TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 4, pp. 1095-1107, June 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON THE CONSTRUCTIONS OF SEQUENTIAL GRAPHS

Hui-Chuan Lu

Abstract. In this paper, we discuss three constructions: attaching construction, adjoining construction and the join of two graphs to obtain larger sequential graphs from small sequential or strongly r-indexable ones.

1. Introduction

All the graphs considered in this paper are finite simple graphs. Let $G = (V(G), E(G))$ be a graph of order p and size q. A *harmonious labeling* of a graph G is an injection $g : V(G) \to \{0, 1, ..., q-1\}$ if the induced labeling on edge set $g^* : E(G) \to \mathbf{Z}_q$ defined by $g^*(uv) \equiv g(u)+g(v) \pmod q$ for each edge $uv \in E(G)$ is 1-1 when G is not a tree. In the case of trees, the vertices are allowed to be labeled from 0 to q without repeating vertex labels, and the resulting edge labels are distinct. A graph G with a harmonious labeling is called a *harmonious graph*.

Harmonious labeling on graphs was introduced in 1980 by Graham and Sloane [8] which is related to a problem of modular versions of certain additive bases for the integers. Results on harmonious graphs can be found in a survey by Gallian [5].

A vertex labeling $f : V(G) \to \{0, 1, ..., q-1\}$ of G ($f : V(G) \to \{0, 1, ..., q\}$ when G is a tree) is called *strongly r-harmonious* [4] or *r-sequential* if f is an injection and the edge labels induced by $f^{\#}(uv) = f(u) + f(v)$ for every edge uv form a sequence of distinct consecutive integers $r, r+1, r+2, \ldots, r+q-1$. A graph G is a *sequential* graph [7] if G is r-sequential for some positive integer r. Note that a sequential graph is clearly a harmonious graph.

In [1], Acharya and Hegde introduced a stronger form of sequential labeling by calling G *strongly r-indexable* if there is an injection function from $V(G)$ to $\{0, 1, \ldots, p-1\}$ such that the set of edge labels induced by adding the vertex labels

Received January 27, 2005; revised May 4, 2005.
Communicated by Xuding Zhu.
2000 *Mathematics Subject Classification*: 05C78.
Key words and phrases: Harmonious labeling, r-Sequential labeling, Strongly r-indexable labeling, 3-Colorable.

is $\{ r, r+1, ..., r+q-1\}$. Notice that for trees and unicycle graphs the notions of r-sequential labelings and strongly r-indexable labelings coincide. Results on sequential graphs and strongly r-indexable graphs can also be found in [5].

Most results on constructing larger sequential graphs focus on particular classes of graphs and methods [cf. 5]. In the present paper, we shall use three kinds of graph operations to cluster sequential or strongly r-indexable graphs together and obtain new classes of sequential graphs as the case may be. It is worth of mentioning that the first two have been utilized to obtain harmonious and sequential trees in [11].

2. Construction I: Attaching Construction

Suppose that $G_1, G_2, ..., G_n$ and H are vertex-disjoint graphs. Let $V(H) = \{w_1, w_2, \ldots, w_n\}$ and let v_i be any vertex of G_i, $i = 1, 2, \ldots, n$. Attaching construction attaches graph G_i to H at w_i by identifying v_i and w_i, for each i = 1, 2, …, n. We denote the resulting graph by $H \oplus [G_1, G_2, ..., G_n]$ at $[v_1, v_2, ..., v_n]$ which is depicted in Figure 1. In particular, if $G_i \cong G$ for all i = 1, 2, …, n and $v_1, v_2, ..., v_n$ are isomorphic to the same vertex v in $V(G)$, then the graph is denoted by $H \oplus [G]_v$. Furthermore, we use $H \oplus [G]$ to denote the collection of $|V(G)|$ graphs $H \oplus [G]_v$ where $v \in V(G)$.

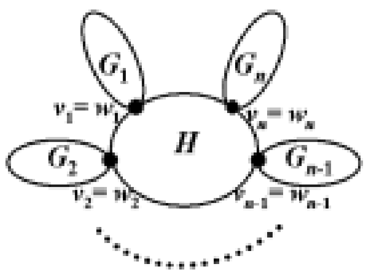


Fig. 1. $H \oplus [G_1, G_2, ..., G_n]$ at $[v_1, v_2, ..., v_n]$

We consider the graph $H \oplus [G]$ where H is of order $2n$+1 and there are $2n$+1 isomorphic copies of G. In the following several theorems, graph G is required to be 3-*colorable* whose definition is given as follows.

An *m-coloring* of a graph G is a vertex labeling $\varphi : V(G) \to \{1, 2, \ldots, m\}$. The labels are *colors.* An m-coloring is *proper* if adjacent vertices in G have different labels. We call a graph *m-colorable* if it has a proper m-coloring. Graphs that can be properly colored by less than four colors may serve as good candidates of G to obtain good labelings of $H \oplus [G]$.

