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ITERATION SCHEME FOR A PAIR OF SIMULTANEOUSLY ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

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Abstract. We introduce the notion of a pair of simultaneously asymptotically quasi-nonexpansive type mappings and prove a general strong convergence theorem of the iteration scheme with errors for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in Banach spaces. The result of this paper is an extension and an improvement of the corresponding well known results.

1. Introduction

The concepts of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. The concepts of asymptotically nonexpansive mapping and the asymptotically nonexpansive type mapping were introduced by Goebel-Kirk [4] and Kirk [8], respectively, which are closely related to the theory of fixed points in Banach spaces. Recently, the iterative approximating problem of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings has been studied by many authors (see, for example, [1, 2, 3, 5-7, 9-16] and the references therein).

In this paper, we introduce the notion of a pair of simultaneously asymptotically quasi-nonexpansive type mappings and prove a general strong convergence theorem of the iteration scheme with errors for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in Banach spaces. Our result is an extension and an improvement of the corresponding well known results [1, 2, 3, 5, 6, 7, 9-16].

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2. Preliminaries

Throughout this paper, let X be a real Banach space, D be a nonempty subset of X $S, T : D \to X$ a couple of mappings, $F(T)$ and $F(S)$ the set of fixed points of T and S respectively, that is, $F(T) = \{x \in D : Fx = x\}$ and $F(S) = \{y \in D : Sy = y\}$. Let m and n denote the nonnegative integers.

Definition 2.1. [2, 3, 4, 8, 10] Let $T : D \to X$ be a mapping,

(1) T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in D$;

(2) T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|$$

for all $x \in D$ and $x^* \in F(T)$;

(3) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in D$ and $n \geq 0$;

(4) T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - x^*\| \leq k_n \|x - x^*\|$$

for all $x \in D$, $x^* \in F(T)$ and $n \geq 0$;

(5) T is said to be *asymptotically nonexpansive type* if

$$\limsup_{n\to\infty} \sup_{x\in D} \{\|T^n x - T^n y\|^2 - \|x - y\|^2\} \leq 0$$

for all $y \in D$ and $n \geq 0$;

(6) T is said to be *asymptotically quasi-nonexpansive type* if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in D} \{\|T^n x - x^*\|^2 - \|x - x^*\|^2\} \leq 0$$

for all $x^* \in F(T)$ and $n \geq 0$.

Remark 2.1. It is easy to see that the following implications hold:

$$\begin{array}{ccc} (1) & \stackrel{F(T)\neq\emptyset}{\Longrightarrow} & (2) \\ \Downarrow & & \Downarrow \\ (3) & \stackrel{F(T)\neq\emptyset}{\Longrightarrow} & (4) \\ \Downarrow & & \Downarrow \\ (5) & \stackrel{F(T)\neq\emptyset}{\Longrightarrow} & (6). \end{array}$$

Definition 2.2. Let $S, T : D \to X$ be two mappings. (S, T) is said to be a pair of *simultaneously asymptotically quasi-nonexpansive type mappings* if $F(T) \neq \emptyset$, $F(S) \neq \emptyset$,

$$\limsup_{n\to\infty} \sup_{x\in D} \{\|T^n x - y^*\|^2 - \|x - y^*\|^2\} \leq 0$$

for all $y^* \in F(S)$ and $n \geq 0$, and

$$\limsup_{n\to\infty} \sup_{y\in D} \{\|S^n y - x^*\|^2 - \|y - x^*\|^2\} \leq 0$$

for all $x^* \in F(T)$ and $n \geq 0$.

For our main result, we need the following lemma.

Lemma 2.1. [15] *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two nonnegative sequences satisfying*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq n_0$, *where* $\sum_{n=0}^{\infty} b_n < \infty$ *and* n_0 *is some positive integer. Then the* $\lim_{n\to\infty} a_n$ *exists.*

3. Main Results

The distance from x to the set A denotes by $D(x, A)$, that is, $D(x, A) = \inf_{a\in A} \|x - a\|$ for each $x \in A$.

