} \circ \cdots \circ f_{w_1}(U) = f_{w_1}^{-1} \circ \cdots \circ f_{w_n}^{-1}(U).$$

Proof. If there exists a multiply connected component V of F_w such that $V \cap U = \emptyset$, then by Lemma 2, for some sufficiently large n, $f_{w_n} \circ \cdots \circ f_{w_1}(V) \cap U \neq \emptyset$. This is impossible. So, V is simple connected.

If U is multiply connected, then for all n, $f_{w_n} \circ \cdots \circ f_{w_1}(U)$ and $f_{w_1}^{-1} \circ \cdots \circ f_{w_n}^{-1}(U)$ are a multiply connected component of F_w. Thus by the above argument, we have

$$U = f_{w_n} \circ \cdots \circ f_{w_1}(U) = f_{w_1}^{-1} \circ \cdots \circ f_{w_n}^{-1}(U).$$

Lemma 3 was proved by Töpler [14] for the case of a single entire function.

Lemma 4. ([4], Lemma 5) *In a domain D the analytic functions g of the family Ⓢ omit the value 0, 1. K is a compact connected subset of D on which the functions all satisfy $|g(z)| \geq 1$. Then there exist constants B, C depending only on K and D and such that for any z, z' in K and any $g \in$ Ⓢ we have $|g(z')| < B|g(z)|^C$.*

Theorem 3. *Under the hypotheses of Lemma 1, let U be an unbounded component of F_w. Then U is simply connected.*

Proof. Without loss of generality, we assume that $0, 1 \in J_w$ and $f_{w_1}(0) = 0$, by Lemma 1. Suppose that U is multiply connected by contradiction. Let γ be an un-contractible Jordan curve in U. Then $f_{w_n} \circ \cdots \circ f_{w_1}(z)|_\gamma \to \infty$ as $n \to \infty$ and $f_{w_n} \circ \cdots \circ f_{w_1}(\gamma)$ winds 0 at least once when n is sufficiently large by Lemma 2. Take a sufficiently large k such that $f_{w_k} \circ \cdots \circ f_{w_1}(\gamma)$ winds 0 and

$$M(\frac{1}{4}r, f_{w_1}) > r, r > r_0,$$

where r_0 is the minimum distance between $f_{w_k} \circ \cdots \circ f_{w_1}(\gamma)$ and 0. Take a sufficiently large p such that $f_{w_p} \circ \cdots \circ f_{w_1}(\gamma)$ winds 0 and $t(> r)$ is the minimum distance between $f_{w_p} \circ \cdots \circ f_{w_1}(\gamma)$ and 0. By Lemma 3, $f_{w_k} \circ \cdots \circ f_{w_1}(\gamma) \subset U$ and $f_{w_p} \circ \cdots \circ f_{w_1}(\gamma) \subset U$. Take a Jordan arc γ' in U connecting $f_{w_k} \circ \cdots \circ f_{w_1}(\gamma)$ and $f_{w_p} \circ \cdots \circ f_{w_1}(\gamma)$. Set

$$\Gamma = \gamma' \cup f_{w_k} \circ \cdots \circ f_{w_1}(\gamma) \cup f_{w_p} \circ \cdots \circ f_{w_1}(\gamma).$$

Γ is a compact subset of U. Since $f_{w_1}(U) \subset U$ by Lemma 3, we may assume that $|(f^n_{w_1}(z))| > 1$ on Γ and $f^n_{w_1}(z) \neq 0, 1$ in U, $n = 0, 1, 2, \cdots$. By Lemma 4, there exist constants B, C which are only dependent on Γ and U such that

$$|f^n_{w_1}(z_2)| < B|f^n_{w_1}(z_1)|^C, z_1, z_2 \in \Gamma,$$

$n = 1, 2, \cdots$. By the same method of Baker's in [3], using a result of Pólya [10], we obtain that

$$|f^n_{w_1}(z_2)| > B|f^n_{w_1}(z_1)|^C$$

for all sufficiently large n. This is impossible. Theorem 3 follows.

Following Zheng and Wang [19], we shall prove the main results below.