Theorem 2.1. *Let H be strongly r-indexable of order* $2n{+}1$ *and size t, and let G be a* 3*-colorable graph of size q having a k-sequential labeling g. Then we have the following:*

(a) *If* $r{+}t = 3n{+}1$ *and* $g(u_0) = (k-1)/2$, *then* $H{\oplus}[G]_{u_0}$ *is* $(k\,(2n+1){+}n{-}t)$*-sequential.*

(b) *If* $r = n$ *and* $g(u_0) = (k+q)/2$, *then* $H \oplus [G]_{u_0}$ *is* $(k\,(2n+1) + n)$*-sequential.*

Proof. Since H with its labeling h is strongly r-indexable and G with its labeling g is k-sequential, we have $h(V(H))= \{0, 1, 2, \ldots, 2n\}$, $h^{\#}(E(H))= \{r, r{+}1, \ldots, r+t-1\}$ and $g^{\#}(E(G))= \{k, k{+}1, \ldots, k+q-1\}$. Let $\{w_1, w_2, \ldots, w_{2n+1}\}$ be the vertex set of H with $h(w_i) = i-1$, i = 1, 2, $\ldots$, $2n+1$, and $\{u_0, u_1, \ldots, u_{p-1}\}$ be the vertex set of G. For the sake of convenient notation, the i-th copy of G is called G_i which is attached to the vertex w_i of H. Let $\{u_{i,0}, u_{i,1}, \ldots, u_{i,p-1}\}$ be the vertex set of G_i such that $u_{i,j}$ is the isomorphic image of u_j for j= 0, 1, 2, $\ldots$, $p-1$ and i = 1, 2, $\ldots$, $2n+1$. Furthermore, let φ be a proper 3-coloring of G.

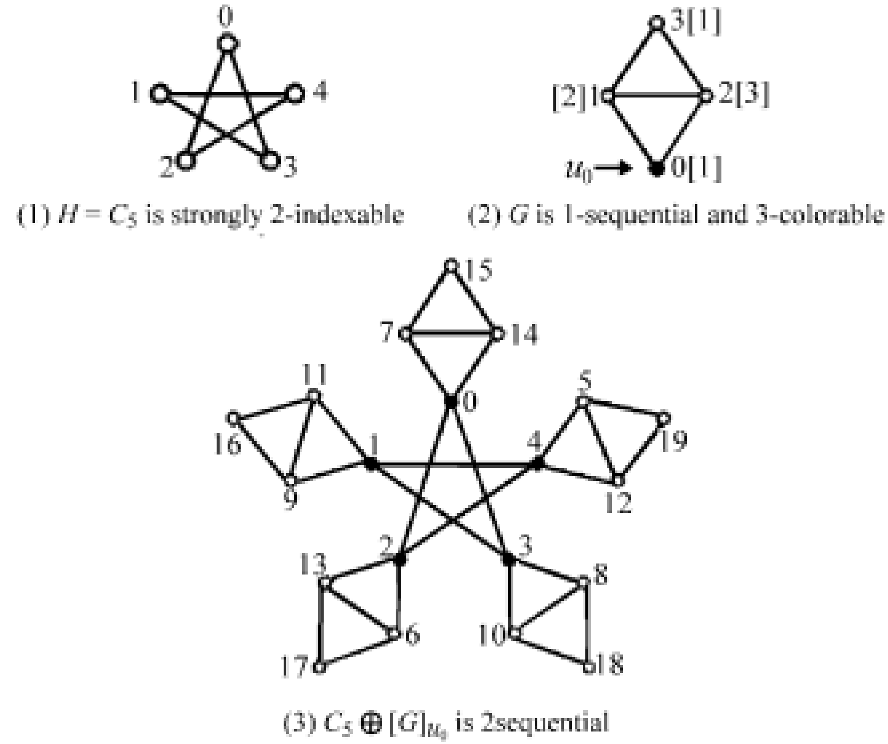
In case (a), when $r+t = 3n+1$, we identify $u_{i,0}$ with w_i, i = 1, 2, $\ldots$, $2n+1$, where $g(u_0) = (k-1)/2$. Without loss of generality, we may assume that $\varphi(u_0)$ = 1.

Let the labeling f of $H \oplus [G]_{u_0}$ be defined as follows: $(0 \le j \le p-1)$

$$f(u_{i,j}) = \begin{cases} g(u_j)(2n+1)+i-1, & \text{if } \varphi(u_j)=1 \quad \text{and } 1 \le i \le 2n+1; \\ g(u_j)(2n+1)+n-(i-1)/2, & \text{if } \varphi(u_j)=2 \quad \text{and } i=1,3,...,2n+1; \\ g(u_j)(2n+1)+2n+1-i/2, & \text{if } \varphi(u_j)=2 \quad \text{and } i=2,4,...,2n; \\ g(u_j)(2n+1)+n-i/2, & \text{if } \varphi(u_j)=3 \quad \text{and } i=2,4,...,2n; \text{ and} \\ g(u_j)(2n+1)+2n-(i-1)/2, & \text{if } \varphi(u_j)=3 \quad \text{and } i=1,3,...,2n+1. \end{cases}$$