Theorem 3.1. *Let* X *be a real Banach space,* D *be a nonempty subset of* X, (S, T) *be a pair of simultaneously asymptotically quasi-nonexpansive type mappings on* D. *Assume that there exist constants* L_1, L_2, α' *and* $\alpha'' > 0$ *such that*

$$\|Tx - y^*\| \leq L_1 \|x - y^*\|^{\alpha'}, \quad \forall x \in D, \quad \forall y^* \in F(S), \tag{3.1}$$

and

$$\|Sx - x^*\| \leq L_2 \|x - x^*\|^{\alpha''}, \quad \forall x \in D, \quad \forall x^* \in F(T). \tag{3.2}$$

For any given $x_0 \in D$, the iteration scheme $\{x_n\}$ with errors is defined by

$$\left\{\begin{array}{ll} z_n = (1-\beta_n)x_n + \beta_n S^n x_n + \beta_n v_n, & n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n z_n + \alpha_n u_n, & n \geq 0, \end{array}\right. \tag{3.3}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in D, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n < \infty$. Suppose that $\{y_n\}$ is a sequence in D and define $\{\varepsilon_n\}$ by

$$\left\{\begin{array}{ll} w_n = (1-\beta_n)y_n + \beta_n S^n y_n + \beta_n v_n, & n \geq 0, \\ \varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T^n w_n - \alpha_n u_n\|, & n \geq 0. \end{array}\right. \tag{3.4}$$

If $F(S) \cap F(T) \neq \emptyset$, then we have the following:

(i) *$\{x_n\}$ converges strongly to some common fixed point y^* of S and T if and only if*

$$\liminf_{n\to\infty} D(x_n, F(S)\cap F(T)) = 0.$$

(ii) *$\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\liminf_{n\to\infty} D(y_n, F(S)\cap F(T)) = 0$ imply that $\{y_n\}$ converges strongly to some common fixed point y^* of S and T.*

(iii) *If $\{y_n\}$ converges strongly to some common fixed point y^* of S and T, then $\lim_{n\to\infty} \varepsilon_n = 0$.*

To prove Theorem 3.1, we first give the following lemma.

Lemma 3.1. *Assume all the assumptions in Theorem 3.1 hold and $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. Then for any given $\varepsilon > 0$, there exist a positive integer n_0 and a constant $M > 0$ such that*

(i) $\|y_{n+1} - y^*\| \leq \|y_n - y^*\| + \alpha_n M + \varepsilon_n$, $\forall y^* \in F(S)\cap F(T)$, $n \geq n_0$, *where* $M = 2\varepsilon + \sup_{n\geq 0}\{\|u_n\| + \|v_n\|\} < \infty$,

(ii) $\|y_m - y^*\| \leq \|y_n - y^*\| + M\sum_{k=n}^{m-1} \alpha_k + \sum_{k=n}^{m-1} \varepsilon_k$, $\forall y^* \in F(S)\cap F(T)$, $n \geq n_0$, $m > n$, *where* $M = 2\varepsilon + \sup_{n\geq 0}\{\|u_n\| + \|v_n\|\} < \infty$,

(iii) $\lim_{n\to\infty} D(y_n, F(S)\cap F(T))$ *exists.*

Proof. Take any $y^* \in F(S)\cap F(T)$, it follows from (3.4) that

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq \varepsilon_n + \|(1-\alpha_n)(y_n - y^*) + \alpha_n(T^n w_n - y^*) + \alpha_n u_n\| \\ &\leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n(\|T^n w_n - y^*\| - \|w_n - y^*\|) \\ &\quad + \alpha_n\|w_n - y^*\| + \alpha_n\|u_n\| + \varepsilon_n \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|w_n - y^*\| &= \|(1-\beta_n)(y_n - y^*) + \beta_n(S^n y_n - y^*) + \beta_n v_n\| \\ &\le (1-\beta_n)\|y_n - y^*\| + \beta_n(\|S^n y_n - y^*\| - \|y_n - y^*\|) \\ &\quad + \beta_n\|y_n - y^*\| + \beta_n\|v_n\|.\end{aligned} \tag{3.6}$$

Since (S, T) is a pair of simultaneously asymptotically quasi-nonexpansive type mappings, from Definition 2.2, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{x\in D}\{\|T^n x - y^*\|^2 - \|x - y^*\|^2\} \\ &= \limsup_{n\to\infty}\sup_{x\in D}\{(\|T^n x - y^*\| - \|x - y^*\|)(\|T^n x - y^*\| + \|x - y^*\|)\} \\ &\le 0.\end{aligned}$$

Then we have

$$\limsup_{n\to\infty}\sup_{x\in D}\{\|T^n x - y^*\| - \|x - y^*\|\} \le 0.$$

Let $\varepsilon > 0$, be given. Then there exists a positive integer n_0' such that for any $n \ge n_0'$, we have

$$\sup_{x\in D}\{\|T^n x - y^*\| - \|x - y^*\|\} < \varepsilon. \tag{3.7}$$

Since $\{w_n\}$ is in D, from (3.7) we obtain

$$\|T^n w_n - y^*\| - \|w_n - y^*\| < \varepsilon \tag{3.8}$$

for all $n \ge n_0'$. As the inequality (3.8), there exists a positive integer n_0'' such that for any $n \ge n_0''$,

$$\|S^n y_n - y^*\| - \|y_n - y^*\| < \varepsilon \tag{3.9}$$

for the mapping S.

