TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1557-1573, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

EXISTENCE THEOREMS OF POSITIVE SOLUTIONS FOR A FOURTH-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

De-xiang Ma, Yu Tian and Wei-gao Ge

Abstract. In this paper, the following fourth-order three-point boundary value problem with p-Laplacian operator is studied:

$$\begin{cases} (\phi_p(u''(t)))'' = a(t)f(u(t)), \quad t \in (0,1), \\ u(0) = \xi u(1),\ u'(1) = \eta u'(0), \\ u''(0) = \alpha_1 u''(\delta),\ u''(1) = \beta_1 u''(\delta), \end{cases}$$

where $\alpha_1,\ \beta_1 \geq 0$, $\xi \neq 1$, $\eta \neq 1$, $0 < \delta < 1$ and $\phi_p(z) = |z|^{p-2}z$ for $p > 1$. We impose growth conditions on f which guarantee the existence of at least three positive solutions for the problem.

1. Introduction

In the last ten years, a great deal of work has been done to study the positive solutions of two point boundary value problems for differential equations which are used to describe a number of physical, biological and chemical phenomena. For additional background and results, we refer the reader to the monograph by Agarwal, O'Regan and Wong [1] as well as the recent contributions by [2-8].

Boundary value problems for even order differential equations can arise, especially for fourth-order equations. Recently, three-point or multiple-point boundary value problems of the differential equations were presented and studied by many authors, see [9-10].

Received April 8, 2004; accepted May 4, 2005.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: 34B10, 34B15.
Key words and phrases: Boundary value problem, Multiple positive solutions, Cone, Avery-Henderson fixed point theorem (generalized Leggett-Williams fixed Theorem).
Supported by National Natural Sciences Foundation of China(10371006)

In this paper, we are concerned with the existence of three positive solutions for the fourth-order three-point boundary value problem (BVP for short) consisted of the p-Laplacian differential equation

$$(\phi_p(u''(t)))'' - a(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1}$$

and the following boundary value conditions

$$u(0) = \xi u(1),\ u'(1) = \eta u'(0),\ u''(0) = \alpha_1 u''(\delta),\ u''(1) = \beta_1 u''(\delta), \tag{2}$$

where $f : \ R \to [0, +\infty)$ and $a : \ (0,1) \to [0, +\infty)$ are continuous functions, $\alpha_1,\ \beta_1 \geq 0$, $\xi \neq 1$, $\eta \neq 1$, $0 < \delta < 1$ and $\phi_p(z) = |z|^{p-2}z$ for $p > 1$.

When $p = 2$, (1) becomes $u^{(4)}(t) - a(t)f(u(t)) = 0, \quad t \in (0,1)$.

The fourth-order three-point boundary value problem $(1)-(2)$ has not received as much attention in the literature as lidstone condition boundary value problem:

$$\begin{cases} u^{(4)}(t) = a(t)f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{3}$$

and as the three-point boundary value problem for the second-order differential equation

$$\begin{cases} u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \alpha u(\eta) \end{cases} \tag{4}$$

that were extensively considered in [2-5] and [9-10], respectively. The results of existence of positive solutions of BVP (1)-(2) are relatively scarce. Recently, there is an increasing interest in obtaining twin or three positive solutions for two-point boundary value problems by using multiple fixed points theorems on cones. The purpose of this paper is to establish the existence of at least three positive solutions of (1)-(2). Our arguments involve the use of the concavity and integral representation of solutions and a fixed point theorem (Theorem 2.1) which is a nice generalization of the well-known Leggett-Williams fixed point Theorem. We will impose growth conditions on f which ensure the existence of at least three positive solutions of (1)-(2).

For the remainder of the paper, we assume that

(i) $0 < \int_0^1 a(s)ds < +\infty$.

(ii) q satisfies $\frac{1}{p} + \frac{1}{q} = 1$ and $(\phi_p)^{-1}(z) = \phi_q(z) = |z|^{q-2}z$.

2. PRELIMINARY

In this section, we present two definitions in Banach space, an appreciate generalized form of Leggett-Williams fixed point theorem by Avery and Henderson [7] and four lemmas.

Definition 2.1. Let X be a real Banach space and P be a cone of X. A map $\psi : P \to [0, +\infty)$ is called nonnegative continuous concave functional map if ψ is nonnegative, continuous and satisfies $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.2. Let X be a real Banach space and P be a cone of X. A map $\beta : P \to [0, +\infty)$ is called nonnegative continuous convex functional map if β is nonnegative, continuous and satisfies $\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Let γ, β and θ be nonnegative continuous convex functionals on P, and let α and ψ be nonnegative continuous concave functionals on P. For nonnegative numbers h, a, b, d and c, we define the following sets:

$$P(\gamma, c) = \{x \in P : \ \gamma(x) < c\},$$

$$P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x), \ \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, d, c) = \{x \in P : \ \beta(x) \leq d, \ \gamma(x) \leq c\},$$

$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : \ a \leq \alpha(x), \ \theta(x) \leq b, \ \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : \ h \leq \psi(x), \ \beta(x) \leq d, \ \gamma(x) \leq c\}.$$

To obtain multiple positive solutions of BVP $(1)-(2)$, the following fixed point theorem in [7] is needed.