It's easy to verify that all vertex labels are distinct. To complete the proof, we need to show that the edge labels are consecutive integers from $(k(2n{+}1){+}n{-}t)$ to $((k+q-1)(2n+1)+3n)$. First note that $f^{\#}(E(H)) = \{\,2g(u_0)(2n+1)+j \mid j = r, r+1, \ \ldots, \ r+t-1\,\}$. The facts $r+t = 3n+1$ and $g(u_0) = (k-1)/2$ imply that $f^{\#}(E(H)) = \{k(2n{+}1){+}n{-}t,\ k(2n{+}1){+}n{-}t{+}1, ...,\ k(2n{+}1){+}n{-}1\}$. Next, let's take an arbitrary edge, say $u_a u_b$, $0 \le a < b \le p-1$, of G and investigate the edge labels of all its isomorphic images $u_{i,a}u_{i,b}$, i = 1, 2, $\ldots$, $2n+1$. A routine computation shows, no matter what colors the vertices u_a and u_b are, the labels of these edges are $g^{\#}(u_a u_b)(2n+1)+n$, $g^{\#}(u_a u_b)(2n+1)+n+1$, $\ldots$, $g^{\#}(u_a u_b)(2n+1)+3n$. Since $g^{\#}(E(G)) = \{k,\ k{+}1, \ldots,\ k+q-1\}$, the edge labels of all G_i's are consecutive integers from $(k(2n+1)+n)$ to $((k+q-1)(2n+1)+3n)$. Hence the proof is completed. Case (b) is similar to case (a) and the proof is omitted. ■

We present an example to illustrate the labeling in Figure 2. The value in brackets indicates the color of that vertex.



(1) $H = C_5$ is strongly 2-indexable (2) G is 1-sequential and 3-colorable

(3) $C_5 \oplus [G]_{u_0}$ is 2sequential

Fig. 2.

Remark.

(1) Theorem 2.1 generalizes Theorem 2.1 in [15].

(2) If G is strongly k-indexable instead of just being k-sequential in Theorem 2.1, then $H \oplus [G]_{u_0}$ is also strongly m-indexable for some values of m as indicated in the theorem.

Among known 1-sequential graphs are wheels $W_m(m \equiv 0$ or 1 (mod 3)) [4]; friendship graphs $C_3^{(m)}$ ($m \equiv 0$ or 1 (mod 4)) [9,10]; stars S_m; fans F_m [4]; helms H_m [12]; and $S_m + \overline{K_t}$[6]. All but W_{2m+1} and H_{2m+1} are 3-colorable. Therefore, by Theorem 2.1, the following corollary is straightforward.

Corollary 2.2. *Suppose that H is a strongly r-indexable graph of order* $2n+1$ *and size t with* $r+t = 3n+1$. *If* u_0 *is the vertex of the latter graph in the following that receives the label 0, then*

(a) *If* $m \equiv 0$ *or* 4 (mod 6), *then* $H \oplus [W_m]_{u_0}$ *is* $(3n - t + 1)$*-sequential.*

(b) *If* $m \equiv 0$ *or* 1 (mod 4), *then* $H \oplus [C_3^{(m)}]_{u_0}$ *is* $(3n - t + 1)$*-sequential.*

(c) $H \oplus [S_m]_{u_0}, H \oplus [F_m]_{u_0}, H \oplus [H_{2m}]_{u_0}$ *and* $H \oplus [S_m + \overline{K_t}]_{u_0}$ *are* $(3n-t+1)$*-sequential , for all* $m \geq 2$.

If we restrict the case to sequential cycles, that is, odd cycles. Theorem 2.1 says that $C_{2n+1} \oplus [C_{4m+3}]$ is sequential and therefore harmonious. But it doesn't work for $C_{2n+1} \oplus [C_{4m+1}]$. In fact, $C_{2n+1} \oplus [C_{4m+1}]$ is not harmonious because it violates the following theorem by Graham and Sloane. So Theorem 2.4 is obvious.

Theorem 2.3.[8, Theorem 11]. *If a harmonious graph has 2t edges and the degree of every vertex is divisible by* 2^k, *then t is divisible by* 2^k.

Theorem 2.4. $C_{2n+1} \oplus [C_{2m+1}]$ *is harmonious if and only if m is odd.*

From the proof of Theorem 2.1, we observe that if we restrict f to the $2n$+1 copies of G, f remains a sequential labeling of $(2n+1)G$(the disjoint union of $2n+1$ copies of G). In the same way, it is harmonious if G is just harmonious.

Theorem 2.5.