Let $n_0 = \max\{n_0', n_0''\}$. Substituting (3.6), (3.8) and (3.9) into (3.5), we have

$$\begin{aligned}&\|y_{n+1} - y^*\| \le \|y_n - y^*\| + \alpha_n(2\varepsilon + \|u_n\| + \|v_n\|) + \varepsilon_n, \\ &\forall y^* \in F(S)\cap F(T), n \ge n_0.\end{aligned} \tag{3.10}$$

Set $M = 2\varepsilon + \sup_{n\ge 0}\{\|u_n\| + \|v_n\|\} < \infty$, it follows from (3.10) that

$$\|y_{n+1} - y^*\| \le \|y_n - y^*\| + \alpha_n M + \varepsilon_n, \;\; \forall y^* \in F(S)\cap F(T), \;\; n \ge n_0.$$

The conclusion (1) holds.

From conclusion (1), we have, for any $m > n$,

$$\begin{aligned}\|y_m - y^*\| &\leq \|y_{m-1} - y^*\| + \alpha_{m-1}M + \varepsilon_{m-1} \\ &\leq \|y_{m-2} - y^*\| + \alpha_{m-2}M + \alpha_{m-1}M + \varepsilon_{m-2} + \varepsilon_{m-1} \\ &\leq \cdots \\ &\leq \|y_n - y^*\| + M\sum_{k=n}^{m-1}\alpha_k + \sum_{k=n}^{m-1}\varepsilon_k, \ \ \forall y^* \in F(S)\cap F(T), \ \ n \geq n_0,\end{aligned}$$

which implies that the conclusion (2) holds.

Again, it follows from conclusion (1) that

$$D(y_{n+1}, F(S)\cap F(T)) \leq D(y_n, F(S)\cap F(T)) + \alpha_n M + \varepsilon_n, \ \ n \geq n_0.$$

Since $M < \infty$, $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\sum_{n=0}^{\infty}\varepsilon_n < \infty$, we have

$$\sum_{n=0}^{\infty}(\alpha_n M + \varepsilon_n) < \infty.$$

Thus, Lemma 2.1 implies that the conclusion (3) holds. This completes the proof of Lemma 3.1. ■

Since the Lemma 3.1 holds for an arbitrary sequence $\{y_n\}$ in D, we have the following corollary as the proof of Lemma 3.1.

Corollary 3.1. *Assume all assumptions in Theorem* 3.1 *hold. Then for any given* $\varepsilon > 0$, *there exist a positive integer* n_0 *and a constant* $M > 0$ *such that*

(i) $\|x_{n+1} - y^*\| \leq \|x_n - y^*\| + \alpha_n M$, $\forall y^* \in F(S)\cap F(T)$, $n \geq n_0$, *where* $M = 2\varepsilon + \sup_{n\geq 0}\{\|u_n\| + \|v_n\|\} < \infty$,

(ii) $\|x_m - y^*\| \leq \|x_n - y^*\| + M\sum_{k=n}^{m-1}\alpha_k$, $\forall y^* \in F(S)\cap F(T)$, $n \geq n_0$, $m > n$, *where* $M = 2\varepsilon + \sup_{n\geq 0}\{\|u_n\| + \|v_n\|\} < \infty$,

(iii) $\lim_{n\to\infty} D(x_n, F(S)\cap F(T))$ *exists.*

The Proof of the Theorem 3.1

It is easy to see that the necessity of conclusion (i) is obvious and the sufficiency follows from conclusion (ii). Now, we prove the conclusion (ii). It follows from Lemma 3.1 (3) that $\lim_{n\to\infty} D(y_n, F(S)\cap F(T))$ exists. Since