Theorem 2.1. [7] *Let X be a real Banach space and P be a cone of X. Suppose that γ, β and θ are three nonnegative continuous convex functionals on P and α, ψ are two nonnegative continuous concave functionals on P such that for some positive numbers c and M,*

$$\alpha(x) \leq \beta(x), \quad ||x|| \leq M\gamma(x) \quad \textit{for} \quad x \in \overline{P(\gamma, c)}.$$

Suppose further that $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist $h, d, a, b \geq 0$ with $0 < d < a$ such that each of the following is satisfied:

(*i*) $\{x \in P(\gamma, \theta, \alpha, a, b, c) : \ \alpha(x) > a\} \neq \emptyset$ *and* $x \in P(\gamma, \theta, \alpha, a, b, c)$ *implies* $\alpha(Tx) > a$,

(ii) $\{x \in Q(\gamma, \beta, \psi, h, d, c) : \ \beta(x) < d\} \neq \emptyset$ *and* $x \in Q(\gamma, \beta, \psi, h, d, c)$ *implies* $\beta(Tx) < d$,

(iii) $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Tx) > b$ *implies* $\alpha(Tx) > a$,

(iv) $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Tx) < h$ *implies* $\beta(Tx) < d$.
Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *such that*

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad d < \beta(x_3), \quad \text{with} \quad \alpha(x_3) < a.$$

Lemma 2.1. *If* $f \in C(R, R)$, $M = 1 - \phi_p(\alpha_1) - [\phi_p(\beta_1) - \phi_p(\alpha_1)]\delta \neq 0$. *Then the unique solution of the following second-order three-point boundary value problem*

$$\begin{cases} -y'' = f(t), \quad t \in (0,1), \\ y(0) = \phi_p(\alpha_1) y(\delta), \quad y(1) = \phi_p(\beta_1) y(\delta) \end{cases} \tag{5}$$

is

$$y(t) = \frac{1}{M} \int_0^1 g(t,s) a(s) ds, \quad t \in (0,1),$$

where

$$g(t,s) = \begin{cases} s(1-t) + \phi_p(\beta_1) s(t-\delta), & 0 \leq s \leq t < \delta < 1 \text{ or} \\ & 0 \leq s \leq \delta \leq t \leq 1, \\ t(1-s) + \phi_p(\beta_1) t(s-\delta) + \phi_p(\alpha_1)(1-\delta)(s-t), & 0 \leq t \leq s \leq \delta < 1, \\ s(1-t) + \phi_p(\beta_1)\delta(t-s) + \phi_p(\alpha_1)(1-t)(\delta - s), & 0 \leq \delta \leq s \leq t \leq 1, \\ (1-s)(t - \phi_p(\alpha_1) t + \phi_p(\alpha_1)\delta), & 0 < \delta \leq t \leq s \leq 1 \text{ or} \\ & 0 \leq t < \delta \leq s \leq 1. \end{cases}$$

Proof. In fact, if $y(t)$ is a solution of (5), then we suppose that

$$y(t) = -\int_0^t (t-s) f(s) ds + At + B, \quad t \in (0,1).$$

By the boundary conditions of (5), it follows that

$$B = -\phi_p(\alpha_1) \int_0^\delta (\delta - s) f(s) ds + \phi_p(\alpha_1)\delta A + \phi_p(\alpha_1) B$$

and

$$-\int_0^1 (1-s)f(s)ds + A + B = -\phi_p(\beta_1)\int_0^\delta (\delta-s)f(s)ds + \phi_p(\beta_1)\delta A + \phi_p(\beta_1)B.$$

Hence,

$$\begin{aligned} y(t) = & -\int_0^t (t-s)f(s)ds + \frac{[1-\phi_p(\alpha_1)]t}{M}\int_0^1 (1-s)f(s)ds \\ & -\frac{[\phi_p(\beta_1)-\phi_p(\alpha_1)]t}{M}\int_0^\delta (\delta-s)f(s)ds \\ & +\frac{\phi_p(\alpha_1)\delta}{M}\int_0^1 (1-s)f(s)ds - \frac{\phi_p(\alpha_1)}{M}\int_0^\delta (\delta-s)f(s)ds \\ = & \frac{1}{M}\int_0^1 g(t,s)f(s)ds. \end{aligned}$$

■

We may verify that $g(t,s) \geq 0$ for $(t,s) \in [0,1]\times[0,1]$ if $M > 0$.