(*1*) *If G is harmonious and 3-colorable, then* $(2n+1)G$ *is harmonious and 3-colorable.*

(*2*) *If G is k-sequential and 3-colorable, then* $(2n+1)G$ *is* $(k(2n+1)+n)$*-sequential and 3-colorable.*

(*3*) *If G is strongly k-indexable and 3-colorable, then* $(2n+1)G$ *is strongly (*$k(2n+1)+n$*)-indexable and 3-colorable.*

The case when H is of even order is more challenging. We start with showing that $P_2 \oplus [K_n]$ is not sequential and $2K_n$ is not harmonious for infinitely many n.

Lemma 2.6. [2, §18, Theorem 1]. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$,*where* $p_1, p_2, \ldots, p_m$ *are distinct primes. Then n is not a sum of two squares if and only if there is a prime*

(*) $$p_i \equiv 3 \text{ (mod 4) } \textit{and } \alpha_i \textit{ is odd.}$$

Theorem 2.7. *If* $n \equiv 1 \pmod 4$ *and satisfies condition* $(*)$, *then* $P_2 \oplus [K_n]$ *is not sequential.*

Proof. Let $\{ u_{i,j} | \, j = 0, 1, ..., n-1\}$ be the vertex set of the i-th copy of K_n, i =1, 2, and $E(P_2) = \{u_{1,0}u_{2,0}\}$. Suppose that f is an r-sequential labeling of $P_2 \oplus [K_n]$. Then

$$\begin{aligned} & n(f(u_{1,0}) + f(u_{2,0})) + (n-1)(f(u_{1,1}) + f(u_{1,2}) + \cdots \\ & \quad + f(u_{1,n-1}) + f(u_{2,1}) + f(u_{2,2}) + \cdots + f(u_{2,n-1})) \\ & = r + (r+1) + \cdots + (r + n(n-1)) \\ & = (n(n-1)+1)r + \frac{n(n-1)(n(n-1)+1)}{2} \end{aligned}$$

By taking modulo $(n-1)/2$, we have $f^{\#}(u_{1,0}u_{2,0}) \equiv r \; (\bmod \; (n-1)/2)$. Since $(n-1)/2$ is even, $f^{\#}(u_{1,0}u_{2,0})$ and r have the same parity.

Let $E_i(O_i)$ be the cardinality of the set of vertices of the ith copy of K_n labeled even (odd), i= 1, 2. Then $E_1 + O_1 = E_2 + O_2 = n$ and, in $2K_n$, the numbers of edges labeled odd and even are equal. That is, $E_1(E_1-1)/2 + O_1(O_1-1)/2 + E_2(E_2-1)/2 + O_2(O_2-1)/2 = E_1O_1 + E_2O_2$, or equivalently, $(E_1 - O_1)^2 + (E_2 - O_2)^2 = 2n$. Since n satisfies condition $(*)$, $2n$ is not a sum of two squares. This gives a contradiction. ■

By similar argument we come to the conclusion as follows:

Lemma 2.8. *If n satisfies condition $(*)$, then $2K_n$ is not harmonious.*

Next, some results on graphs obtained by attaching an even number of particular classes of graphs to a path are given.

Theorem 2.9.

(*1*) $P_{2n} \oplus [C_{2m+1}]$ *is $2mn$-sequential, for all $n \geq 1$ and m = 1, 2.*

(*2*) $P_{2n} \oplus [C_{2m+1} + K_1]_w$ *is harmonious where $V(K_1) = \{w\}$, for all $m \geq 1$ and n =1, 2.*

Proof. Suppose that the vertices of C_{2m+1} run consecutively u_0, u_1, $\ldots$, u_{2m} with u_{2m} joined to u_0 and the vertex set of the i-th copy of C_{2m+1} is $\{u_{i,j} \, | j \in \mathbf{Z}_{2m+1}\}$ where $u_{i,j}$ is the isomorphic image of u_j, $i = 1, 2, ..., 2n$.

(1) Let $E(P_{2n}) = \{ u_{i,0}u_{i+1,0} | \, i = 1, 2, ..., 2n-1\}$. We start with defining a labeling $\overline{f}_m$ of $2nC_{2m+1}$ as follows, m = 1, 2.

$$\overline{f}_1(u_{i,j}) = \begin{cases} i - 1 + jn, & \text{if } i = 1, 2, ..., 2n, \quad \text{and} \quad j = 0, 2; \text{ and} \\ 8n - 2i, & \text{if } i = 1, 2, ..., 2n, \quad \text{and} \quad j = 1; \end{cases}$$

and

$$\overline{f}_2(u_{i,j}) = \begin{cases} i-1+jn, & \text{if } i=1,2,...,2n, \text{ and } j=0,2,4; \\ 8n+i-2-(2n-1)(j-1)/2, & \text{if } i=1,2,...,n, \text{ and } j=1,3; \text{ and} \\ 10n+i-2-(2n-1)(j-1)/2, & \text{if } i=n+1,n+2,...,2n, \text{ and } j=1,3. \end{cases}$$

Then we convert $\overline{f}_m$ into the desired labeling f_m of $P_{2n} \oplus [C_{2m+1}]$, m= 1, 2.