$$\liminf_{n\to\infty} D(y_n, F(S)\cap F(T)) = 0,$$

we have

$$\lim_{n\to\infty} D(y_n, F(S)\cap F(T)) = 0. \tag{3.11}$$

First, we have to prove that $\{y_n\}$ is a Cauchy sequence in X. In fact, it follows from (3.11), the assumptions $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\sum_{n=0}^{\infty}\varepsilon_n < \infty$ that for any given $\varepsilon > 0$ there exists a positive integer $n_1 \geq n_0$ (where n_0 and M are the positive integers appeared in Lemma 3.1) such that

$$D(y_n, F(S)\cap F(T)) < \varepsilon, \;\; n \geq n_1, \tag{3.12}$$

$$\sum_{n=n_1}^{\infty}\alpha_n < \varepsilon \tag{3.13}$$

and

$$\sum_{n=n_1}^{\infty}\varepsilon_n < \varepsilon. \tag{3.14}$$

By the definition of infimum, it follows from (3.12) that for any given $n \geq n_1$ there exists an $y^*(n) \in F(S)\cap F(T)$ such that

$$\|y_n - y^*(n)\| < 2\varepsilon. \tag{3.15}$$

On the other hand, for any $m, n \geq n_1$, without loss of generality $m > n$, it follows from Lemma 3.1 (2) that

$$\begin{aligned}\|y_m - y_n\| &\leq \|y_m - y^*(n)\| + \|y_n - y^*(n)\| \\ &\leq 2\|y_n - y^*(n)\| + M\sum_{k=n}^{m-1}\alpha_k + \sum_{k=n}^{m-1}\varepsilon_k.\end{aligned} \tag{3.16}$$

Therefore by (3.13)-(3.16), for any $m > n \geq n_1$, we have

$$\|y_m - y_n\| < 4\varepsilon + M\varepsilon + \varepsilon = \varepsilon(5+M),$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$.

Now, we prove that y^* is a fixed point of T. Since $y_n \to y^*$ and $D(y_n, F(S)\cap F(T)) \to 0$ as $n \to \infty$, for any given $\varepsilon > 0$, there exists a positive integer $n_2 \geq n_1 \geq n_0$ such that

$$\|y_n - y^*\| < \varepsilon, \quad D(y_n, F(S)\cap F(T)) < \varepsilon \tag{3.17}$$

for all $n \geq n_2$. The second inequality in (3.17) implies that there exists $y_1^* \in F(S) \cap F(T)$ such that

$$\|y_{n_2} - y_1^*\| < 2\varepsilon. \tag{3.18}$$

Moreover, it follows from (3.7) that

$$\|T^n y^* - y_1^*\| - \|y^* - y_1^*\| < \varepsilon \tag{3.19}$$

for all $n \geq n_2$. Thus, from (3.17)-(3.19), for any $n \geq n_2$, we have

$$\begin{aligned}
\|T^n y^* - y^*\| &\leq \{\|T^n y^* - y_1^*\| - \|y^* - y_1^*\|\} + 2\|y^* - y_1^*\| \\
&< \varepsilon + 2\{\|y^* - y_{n_2}\| + \|y_1^* - y_{n_2}\|\} \\
&< \varepsilon + 2(\varepsilon + 2\varepsilon) = 7\varepsilon,
\end{aligned}$$

which implies that $T^n y^* \to y^*$ as $n \to \infty$. Again since

$$\|T^n y^* - Ty^*\| \leq \{\|T^n y^* - y_1^*\| - \|y^* - y_1^*\|\} + \|y^* - y_1^*\| + \|Ty^* - y_1^*\|$$

for all $n \geq n_2$, by assumption (3.1) and (3.17)-(3.19), we obtain

$$\begin{aligned}
\|T^n y^* - Ty^*\| &< \varepsilon + \|y^* - y_1^*\| + L_1 \|y^* - y_1^*\|^{\alpha'} \\
&\leq \varepsilon + \|y^* - y_{n_2}\| + \|y_1^* - y_{n_2}\| + L_1\{\|y^* - y_{n_2}\| + \|y_1^* - y_{n_2}\|\}^{\alpha'} \\
&< \varepsilon + 3\varepsilon + L_1\,(3\varepsilon)^{\alpha'} = 4\varepsilon + L_1\,(3\varepsilon)^{\alpha'},
\end{aligned}$$

which shows that $T^n y^* \to Ty^*$ as $n \to \infty$. By the uniqueness of limit, we have $Ty^* = y^*$, that is, y^* is a fixed point of T.