Lemma 2.2. *If* $f \in C(R,R)$, $M_1 = (1-\xi)(1-\eta) \neq 0$. *Then the unique solution of the following second-order boundary value problem*

$$\begin{cases} -y'' = f(t), & t \in (0,1) \\ u(0) = \xi y(1), u'(1) = \eta y'(0) \end{cases} \tag{6}$$

is

$$y(t) = \frac{1}{M_1}\int_0^1 h(t,s)f(s)ds, \quad t \in [0,1],$$

where

$$h(t,s) = \begin{cases} s + \eta(t-s) + \xi\eta(1-t), & 0 \leq s \leq t \leq 1, \\ t + \xi(s-t) + \xi\eta(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. In fact, if $y(t)$ is a solution of (6), then we suppose that

$$y(t) = -\int_0^t (t-s)f(s)ds + At + B, \quad t \in [0,1].$$

By the boundary conditions (6), we get

$$B = \xi\left[B + A - \int_0^1 (1-s)f(s)ds\right]$$

and

$$A - \int_0^1 f(s)ds = \eta A.$$

Hence,

$$\begin{aligned} y(t) = &- \int_0^t (t-s)f(s)ds + t\frac{\int_0^1 f(s)ds}{1-\eta} \\ &+ \frac{\xi}{1-\xi}\left[\frac{\int_0^1 f(s)ds}{1-\eta} - \int_0^1 (1-s)f(s)ds\right] \\ = &\frac{1}{M_1}\int_0^1 h(t,s)f(s)ds. \end{aligned}$$

■

Obviously, if $\xi, \eta \geq 0$, then $h(t,s) \geq 0$.

Suppose that $u(t)$ is solution of problem (1)-(2). By Lemma 2.1 and (5),

$$u''(t) = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,s)a(s)f(u(s))ds\right). \tag{7}$$

By Lemma 2.2 and (6),

$$u(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\right)ds.$$

Lemma 2.3. *Suppose that* $0 \leq \xi, \eta < 1$, $0 < t_1 < t_2 < 1$ *and* $\delta \in (0,1)$. *If* $s \in [0,1]$, *then*

$$\frac{h(t_1,s)}{h(t_2,s)} \geq \frac{t_1}{t_2}, \tag{8}$$

and

$$\frac{h(1,s)}{h(\delta,s)} \leq \frac{1}{\delta}. \tag{9}$$

Proof. Let $s \in [0,1]$. Firstly, we prove (8).

If $s \leq t_1 < t_2$, then

$$\begin{aligned} &\frac{h(t_1,s)}{h(t_2,s)} = \frac{s+\eta(t_1-s)+\xi\eta(1-t_1)}{s+\eta(t_2-s)+\xi\eta(1-t_2)} = \frac{s(1-\eta)+\xi\eta+\eta t_1(1-\xi)}{s(1-\eta)+\xi\eta+\eta t_2(1-\xi)} \\ &\geq \frac{\eta t_1(1-\xi)}{\eta t_2(1-\xi)} = \frac{t_1}{t_2}. \end{aligned}$$

If $t_1 < t_2 \le s$, then

$$\frac{h(t_1,s)}{h(t_2,s)} = \frac{t_1+\xi(s-t_1)+\xi\eta(1-s)}{t_2+\xi(s-t_2)+\xi\eta(1-s)} \ge \frac{t_1+\xi(s-t_1)}{t_2+\xi(s-t_2)} \ge \frac{t_1}{t_2}.$$

If $t_1 < s < t_2$, then

$$\frac{h(t_1,s)}{h(t_2,s)} = \frac{t_1+\xi(s-t_1)+\xi\eta(1-s)}{s+\eta(t_2-s)+\xi\eta(1-t_2)}.$$

Since $[\xi(s-t_1)+\xi\eta(1-s)]-[\xi\eta(1-t_2)] = \xi(s-t_1)+\xi\eta(t_2-s) \ge 0$ and $\frac{t_1}{s+\eta(t_2-s)} - \frac{t_1}{t_2} = \frac{t_1(t_2-s)(1-\eta)}{t_2[s+\eta(t_2-s)]} \ge 0$, it follows that

$$\frac{h(t_1,s)}{h(t_2,s)} \ge \frac{t_1+\xi\eta(1-t_2)}{s+\eta(t_2-s)+\xi\eta(1-t_2)} \ge \frac{t_1}{s+\eta(t_2-s)} \ge \frac{t_1}{t_2}.$$

Now, we prove (9).
If $\delta \le s$, then

$$\begin{aligned}\frac{h(1,s)}{h(\delta,s)} - \frac{1}{\delta} &= \frac{s+\eta(1-s)}{\delta+\xi(s-\delta)+\xi\eta(1-s)} - \frac{1}{\delta} \\ &\le \frac{s+\eta(1-s)}{\delta+\xi\eta(1-s)} - \frac{1}{\delta} = \frac{\eta(1-s)(\eta-1)-\xi\eta(1-s)}{\delta[\delta+\xi\eta(1-s)]} \le 0.\end{aligned}$$