$$f_m(u_{i,j}) = \begin{cases} \overline{f}_m(u_{i,j}), & \text{if } i=1,3,...,2n-1, \quad \text{and} \quad j \in \mathbf{Z}_{2m+1}; \quad \text{and} \\ \overline{f}_m(u_{i,j-1}), & \text{if } i=2,4,...,2n. \quad \text{and} \quad j \in \mathbf{Z}_{2m+1}. \end{cases}$$

(2) Let the vertex set of the i-th copy of $C_{2m+1}+K_1$ be $\{w_i, u_{i,j} | j \in \mathbf{Z}_{2m+1}\}$ where w_i is the isomorphic image of w. The labeling f_n of $P_{2n} \oplus [C_{2m+1}+K_1]_w$, n= 1, 2, is defined by

$$f_1(u) = \begin{cases} \overline{f}(u_j)+(3m+2)(i-1), & \text{if } u=u_{i,j}, \quad i=1,2, \quad \text{and } j \in \mathbf{Z}_{2m+1}; \\ (4-i)m+1, & \text{if } u=w_i, \quad i=1,2. \end{cases}$$

and

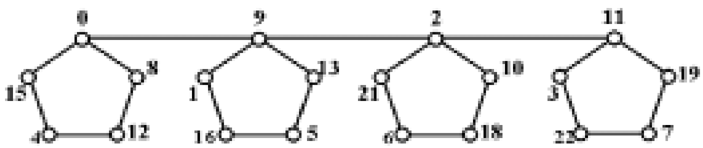
$$f_2(u) = \begin{cases} \overline{f}(u_j)+(2m+1)(i-1)/2, & \text{if } u=u_{i,j}, \quad i=1,3,...,2n-1, j \in \mathbf{Z}_{2m+1}; \\ \overline{f}(u_j)+3m+3+(2m+1)i/2, & \text{if } u=u_{i,j}, \quad i=2,4,...,2n, j \in \mathbf{Z}_{2m+1}; \\ 9m+6+(m+1)(i-1), & \text{if } u=w_i, \quad i=1,3,...,2n-1; \text{ and} \\ 14m+8+(m+1)(i-2), & \text{if } u=w_i, \quad i=2,4,...2n. \end{cases}$$

where $\overline{f}$ is a sequential labeling of C_{2m+1} giving by

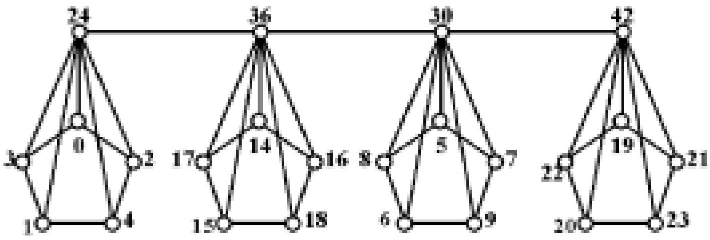
$$\overline{f}(u_j) = \begin{cases} j/2, & \text{if } j=0,2,...,2m; \text{ and} \\ m+(j+1)/2, & \text{if } j=1,3,...,2m-1. \end{cases}$$

The remainder of the verification is just a routine computation. ■

For clarity, we present examples in Figure 3.



(1) $P_4 \oplus [C_5]$ is 8-sequential.



(2) $P_4 \oplus [C_5 + K_1]_w$ is harmonious.

Fig. 3.

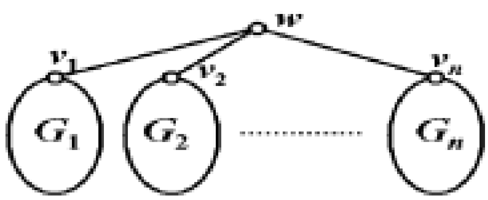
3. Construction II : adjoining Construction

The adjoining construction is in fact a special attaching-type operation. It attaches a collection of vertex-disjoint graphs at each pendant vertex of a star instead of each vertex of H. In other words, we adjoin a vertex w to a vertex u_i of each graph G_i in the collection of vertex-disjoint graphs $\{G_1, G_2, ..., G_n\}$ and denote the resulting graph by $\oplus(G_1, G_2, ..., G_n)$ at $(v_1, v_2, ..., v_n)$, see Figure 4. In the case that each $G_i \cong G$, i =1, 2,. . . ,n, and $v_1, v_2, ..., v_n$ are isomorphic images of the vertex v in $V(G)$, the graph is denoted by $\oplus G^n\,|_v$. Furthermore, we use $\oplus G^n$ to denote the collection of $|V(G)|$ graphs $\oplus G^n\,|_v$ where $v \in V(G)$.

Theorem 3.1. *Suppose that G is* 3*-colorable and has a k-sequential labeling g. If* $g(u_0) = k$, *then* $\oplus G^{2n+1}\,|_{u_0}$ *is* $(k(2n+1)+n)$*-sequential.*

Proof. By labeling the $2n+1$ copies of G the same as in Theorem 2.1 and labeling the new vertex w by $q(2n+1)+n$, we have the desired labeling. ∎

Remark. By the same way of labeling the $2n$ copies of C_3 in $P_{2n} \oplus [C_3]$ in Theorem 2.9 (1) and labeling the new vertex by $6n-1$, we also have that $\oplus\,(C_3)^{2n}$ is harmonious.