3. Proofs of Main Results

In order to prove the Theorems, we need to recall some properties on hyperbolic domains, see [1,9]. Let W and Y be hyperbolic domains. For any $z_1, z_2 \in W$, $\rho_W(z_1, z_2)$ denotes the hyperbolic distance between z_1 and z_2 on W, i.e.

$$\rho_W(z_1, z_2) = \inf_{\gamma \in W} \int_\gamma \lambda_W(z)|dz|, \tag{2}$$

where γ denote all Jordan curves connecting z_1 to z_2 in W, $\lambda_W(z)$ is hyperbolic metric of the domain W. Let $f : W \to Y$ be analytic. Then

$$\rho_Y(f(z_1), f(z_2)) \leq \rho_W(z_1, z_2), \qquad z_1, z_2 \in W. \tag{3}$$

Proof of Theorem 1

By contradiction, assume that U is an unbounded component of F_w. By Theorem 3, U is simply connected.

Fixed a point $z_0 \in U$. For a sufficiently large $r > |z_0|$, there exists $r_1 \in (r, r^d)$ such that

$$|f_{w_1}(z)| \geq L(r_1, f_{w_1}) > M(r, f_{w_1})^d > |f_{w_1}(z_0)|^d, |z| = r_1.$$

Take a Jordan arc γ joining z_0 to a point of $\{z : |z| = r_1\}$ such that $\gamma \subset U \cap \{z : |z| \leq r_1\}$. Put $\tilde{r}_1 = M(r, f_{w_1})$. Then

$$f_{w_1}(\gamma) \cap \{z : |z| = \tilde{r}_1^d\} \neq \emptyset,$$
$$f_{w_1}(\gamma) \cap \{z : |z| = \tilde{r}_1\} \neq \emptyset.$$

There exists $r_2 \in (\tilde{r}_1, \tilde{r}_1^d)$ such that

$$\begin{aligned}|f_{w_2}(z)| \geq L(r_2, f_{w_2}) &\geq M(\tilde{r}_1, f_{w_2})^d = M(M(r, f_{w_1}), f_{w_2})^d \\ &\geq M(r, f_{w_2} \circ f_{w_1})^d > |f_{w_2} \circ f_{w_1}(z_0)|^d, |z| = r_2.\end{aligned}$$

Put $\tilde{r}_2 = M(\tilde{r}_1, f_{w_2})$. Then

$$f_{w_2} \circ f_{w_1}(\gamma) \cap \{z : |z| = \tilde{r}_2^d\} \neq \emptyset,$$

$$f_{w_2} \circ f_{w_1}(\gamma) \cap \{z : |z| = \tilde{r}_2\} \neq \emptyset.$$

So, there is a point $z_2 \in \gamma$ satisfying

$$|f_{w_2} \circ f_{w_1}(z_2)| \geq M(r, f_{w_2} \circ f_{w_1})^d > |f_{w_2} \circ f_{w_1}(z_0)|^d.$$

By the Mathematical Induction, for all sufficiently large n, there is a point $z_n \in \gamma$ satisfying

$$|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)| \geq M(r, f_{w_n} \circ \cdots \circ f_{w_1})^d > |f_{w_n} \circ \cdots \circ f_{w_1}(z_0)|^d.$$

Clearly, $f_{w_n} \circ \cdots \circ f_{w_1}(z_n) \to \infty$ as $n \to \infty$, $z_n \in \gamma$.

Note that $f_{w_n} \circ \cdots \circ f_{w_1}(U) \subset U_n$, where U_n is a simply connected and unbounded component of F_w. From [9], for any $a \in \partial U_n$,

$$\lambda_{U_n}(z) \geq \frac{1}{2d(z, \partial U_n)} \geq \frac{1}{2(|z| + |a|)},$$

where $d(z, \partial U_n)$ is the Euclidean distance from z to ∂U_n. Therefore

$$\begin{aligned}\rho_{U_n}(f_{w_n} \circ \cdots \circ f_{w_1}(z_0), f_{w_n} \circ \cdots \circ f_{w_1}(z_n)) &\geq \int_{|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)|}^{|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)|} \frac{|dz|}{2(|z| + |a|)} \\ &\geq \frac{1}{2} \log \frac{|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)| + |a|}{|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| + |a|}.\end{aligned}$$