If $\delta \ge s$, then

$$\begin{aligned}&\frac{h(1,s)}{h(\delta,s)} - \frac{1}{\delta} = \frac{s+\eta(1-s)}{s+\eta(\delta-s)+\xi\eta(1-\delta)} - \frac{1}{\delta} \\ &\le \frac{s+\eta(1-s)}{s+\eta(\delta-s)} - \frac{1}{\delta} = \frac{s(1-\delta)(\eta-1)}{\delta[s+\eta(\delta-s)]} \le 0.\end{aligned}$$

■

Lemma 2.4. *Suppose that* $\xi, \eta > 1$, $0 < t_1 < t_2 < 1$ *and* $\delta \in (0,1)$. *If* $s \in [0,1]$, *then*

$$\frac{h(t_2,s)}{h(t_1,s)} \ge \frac{1-t_2}{1-t_1}, \tag{10}$$

and

$$\frac{h(0,s)}{h(\delta,s)} \le \frac{1}{1-\delta}. \tag{11}$$

Proof. Let $s \in [0,1]$. Firstly, we prove (10).

If $s \leq t_1 < t_2$, then

$$\begin{aligned}\frac{h(t_2,s)}{h(t_1,s)} - \frac{1-t_2}{1-t_1} &= \frac{s+\eta(t_2-s)+\xi\eta(1-t_2)}{s+\eta(t_1-s)+\xi\eta(1-t_1)} - \frac{1-t_2}{1-t_1} \\ &\geq \frac{\eta(t_2-s)+\xi\eta(1-t_2)}{\eta(t_1-s)+\xi\eta(1-t_1)} - \frac{1-t_2}{1-t_1} \\ &= \frac{\eta(t_2-t_1)(1-s)}{(1-t_1)[\eta(t_1-s)+\xi\eta(1-t_1)]} > 0.\end{aligned}$$

If $t_1 < t_2 \leq s$, then

$$\begin{aligned}\frac{h(t_2,s)}{h(t_1,s)} - \frac{1-t_2}{1-t_1} &= \frac{t_2+\xi(s-t_2)+\xi\eta(1-s)}{t_1+\xi(s-t_1)+\xi\eta(1-s)} - \frac{1-t_2}{1-t_1} \\ &= \frac{(t_2-t_1)[1+\xi(1-s)(\eta-1)]}{(1-t_1)[t_1+\xi(s-t_1)+\xi\eta(1-s)]} > 0.\end{aligned}$$

If $t_1 < s < t_2$, then

$$\begin{aligned}\frac{h(t_2,s)}{h(t_1,s)} - \frac{1-t_2}{1-t_1} &= \frac{s+\eta(t_2-s)+\xi\eta(1-t_2)}{t_1+\xi(s-t_1)+\xi\eta(1-s)} - \frac{1-t_2}{1-t_1} \\ &\geq \frac{s+\xi\eta(1-t_2)}{t_1+\xi(s-t_1)+\xi\eta(1-s)} - \frac{1-t_2}{1-t_1} \\ &= \frac{(s-t_1)+t_1(t_2-s)+\xi(1-t_2)(s-t_1)(\eta-1)}{(1-t_1)[t_1+\xi(s-t_1)+\xi\eta(1-s)]} > 0.\end{aligned}$$

Now, we prove (11).
If $\delta \leq s$, then

$$\begin{aligned}\frac{h(0,s)}{h(\delta,s)} - \frac{1}{1-\delta} &= \frac{\xi s+\xi\eta(1-s)}{\delta+\xi(s-\delta)+\xi\eta(1-s)} - \frac{1}{1-\delta} \\ &\leq \frac{\xi s+\xi\eta(1-s)}{\delta+\xi\eta(1-s)} - \frac{1}{1-\delta} = \frac{\xi s(1-\delta)(1-\eta)-\delta}{(1-s)[\delta+\xi\eta(1-\delta)]} \leq 0.\end{aligned}$$

If $\delta \geq s$, then

$$\begin{aligned}\frac{h(0,s)}{h(\delta,s)} - \frac{1}{1-\delta} &= \frac{\xi s+\xi\eta(1-s)}{s+\eta(\delta-s)+\xi\eta(1-\delta)} - \frac{1}{1-\delta} \\ &\leq \frac{\xi s+\xi\eta(1-s)}{s+\xi\eta(1-\delta)} - \frac{1}{1-\delta} \\ &= \frac{s\xi(1-\delta)(1-\eta)-s}{(1-\delta)[s+\xi\eta(1-\delta)]} \leq 0.\end{aligned}$$