$\oplus(G_1, G_2, ..., G_n)$ at $(v_1, v_2, ..., v_n)$

Fig. 4.

4. Construction III: The Join of Two Graphs

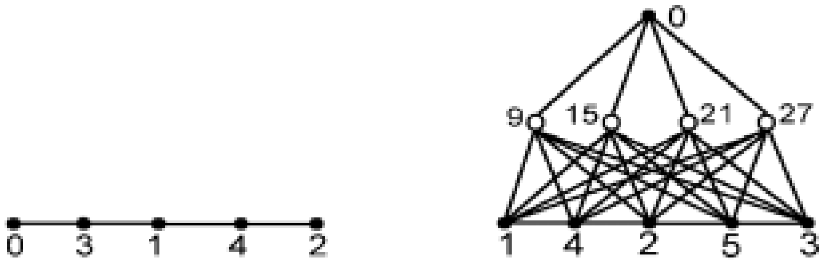
The join of two graphs H and G, written $H + G$, is obtained from the disjoint union of G and H by adding the edges $\{uv \mid u \in V(G),\ v \in V(H)\}$.

Not many results on the joins of graphs have been obtained so far. Most of them consider the join of two particular classes of graphs, such as $P_n + \overline{K_t}$[6], $S_n + \overline{K_t}$[6], $C_{2n+1} + \overline{K_t}$[13] and $P_n + K_2$[14]. We give some generalized results in the following.

Theorem 4.1. *If G is strongly r-indexable, then* $(G \cup \overline{K_t}) + \overline{K_m}$ *is* $(r + 2t)$-*sequential, for all* $m \geq 1$ *and* $t \geq 0$.

Proof. Suppose that $V(\overline{K_m}) = \{v_0, v_1, ..., v_{m-1}\}$, $V(\overline{K_t}) = \{w_0, w_1, ..., w_{t-1}\}$, $V(G) = \{u_0, u_1, ..., u_{p-1}\}$, $|E(G)| = q$ and g is a labeling of G which is strongly r-indexable. The labeling $f$of $(G \cup \overline{K_t}) + \overline{K_m}$ defined in the following is $(r + 2t)$-sequential. (An example is presented in Figure 5.)

$$f(u) = \begin{cases} i, & \text{if } u = w_i, \quad \text{and} \quad i = 0, 1, ..., t-1; \\ g(u_i) + t, & \text{if } u = u_i, \quad \text{and} \quad i = 0, 1, ..., p-1; \text{ and} \\ r + q + 2t + i(p + t), & \text{if } u = v_i, \quad \text{and} \quad i = 0, 1, ..., m-1. \end{cases}$$ ■



(1) $G = P_5$ (2) $(P_5 \cup \overline{K_1}) + \overline{K_4}$ is 5-sequential.

Fig. 5.

Combining Theorem 4.1 with Theorem 2.5, we can construct larger harmonious graphs as follows.

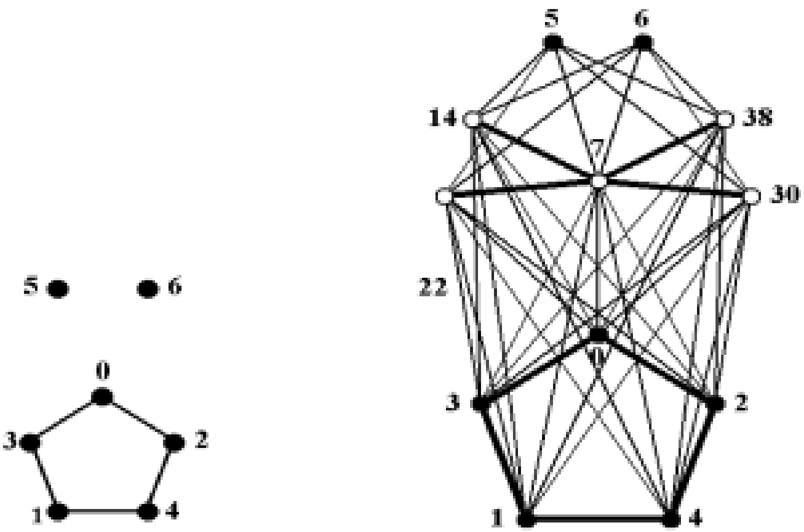
Corollary 4.2. $((2n+1)G \cup \overline{K_t}) + \overline{K_m}$ *is* $(r(2n+1)+n+2t)$-*sequential for all* $m \geq 1$ *and* $t \geq 0$ *provided that G is strongly r-indexable and* 3-*colorable.*

We next try to replace the $\overline{K_m}$ by S_m in $(G \cup \overline{K_t}) + \overline{K_m}$. In this case, one more condition has to be satisfied.