Next, we prove that y^* is also a fixed point of S. Since $y_n \to y^*$ and $y^* \in F(T)$, $D(y_n, F(T)) \to 0$ (also follows from $D(y_n, F(S) \cap F(T)) \to 0$ and $D(y_n, F(T)) \leq D(y_n, F(S) \cap F(T))$). Thus, for any given $\varepsilon > 0$, there exists a positive integer $n_3 \geq n_2 \geq n_1 \geq n_0$ such that

$$\|y_n - y^*\| < \varepsilon, \quad D(y_n, F(T)) < \varepsilon \tag{3.20}$$

for all $n \geq n_3$. The second inequality in (3.20) implies that there exists $y_2^* \in F(T)$ such that

$$\|y_{n_3} - y_2^*\| < 2\varepsilon. \tag{3.21}$$

Since (S, T) is a pair of simultaneously asymptotically quasi-nonexpansive type mappings, as the inequality (3.9), we have

$$\|S^n y^* - y_2^*\| - \|y^* - y_2^*\| < \varepsilon \tag{3.22}$$

for all $n \ge n_3$. Thus, from (3.20)-(3.22), for any $n \ge n_3$, we have

$$\begin{aligned}\|S^n y^* - y^*\| &\le \{\|S^n y^* - y_2^*\| - \|y^* - y_2^*\|\} + 2\|y^* - y_2^*\| \\ &< \varepsilon + 2\{\|y^* - y_{n_3}\| + \|y_2^* - y_{n_3}\|\} \\ &< \varepsilon + 2\,(\varepsilon + 2\varepsilon) = 7\varepsilon,\end{aligned}$$

which implies that $S^n y^* \to y^*$ as $n \to \infty$. Again since

$$\|S^n y^* - Sy^*\| \le \{\|S^n y^* - y_2^*\| - \|y^* - y_2^*\|\} + \|y^* - y_2^*\| + \|Sy^* - y_2^*\|$$

for all $n \ge n_3$, by assumption (3.2) and (3.20)-(3.22), we obtain

$$\begin{aligned}\|S^n y^* - Sy^*\| &\le \varepsilon + \|y^* - y_2^*\| + L_2\|y^* - y_2^*\|^{\alpha''} \\ &\le \varepsilon + \|y^* - y_{n_3}\| + \|y_2^* - y_{n_3}\| + L_2\{\|y^* - y_{n_3}\| + \|y_2^* - y_{n_3}\|\}^{\alpha''} \\ &< \varepsilon + 3\varepsilon + L_2\,(3\varepsilon)^{\alpha''} = 4\varepsilon + L_2\,(3\varepsilon)^{\alpha''},\end{aligned}$$

which shows that $S^n y^* \to Sy^*$ as $n \to \infty$. By the uniqueness of limit, we have $Sy^* = y^*$, that is, y^* is also a fixed point of S. Thus, the conclusion (ii) holds.

From (3.4), (3.6), (3.8) and (3.9), we have, for any given $\varepsilon > 0$,

$$\begin{aligned}\varepsilon_n &\le \|y_{n+1} - y^*\| + \|(1-\alpha_n)(y_n - y^*) + \alpha_n(T^n w_n - y^*) + \alpha_n u_n\| \\ &\le \|y_{n+1} - y^*\| + (1-\alpha_n)\|y_n - y^*\| + \alpha_n(\|T^n w_n - y^*\| - \|w_n - y^*\|) \\ &\quad + \alpha_n\|w_n - y^*\| + \alpha_n\|u_n\| \\ &\le \|y_{n+1} - y^*\| + \|y_n - y^*\| + \alpha_n(2\varepsilon + \|u_n\| + \|v_n\|) \\ &\le \|y_{n+1} - y^*\| + \|y_n - y^*\| + \alpha_n M\end{aligned}$$

for all $n \ge n_0$, where $M = 2\varepsilon + \sup_{n\ge 0}\{\|u_n\| + \|v_n\|\} < \infty$. Since $y_n \to y^*$, $M < \infty$ and $\sum_{n=0}^{\infty} \alpha_n < \infty$, it follows that $\lim_{n\to\infty} \varepsilon_n = 0$. Thus the conclusion (iii) holds. This completes the proof of Theorem 3.1. ■

Remark 3.1.

(1) If let $T = S$ in Theorem 3.1, then we obtain the main result of Chang-Kim-Kang [2] for the asymptotically quasi-nonexpansive type mapping.

(2) Theorem 3.1 extends, improves and unifies the corresponding well known results in [1, 2, 3, 5, 6, 7, 9-16].

(3) It is not difficult to extend for the case of finite family of asymptotically quasi-nonexpansive type mappings.

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