■

3. THREE POSITIVE SOLUTIONS OF (1)-(2)

Now, let the classical Banach space $X = C([0,1])$ be endowed with the norm $||x|| = \max_{0\le t\le 1}|x(t)|$. The cones $P_1, P_2 \subset X$ are defined as follows:

$$P_1 = \{u \in X : u(t) \text{ is nonnegative concave and nondecreasing on } (0,1)\},$$

$$P_2 = \{u \in X : u(t) \text{ is nonnegative concave and nonincreasing on } (0,1)\}.$$

Next, let $t_1, t_2, t_3 \in (0,1)$ with $t_1 < t_2$. Define nonnegative continuous concave functionals α, ψ and nonnegative convex functionals β, θ, γ on P_1 by

$$\begin{aligned}
&\gamma(x) = \max\nolimits_{t\in[0,t_3]} x(t) = x(t_3), \quad x \in P_1,\\
&\psi(x) = \min\nolimits_{t\in[\delta,1]} x(t) = x(\delta), \quad x \in P_1,\\
&\beta(x) = \max\nolimits_{t\in[\delta,1]} x(t) = x(1), \quad x \in P_1,\\
&\alpha(x) = \min\nolimits_{t\in[t_1,t_2]} x(t) = x(t_1), \quad x \in P_1,\\
&\theta(x) = \max\nolimits_{t\in[t_1,t_2]} x(t) = x(t_2), \quad x \in P_1.
\end{aligned}$$

It is easy to prove that $\alpha(x) = x(t_1) \le x(1) = \beta(x)$ and $||x|| = x(1) \le \frac{1}{t_3}x(t_3) = \frac{1}{t_3}\gamma(x)$ for $x \in P_1$.

Theorem 3.1. *Suppose that $0 \le \xi,\ \eta < 1$ and $M > 0$. There exist positive numbers $0 < a < b < c$ such that $0 < a < b < \frac{t_1}{t_2}b \le c$ and $f(w)$ satisfies the following conditions:*

$$f(w) < \phi_p\left(\frac{a}{C}\right), \quad 0 \le w \le a, \tag{12}$$

$$f(w) > \phi_p\left(\frac{b}{B}\right), \quad b \le w \le \frac{t_2}{t_1}b, \tag{13}$$

$$f(w) \le \phi_p\left(\frac{c}{A}\right), \quad 0 \le w \le \frac{1}{t_3}c, \tag{14}$$

where A, B and C are defined as follows:

$$\begin{aligned}
A &= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\left[\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)\right]ds,\\
B &= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_1,s)\left[\phi_q\left(\int_{t_1}^{t_2} g(s,r)a(r)dr\right)\right]ds,\\
C &= \frac{1}{M_1\phi_q(M)}\int_0^1 h(1,s)\left[\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)\right]ds.
\end{aligned}$$

Then BVP (1)-(2) has at least three positive solutions $x_1, x_2, x_3 \in \overline{P_1(\gamma, c)}$ *such that*

(15) $\quad x_1(t_1) > b,\ x_2(1) < a,\ x_3(t_1) < b,\ x_3(1) > a$ *and* $x_i(\delta) \le c$ *for* $i = 1, 2, 3.$

Proof. Define the completely continuous operator $T:\ P_1 \to X$ by

$$Tu(t) = \frac{1}{M_1\phi_q(M)} \int_0^1 h(t,s)\left[\phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)\right] ds.$$

It is easy to know that u is a positive solution of (1)-(2) if and only if u is a fixed point of T on cone P_1.

Firstly, we prove $T:\ \overline{P_1(\gamma, c)} \to \overline{P_1(\gamma, c)}$.

For $u \in P_1$, since $M > 0$ and $M_1 = (1-\xi)(1-\eta) > 0$, it follows that $Tu \ge 0$. Furthermore,

$$\begin{aligned}(Tu)'(t) &= \frac{1-\xi}{M_1\phi_q(M)}\left[\eta\int_0^t \phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds\right.\\ &\quad\left. + \int_t^1 \phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds\right] \ge 0,\end{aligned}$$

$$(Tu)''(t) = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,r)f(u(r))a(r)dr\right) \le 0.$$

So, $TP_1 \subset P_1$.

For $u \in \overline{P_1(\gamma, c)}$, $0 \le u(t) \le ||u|| \le \frac{1}{t_3}\gamma(u) \le \frac{1}{t_3}c$. By (14),

$$\begin{aligned}\gamma(Tu) &= \max_{t\in[0,t_3]} Tu(t) = Tu(t_3)\\ &= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds\\ &\le \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)\phi_p(\frac{c}{A})a(r)dr\right)ds\\ &\le \frac{c}{A}\frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)ds = c.\end{aligned}$$

Therefore, $T:\ \overline{P_1(\gamma, c)} \to \overline{P_1(\gamma, c)}$.