Theorem 4.3. *Let G be a strongly r-indexable graph of order p and size q. If* $p = r + q$, *then* $G + S_m$ *is r-sequential for all* $m \geq 1$.

Proof. Suppose that $V(G) = \{u_0, u_1, ..., u_{p-1}\}$, g is a labeling of G which is strongly r-indexable, and $V(S_m) = \{v_0, v_1, ..., v_m\}$ where v_0 is the center of S_m. Then the labeling f of $G + S_m$ defined as follows is an r-sequential labeling. (See Figure 6 for illustration)

$$f(u) = \begin{cases} g(u_i), & \text{if } u = u_i, \text{ and } i = 0, 1, ..., p-1; \\ p, & \text{if } u = v_0; \text{ and} \\ 2p + (i-1)(p+1), & \text{if } u = v_i, \text{ and } i = 1, ..., m. \end{cases}$$ ■



(1) $G = C_5 \cup \overline{K_2}$ is strongly 2-indexable. (2) $(C_5 \cup \overline{K_2}) + S_4$ is 2-sequential.

Fig. 6.

As in the example in Figure 6, most graphs that satisfy the condition $p = r + q$ are in fact disconnected. We can make a known strongly r-indexable graph satisfy this condition by adding some extra isolated vertices to it.

Corollary 4.4. *If G is a strongly r-indexable graph of order p and size q, then* $(G \cup \overline{K_{r+q-p}}) + S_m$ *is r-sequential for all* $m \geq 1$.

Corollary 4.5. $S_m + S_n$ *is* 1*-sequential for all* $m,\ n \geq 1$.

Remark. $K_{1,m,n}$in [3] is isomorphic to $S_m + \overline{K_n}$ and $K_{1,1,m,n}$ in [6] is exactly $S_m + S_n$. Both graphs are shown to be 1-sequential in this paper.

Sethuraman and Selvaraju [14] proved that $P_n + P_2$ is harmonious and they asked whether $P_n + P_m$ or $P_n + S_m$ is harmonious. By Theorem 4.3, we have

$S_n + P_3 = S_n + S_2$ is harmonious. But we could not go any further at this moment. However, we observed that $C_{2n} + P_2$ is harmonious even though C_{2n} is not harmonious.

Theorem 4.6.

(1) $C_{2n} + S_1 = C_{2n} + P_2$ *is harmonious.*

(2) $C_{4n+3} + S_{2m+1}$ *is not harmonious.*

Proof. (1) Let the vertices of C_n run consecutively $u_0,\ u_1,\ \ldots,\ u_{n-1}$ with u_{n-1} joined to u_0 and $V(P_2) = \{w_1,\ w_2\}$. We split the proof into 3 cases:
(a) For $C_{4t} + P_2$, $t \geq 1$, let the labeling f be given by

$$f(u) = \begin{cases} j/2, & \text{if } u = u_j \quad \text{and } j = 0, 2, ..., 2t-2; \\ 2t + (j-1)/2, & \text{if } u = u_j \quad \text{and } j = 1, 3, ..., 2t-1; \\ (j-1)/2, & \text{if } u = u_j \quad \text{and } j = 2t+1, 2t+3, ..., 4t-1; \\ 2t + j/2, & \text{if } u = u_j \quad \text{and } j = 2t, 2t+2, ..., 4t-2; \text{ and} \\ 6t + 4t(i-1), & \text{if } u = w_i \quad \text{and } i = 1, 2. \end{cases}$$

(b) For $C_{8t+6} + P_2$, $t \geq 0$, we define the desired labeling as follows:

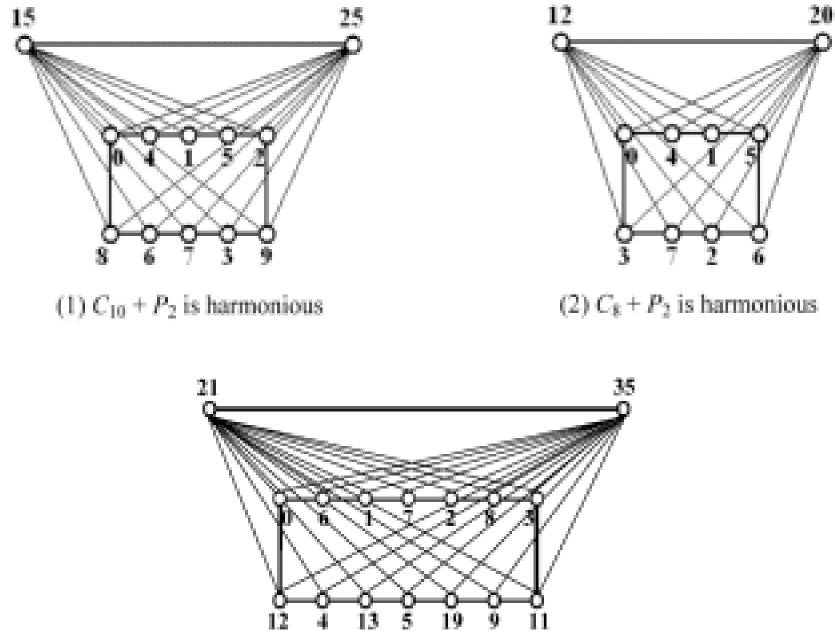
$$f(u) = \begin{cases} j/2, & \text{if } u = u_j, j = 0, 2, ..., 4t+2; \\ 4t + 2 + (j-1)/2, & \text{if } u = u_j, j = 1, 3, ..., 4t+1; \\ 6t + 3, & \text{if } u = u_{4t+4}, \\ 4t + 3 + (j-1)/2 - (-1)^{(j-1)/2}, & \text{if } u = u_j, j = 4t+3, 4t+5, ..., 8t+5; \\ j/2 - 1 - (-1)^{j/2}, & \text{if } u = u_j, j = 4t+6, 4t+8, ..., 8t+4; \\ 12t + 9 + (8t+6)(i-1), & \text{if } u = w_i, i = 1, 2. \end{cases}$$