Set $A = \max\{\rho_U(z_0, z) : z \in \gamma\}$. Obviously $A \in (0, \infty)$. By the Principle of Hyperbolic Metric,

$$\rho_{U_n}(f_{w_n} \circ \cdots \circ f_{w_1}(z_0), f_{w_n} \circ \cdots \circ f_{w_1}(z_n)) \leq \rho_U(z_0, z_n) \leq A.$$

Combining the above,

$$|f_{w_n} \circ \cdots f_{w_1}(z_0)|^d < |f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| + |a| \leq (|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| + |a|)e^{2A}.$$

This is impossible as $n \to \infty$, since $d > 1$ and $f_{w_n} \circ \cdots \circ f_{w_1}(z_0) \to \infty$ as $n \to \infty$. Theorem 1 follows.

In order to prove Corollary 2, we need to prove the following lemma, which is from [15].

Lemma 5. *Let E be a set of values r of positive lower logarithmic density. Then there exists a positive number $t > 1$, for all sufficiently large r, the linear measure of $(r, r^t) \cap E$ is positive, i.e.*

$$mes((r, r^t) \cap E) > 0.$$

Proof. Assume that the conclusion is invalid by contradiction. Then for any positive number $t > 1$, there exists an unbounded set of r, say E_t such that for all $r \in E_t$, $mes([r, r^t] \cap E) = 0$. Choose t satisfying

$$\frac{1}{t} < \underline{\log dense} E,$$

where

$$\underline{\log dense} E = \varliminf_{r\to\infty} \frac{1}{\log r} \int_{(1,r)\cap E} \frac{dt}{t}.$$

We can select an unbounded series $\{r_n\}_{n=1}^{\infty} \subset E_t$ such that

$$mes([r_n, r_n^t] \cap E) = 0, n = 1, 2, \cdots.$$

Since

$$\begin{aligned} \frac{1}{\log r_n^t} \int_{E\cap(1,r_n^t)} \frac{dt}{t} &\le \frac{1}{t\log r_n} \int_1^{r_n} \frac{dt}{t} + \frac{1}{t\log r_n} \int_{(r_n,r_n^t)\cap E} \frac{dt}{t} \\ &= \frac{1}{t} + \frac{1}{t\log r_n} mes((r_n, r_n^t) \cap E) = \frac{1}{t}. \end{aligned}$$

We deduce a contradiction

$$\underline{\log dense} E \le \lim_{n\to\infty} \frac{1}{\log r_n^t} \int_{E\cap(1,r_n^t)} \frac{dt}{t} \le \frac{1}{t}.$$

Proof of Corollary 2

Choose a sufficiently large $d > 1$ such that $\alpha d \ge 1$ and $\epsilon d > \sigma$, where $\sigma = \sup\{\rho(f_i), i = 1, \cdots, m\}$. For all sufficiently large r, by Lemma 5, there exist $r_j' \in (r^d, r^{d^2}) \cap E_j$ and $r_j \in (r, r^d)$ satisfying $r_j^d = r_j'$. Take ζ satisfying $0 < \zeta < \epsilon d - \sigma$. So

$$\log M(r_j^d, f_j) > r_j^{\epsilon d} > r_j^{\zeta} \log M(r_j, f_j) > d^4 \log M(r, f_j).$$

By Lemma 5, there exists $\tilde{r}_j \in (r^{d^2}, r^{d^3})$ such that

$$\begin{aligned}\log L(\tilde{r}_j, f_j) &> \alpha \log M(\tilde{r}_j, f_j)\\ &> \alpha \log M(r_j^d, f_j)\\ &> \alpha d^4 \log M(r, f_j)\\ &> d^3 \log M(r, f_j), j = 1, 2, \cdots, m.\end{aligned}$$

Now, the condition of Theorem 1 are satisfied. Corollary 2 follows.

Proof of Theorem 2.