Secondly, it is immediate that

$$u_1(t) \in \left\{u \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c):\ \alpha(u) > b\right\} \ne \emptyset,$$

$$u_2(t) \in \{u \in Q(\gamma, \beta, \psi, \delta a, a, c) : \ \beta(u) < a\} \neq \emptyset,$$

where

$$u_1(t) = b + \varepsilon_1 \text{ for } 0 < \varepsilon_1 < \frac{t_2}{t_1} b - b,$$
$$u_2(t) = a - \varepsilon_2 \text{ for } 0 < \varepsilon_2 < a - \delta a.$$

In the following steps, we will verify the remaining conditions of Theorem 2.1.

Step 1. We prove that

$$u \in P(\gamma, \theta, \alpha, b, \frac{t_2}{t_1} b, c) \quad \text{implies} \quad \alpha(Tu) > b. \tag{16}$$

In fact, $u(t) \geq u(t_1) = \alpha(u) \geq b$ and $u(t) \leq u(t_2) = \theta(u) \leq \frac{t_2}{t_1} b$ for $t \in [t_1, t_2]$. By (13),

$$\begin{aligned}
\alpha(Tu) &= \min_{t \in [t_1, t_2]} Tu(t) = Tu(t_1) \\
&= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left(\int_0^1 g(s, r) a(r) f(u(r)) dr \right) ds \\
&\geq \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left(\int_{t_1}^{t_2} g(s, r) a(r) f(u(r)) dr \right) ds \\
&> \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left(\int_{t_1}^{t_2} g(s, r) a(r) \phi_p(\frac{b}{B}) dr \right) ds \\
&= \frac{b}{M_1 \phi_q(M) B} \int_0^1 h(t_1, s) \phi_q \left(\int_{t_1}^{t_2} g(s, r) a(r) dr \right) ds = b.
\end{aligned}$$

Step 2. We prove that

$$u \in Q(\gamma, \beta, \psi, \delta a, a, c) \quad \text{implies} \quad \beta(Tu) < a. \tag{17}$$

In fact, $0 \leq u(t) \leq u(1) = \beta(u) \leq a$ for $t \in [0, 1]$. By (12),

$$\begin{aligned}
\beta(Tu) &= \max_{t \in [\delta, 1]} Tu(t) = Tu(1) \\
&= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left(\int_0^1 g(s, r) a(r) f(u(r)) dr \right) ds \\
&< \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left(\int_0^1 g(s, r) a(r) \phi_p(\frac{a}{C}) dr \right) ds \\
&= \frac{a}{M_1 \phi_q(M) C} \int_0^1 h(1, s) \phi_q \left(\int_0^1 g(s, r) a(r) dr \right) ds = a.
\end{aligned}$$

Step 3. We prove that

$$u \in P(\gamma, \alpha, b, c) \quad \text{with} \quad \theta(Tu) > \frac{t_2}{t_1} b \quad \text{implies} \quad \alpha(Tu) > b. \tag{18}$$

By Lemma 2.3,

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[t_1,t_2]} Tu(t) = Tu(t_1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_1,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right) ds \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(t_1,s)}{h(t_2,s)} h(t_2,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right) ds \\
&\geq \frac{t_1}{t_2} Tu(t_2) = \frac{t_1}{t_2}\theta(Tu) > b.
\end{aligned}$$

Step 4. We prove that

$$u \in Q(\gamma, \beta, a, c) \quad \text{with} \quad \psi(Tu) < \delta a \quad \text{implies} \quad \beta(Tu) < a. \tag{19}$$

By Lemma 2.3,

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[\delta,1]} Tu(t) = Tu(1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(1,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right) ds \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(1,s)}{h(\delta,s)} h(\delta,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right) ds \\
&\leq \frac{1}{\delta} Tu(\delta) = \frac{1}{\delta}\psi(Tu) < a.
\end{aligned}$$

Therefore, the hypotheses of Theorem 2.1 are satisfied and there exist three positive solutions x_1, x_2 and x_3 for BVP $(1)-(2)$ satisfying (15). ■

Similar to Theorem 3.1, let $t_1, t_2, t_3 \in (0,1)$ with $t_1 < t_2$. Define nonnegative continuous concave functionals α, ψ and nonnegative convex functionals β, θ, γ on P_2 by

$$\gamma(u) = \max_{t\in[t_3,1]} u(t) = u(t_3), \quad u \in P_2,$$

$$\psi(u) = \min_{t\in[0,\delta]} u(t) = u(\delta), \quad u \in P_2,$$

$$\beta(u) = \max_{t\in[0,\delta]} u(t) = u(0), \quad u \in P_2,$$

$$\alpha(u) = \min_{t\in[t_1,t_2]} u(t) = u(t_2), \quad u \in P_2,$$

$$\theta(u) = \max_{t\in[t_1,t_2]} u(t) = u(t_1), \quad u \in P_2.$$

by observation, $\alpha(u) = u(t_2) \le u(0) = \beta(u)$ and $||u|| = u(0) \le \frac{1}{t_3}u(t_3) = \frac{1}{t_3}\gamma(u)$ for $u \in P_2$.