(c) For $C_{8t+2} + P_2$, the labeling of $C_{10} + P_2$ is given in Figure 7. When $t \geq 2$, the labeling is defined by

$$f(u) = \begin{cases} j/2, & \text{if } u = u_j, \text{ and } j = 0, 2, ..., 4t; \\ 4t + (j-1)/2, & \text{if } u = u_j, \text{ and } j = 1, 3, ..., 4t-1; \\ 14t - 2j + 4, & \text{if } u = u_j, \text{ and } j = 4t+1, 4t+2; \\ 6t + 2 + (-1)^{(j-1)/2}, & \text{if } u = u_j, \text{ and } j = 4t+3, 4t+5; \\ 2t+2, & \text{if } u = u_{4t+4}, \\ 2t + 2 + (-1)^{j/2}, & \text{if } u = u_j, \text{and } j = 4t+6, 4t+8; \\ 4t+1+(j-1)/2 - (-1)^{(j-1)/2}, & \text{if } u = u_j, \text{ and } j = 4t+7, 4t+9, ..., 8t+1; \\ j/2 - 1 - (-1)^{j/2}, & \text{if } u = u_j, \ t \geq 3 \text{and} j = 4t+10, 4t+12, ..., 8t; \\ 12t + 3 + (8t+2)(i-1), & \text{if } u = w_i, \text{ and } i = 1, 2. \end{cases}$$

(2) A direct consequence of Theorem 2.3. ■

We also present examples of the cases $C_{4t} + P_2$ and $C_{8t+6} + P_2$ in Figure 7.



(1) $C_{10} + P_2$ is harmonious (2) $C_8 + P_2$ is harmonious

(3) $C_{14} + P_2$ is harmonious

Fig. 7.

ACKNOWLEDGEMENT

The author would like to thank the referee for his/her helpful suggestions.

REFERENCES

1. B. D. Acharya and S. M. Hegde, Arithmetic graphs, *J. Graph Theory* **14** (1990), 275-299.
2. U. Dudley, "Elementry Number Theory", W.H. Freeman and Co., 1969.
3. R. Aravamudhana ad M. Murugan, numbering of the vertices of $K_{a,1,b}$, preprint.
4. G. J. Chang, D. F. Hsu and D. G. Rogers, Additive variations on a graceful theme: some results on harmonious and other related graphs, *Congress. Numer.* **32** (1981), 181-197.

5. J. A. Gallian, A dynamic survey of graph labeling, *Electronic J. Comb.,* Dynamic Survey DS6, http://www.combinatorics.org .

6. R. B. Gnanajothi, Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, 1991.

7. T. Grace, On sequential labelings of graphs, *J. Graph Theory* **7** (1983),195-201.

8. R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Meth.* **1** (1980), 382-404.

9. D. F. Hsu, Harmonious labelings of windmill graphs and related graphs, *J. Graph Theory* **6** (1982), 85-87.

10. Q. D. Kang and X. Zhao, Strongly harmonious labelings of windmill graphs, *J. Hebei Normal College* 2 (1992), 1-7.

11. Hui-Chuan Lu and Hung-Lin Fu, New results on harmonious trees, *Utilitas Math.*, to appear.

12. I. Rajasingh and P. R. L. Pushpam, Strongly harmonious labelings of helms, preprint.

13. M. Seoud and M. Youssef, The effect of some operations on labeling of graphs, *Proc. Egyptian Phys. Math. Soc.*, to appear.

14. G. Sethuraman and P. Selvaraju, New classes of graphs on graph labeling, preprint.

15. G. S. Singh, A note on labeling of graphs, *graphs and Combin.* **14** (1998), 201-207.

Hui-Chuan Lu
Center of General Education,
National United University,
Miaoli, Taiwan, R.O.C.
E-mail: hht0936@seed.net.tw