Assume that U is unbounded. Then there is a wandering component V of F_w such that $U \subset V$. So V is unbounded. From (1), there exists $M > 1$ such that

$$\log |f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| < M^n, n = 1, 2, \cdots . \tag{4}$$

Take a positive number ρ satisfying $K = \frac{\rho}{d} > 2M$ and $R_0 > \max\{e^M, |z_0|\}$. For all sufficiently large $r \geq R_0$, there is $r_j \in (r, r^d)$ such that

$$\log L(r_j, f_j) > \rho \log r, j = 1, 2, \cdots, m. \tag{5}$$

Since V is unbounded, we shall derive a contradiction from (5). Make a Jordan curve γ connecting z_0 to a point of $\{z : |z| = R_0^d\}$ in $V \cap \{z : |z| \leq R_0^d\}$. Then from (4) and (5), there is $r_0 \in (R_0, R_0^d)$ such that

$$\begin{aligned}\log |f_{w_1}(z)| &\geq \log L(r_0, f_{w_1}) > \rho \log R_0\\ &= dK \log R_0 > 2M\\ &> \log |f_{w_1}(z_0)|, |z| = r_0.\end{aligned}$$

Set $R_1 = R_0^K$. Then

$$R_1 > e^M > |f_{w_1}(z_0)|.$$

So

$$f_{w_1}(\gamma) \cap \{z : |z| = R_1\} \neq \emptyset,$$

and

$$f_{w_1}(\gamma) \cap \{z : |z| = R_1^d\} \neq \emptyset.$$

Furthermore, there is $r_1 \in (R_1, R_1^d)$ such that

$$\log |f_{w_2}(z)| \geq \log L(r_1, f_{w_2}) > \rho \log R_1, |z| = r_1.$$

And then there exists a point $z_1 \in \gamma$, such that $|f_{w_1}(z_1)| = r_1$, namely

$$\begin{aligned}\log|f_{w_2} \circ f_{w_1}(z_1)| &> \rho \log R_1 = \rho K \log R_0 \\ &> 2M^2 > \log|f_{w_2} \circ f_{w_1}(z_0)|.\end{aligned}$$

By induction, similarly we can find a point $z_n \in \gamma$ satisfying

$$(6) \quad \log|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)| > 2M^n > \log|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)|, n = 1, 2, \cdots .$$

Since V is an unbounded wandering component, J_w has an unbounded component, say Γ. Then

$$f_{w_n} \circ \cdots \circ f_{w_1} : V \to \mathbb{C}\backslash\Gamma, n = 1, 2, \cdots ,$$

is analytic. Write

$$A = \max\{\rho_V(z_0, z) : z \in \gamma\},$$

we have $A < \infty$. From (2) and (3), we obtain

$$\rho_{\mathbb{C}\backslash\Gamma}(f_{w_n} \circ \cdots \circ f_{w_1}(z_0), f_{w_n} \circ \cdots \circ f_{w_1}(z_n)) \le \rho_V(z_0, z_n) \le A.$$

It is well known that

$$\lambda_{\mathbb{C}\backslash\Gamma}(z) d_{\mathbb{C}\backslash\Gamma}(z) \ge \frac{1}{4}, \forall z \in \mathbb{C}\backslash\Gamma,$$

where $d_{\mathbb{C}\backslash\Gamma}(z)$ is the Euclidean distance from z to Γ. Take a point $a \in \Gamma$, and have

$$\begin{aligned}\lambda_{\mathbb{C}\backslash\Gamma}(z) &\ge \frac{1}{4d_{\mathbb{C}\backslash\Gamma}(z)} \\ &\ge \frac{1}{4}\frac{1}{|z| + |a|}.\end{aligned}$$

And then, we have

$$\begin{aligned}A &\ge \rho_{\mathbb{C}\backslash\Gamma}(f_{w_n} \circ \cdots \circ f_{w_1}(z_0), f_{w_n} \circ \cdots \circ f_{w_1}(z_n)) \\ &\ge \int_{|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)|}^{|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)|} \frac{1}{4}\frac{1}{|z| + |a|}|dz| \\ &= \frac{1}{4} \log \frac{|f_{w_n} \circ \cdots \circ f_{w_1}(z_n)| + |a|}{|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| + |a|}.\end{aligned}$$

From (6), we have

$$2M^n < \log(|f_{w_n} \circ \cdots \circ f_{w_1}(z_0)| + |a|) + 4A.$$

From (4), when $n \to \infty$, the above inequality can not occur. This is a contradiction. The contradiction means V is bounded. The proof is completed.

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