Theorem 3.2. *Suppose that* $\xi,\ \eta > 1$ *and* $M > 0$. *There exist positive numbers* $0 < a < b < c$ *such that* $0 < a < b < \frac{1-t_1}{1-t_2}b \le c$ *and* $f(w)$ *satisfies following conditions:*

$$f(w) < \phi_p\left(\frac{a}{C}\right), \quad 0 \le w \le a, \tag{20}$$

$$f(w) > \phi_p\left(\frac{b}{B}\right), \quad b \le w \le \frac{1-t_1}{1-t_2}b, \tag{21}$$

$$f(w) \le \phi_p\left(\frac{c}{A}\right), \quad 0 \le w \le \frac{1}{t_3}c, \tag{22}$$

where A, B *and* C *are defined as follows:*

$$A = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)ds,$$
$$B = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} g(s,r)a(r)dr\right)ds,$$
$$C = \frac{1}{M_1\phi_q(M)}\int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)ds.$$

Then BVP (1)-(2) has at least three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$x_1(t_2) > b,\ x_2(0) < a,\ x_3(t_2) < b,\ x_3(0) > a \text{ and } x_i(\delta) \le c \text{ for } i = 1,2,3. \tag{23}$$

Proof. Define the completely continuous operator $T:\ P_2 \to X$ by

$$Tu(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds.$$

It is easy to know that u is a positive solution of (1)-(2) if and only if u is a fixed point of T on cone P_2.

Firstly, we prove $T:\ \overline{P_2(\gamma,c)} \to \overline{P_2(\gamma,c)}$.

For $u \in P_2$, since $M_1 > 0$ and $M = (1-\xi)(1-\eta) > 0$, it follows that $Tu \ge 0$. Furthermore,

$$\begin{aligned}(Tu)'(t) &= \frac{1-\xi}{M_1\phi_q(M)}\left[\eta\int_0^t \phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds\right.\\ &\qquad \left.+\int_t^1 \phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds\right] \le 0,\end{aligned}$$

$$(Tu)''(t) = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,r)f(u(r))a(r)dr\right) \leq 0.$$

So, $TP_2 \subset P_2$.

For $u \in \overline{P_2(\gamma, c)}$, $0 \leq u(t) \leq ||u|| \leq \frac{1}{t_3}\gamma(u) \leq \frac{1}{t_3}c$. By (22),

$$\begin{aligned}
\gamma(Tu) &= \max_{t\in[t_3,1]} Tu(t) = Tu(t_3) \\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)f(u(r))a(r)dr\right)ds \\
&\leq \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)\phi_p(\frac{c}{A})a(r)dr\right)ds \\
&\leq \frac{c}{M_1\phi_q(M)A}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)ds = c.
\end{aligned}$$

Therefore, $T: \overline{P_2(\gamma, c)} \to \overline{P_2(\gamma, c)}$.

Secondly, it is immediate that

$$u_1(t) \in \{u \in P(\gamma, \theta, \alpha, b, \frac{1-t_1}{1-t_2}b, c): \ \alpha(u) > b\} \neq \emptyset,$$

$$u_2(t) \in \{u \in Q(\gamma, \beta, \psi, (1-\delta)a, a, c): \ \beta(u) < a\} \neq \emptyset,$$

where

$$u_1(t) = b + \varepsilon_1 \text{ for } 0 < \varepsilon_1 < \tfrac{1-t_1}{1-t_2}b - b,$$
$$u_2(t) = a - \varepsilon_2 \text{ for } 0 < \varepsilon_2 < a - (1-\delta)a.$$

In the following steps, we will verify the remaining conditions of Theorem 2.1.

Step 1. We prove that

$$u \in P(\gamma, \theta, \alpha, b, \frac{1-t_1}{1-t_2}b, c) \quad \text{implies} \quad \alpha(Tu) > b. \tag{24}$$

In fact, $u(t) \leq u(t_1) = \theta(u) \leq \frac{1-t_1}{1-t_2}b$ and $u(t) \geq u(t_2) = \alpha(u) \geq b$ for $t \in [t_1, t_2]$. Thus by (21),

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[t_1,t_2]} Tu(t) = Tu(t_2) \\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds \\
&\geq \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} g(s,r)a(r)f(u(r))dr\right)ds \\
&> \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} g(s,r)a(r)\phi_p(\frac{b}{B})dr\right)ds \\
&= \frac{b}{M_1\phi_q(M)B}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} g(s,r)a(r)dr\right)ds = b.
\end{aligned}$$

Step 2. We prove that

$$u \in Q(\gamma, \beta, \psi, (1-\delta)a, a, c) \quad \text{implies} \quad \beta(Tu) < a. \tag{25}$$

In fact, $0 \leq u(t) \leq u(0) = \beta(u) \leq a$ for $t \in [0, 1]$. Thus by (20),

$$\begin{aligned}\beta(Tu) &= \max_{t\in[0,\delta]} Tu(t) = Tu(0)\\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds\\ &< \frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)a(r)\phi_p(\frac{a}{C})dr\right)ds\\ &= \frac{a}{M_1\phi_q(M)C} \int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)a(r)dr\right)ds = a.\end{aligned}$$

Step 3. We prove that

$$u \in P(\gamma, \alpha, b, c) \quad \text{with} \quad \theta(Tu) > \frac{1-t_1}{1-t_2}b \quad \text{implies} \quad \alpha(Tu) > b. \tag{26}$$

By Lemma 2.4,

$$\begin{aligned}\alpha(Tu) &= \min_{t\in[t_1,t_2]} Tu(t) = Tu(t_2)\\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_2,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds\\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(t_2,s)}{h(t_1,s)}h(t_1,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds\\ &\geq \frac{1-t_2}{1-t_1}Tu(t_1) = \frac{1-t_2}{1-t_1}\theta(Tu) > b.\end{aligned}$$

Step 4. We prove that

$$u \in Q(\gamma, \beta, a, c) \quad \text{with} \quad \psi(Tu) < (1-\delta)a \quad \text{implies} \quad \beta(Tu) < a. \tag{27}$$

By Lemma 2.4,

$$\begin{aligned}\beta(Tu) &= \max_{t\in[0,\delta]} Tu(t) = Tu(0)\\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds\\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(0,s)}{h(\delta,s)}h(\delta,s)\phi_q\left(\int_0^1 g(s,r)a(r)f(u(r))dr\right)ds\\ &\leq \frac{1}{1-\delta}Tu(\delta) = \frac{1}{1-\delta}\psi(Tu) < a.\end{aligned}$$

Therefore, the hypotheses of Theorem 2.1 are satisfied and there exist three positive solutions x_1, x_2 and x_3 for BVP $(1)-(2)$ satisfying (23). ∎

Remark. When $0 \leq \xi, \eta < 1$ or $\xi, \eta > 1$, similar to Theorem 3.1 and Theorem 3.2, we can discuss the following four-point fourth-order BVP

$$\begin{cases} (\phi_p(u''(t)))'' - a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \xi u(1),\ u'(1) = \eta u'(0), \\ \alpha_2 u''(\lambda) = \beta_2 u''(\delta),\ u'''(0) = 0, \end{cases}$$

where $f: R \to [0, +\infty)$ and $a: (0,1) \to [0, +\infty)$ are continuous functions, $0 \leq \delta,\ \lambda \leq 1$ and $\phi_p(z) = |z|^{p-2}z$ for $p > 1$. The conclusions are similar to Theorem 3.1 and Theorem 3.2.

References

1. R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive solutions of differential, difference, and integral equations,* Kluwer Academic, Dordrecht, 1999.
2. R. Y. Ma and H. Y. Wang, On the existence of positive solutions of fourth-order ordinary differential equations, *Appl. Anal.*, **59** (1995), 225-231.
3. J. R. Graef and B. Yang, On a nonlinear boundary-value problem for fourth-order equations, *Appl. Anal.*, **72** (1999), 439-448.
4. J. R. Graef and B. Yang, Existence and non-existence of positive solutions of fourth-order nonlinear boundary-value problems, *Appl. Anal.*, **74** (2000), 201-214.
5. R. Ma, J. Zhang and S. Fu, The method of lower and upper solutions for fourth order two-point boundary value problems, *J. Math. Anal. Appl.*, **215** (1997), 415-422.
6. B.G.Zhang and L.Z.Kong, Multiple positive solutions of a class of p-Laplacian equations(in Chinese), *Annals Math.*, **6** (2001), 1-10.
7. R.I.Avery and J.Henderson, Three symmetric positive solutions for a second order boundary value problem, *Appl. Math. Lett.*, **13** (2000), 1-7.
8. P.W.Eloe and J.Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problem, *Diff. Equ. Dynam. Syst.*, **6** (1998), 309-317.
9. R.Y.Ma, Positive solutions of nonlinear three point boundary value problem, *Electronic J. Diff. Equs.*, **34** (1998), 1-8.
10. R.Y.Ma, Existence theorems for a second order three point boundary value problems, *J. Math. Anal. Appl.*, **212** (1997), 430-442.
11. W.Feng and J.R.L.Webb, Solvability of a three point nonlinear boundary value problems at resonance, *Nonlinear Anal. TMA*, **30(6)** (1997), 3227-3238.

De-xiang Ma
Department of Mathematics,
North China Electric Power University,
Beijing 102206,
P. R. China
E-mail: madexiang@sohu.com

Yu Tian and Wei-gao Ge
School of Science,
Beijing University of Posts and Telecommunications,
Beijing 100876,
P. R. China
E-mail: tianyu2992@163.com China